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Artículo de investigación

Some Tauberian theorems for Weighted means of triple integrals

Algunos teoremas de Tauber para medias ponderadas de integrales triples

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Abstract

In this paper, we have extend some Tauberian theorems given for (C,α,β,γ) integrability method to the weighted mean method of type (α,β,γ) determined by the functions $p(x),q(y)$ and $u(z)$.

Keywords: Divergent integrals, weighted means of triple integrals, Tauberian theorems, Tauberian conditions.

Resumen

En este artículo, hemos extendido algunos teoremas tauberianos dados para el método de integrabilidad (C,α,β,γ) al método de media ponderada de tipo (α,β,γ) determinado por las funciones $p(x),q(y)$ y $u(z)$.

Palabras Clave: Integrales divergentes, medias ponderadas de integrales triples, teoremas tauberianos, condiciones tauberianas.

1 Introduction

Let $p(x), q(y)$ and $u(z)$ be non-decreasing continuous function on $[0,\infty)$ such that $p(0)=q(0)=u(0)=0$ and $p(x), q(y), u(z) \to \infty$ as $x,y,z \to \infty$. For a locally integrable function $f(x,y,z)$ on $\mathbb{R}^3_+ = [0,\infty)\times[0,\infty)\times[0,\infty)$, we denote its triple integral by $F(x,y,z) = \int_0^x\int_0^y\int_0^z f(t,s,v)dtdsdv$ and its weighted mean of type (α,β,γ) determined by the functions $p(x), q(y)$ and $u(z)$ by

$$t_{\alpha,\beta,\gamma}(x,y,z) = \int_0^x\int_0^y\int_0^z (1-\frac{p(t)}{p(x)})^{\alpha}(1-\frac{q(s)}{p(y)})^{\beta} (1-\frac{u(v)}{u(x)})^{\gamma} f(t,s,v)dtdsdv \quad (1)$$

where $\alpha,\beta,\gamma > -1$. An improper triple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty f(t,s,u)dtdsdu \quad (2)$$

is said to be integrable to L by weighted mean method of type (α,β,γ) determined by the functions $p(x), q(y)$ and $u(z)$ if

$$\lim_{x,y,z\to\infty} t_{\alpha,\beta,\gamma}(x,y,z) = L. \quad (3)$$

We use the notion of convergence in Pringsheim's sense, that is, x, y and z tend to ∞ independently of each other in (3).

If we take $p(x)=x, q(y)=y$ and $u(z)=z$ in (1), we have the definition of (C,α,β,γ) integrability of $f(x,y,z)$ on $[0,\infty)\times[0,\infty)\times[0,\infty)$ given by (3). The $(C,0,0,0)$ integrability of $f(x,y,z)$ is convergence of the improper double integral (2).

It is clear that if $\lim_{x,y,z\to\infty} F(x,y,z) = L$ exists and $F(x,y,z)$ is bounded on $\mathbb{R}^3_+$, then the limit (3) also exists for $\alpha,\beta,\gamma > -1$. The converse of this implication is not true in general. The converse of this implication may be true only by adding some suitable condition which is called a Tauberian condition. Any theorem which states that convergence of (2) follows from the integrability of $f(x,y,z)$ by the weighted mean method of type (α,β,γ) determined by the function $p(x), q(y)$ and $u(z)$ and a Tauberian condition is said to be a Tauberian theorem.

In the last few years, there has been an increasing interest on summability methods for functions of one, two and three variables. First, Laforgia [6] obtained a sufficient condition under which convergence of the improper integral follows from $(C,1)$ integrability. Later, Canak and Totur [1] extended the main results of Laforgia [6] to the (C,α) integrability of functions by weighted mean methods where $\alpha > -1$. Then, taking into account these works, Totur and Canak [10] obtained some Tauberian theorems in terms of the concept of the general control modulo of non-integer order for functions of one-variable. Recently, Ozsarac and Canak [8] obtained Tauberian theorems for the iterations of weighted mean summable integrals. Besides, Totur et al. [9] presented some new Tauberian conditions in terms of the weighted general control modulo for the weighted mean method of integrals. For some interesting Tauberian theorems for Cesáro and weighted integrability in quantum calculus, we refer the readers to Canak et al. [2], Fitouhi and Brahim [5] and Totur et al. [11] and so on. In [7], Moricz obtained one-sided Tauberian conditions which are necessary and suffcient in order that convergence follow from summability $(C,1,1)$ of (2). More generally, Canak and Totur [3] obtained a sufficient condition under which convergence of (2) follows from (C,α,β) integrability of (2) where $\alpha,\beta > -1$. Later, Findik and Canak [4] showed some new Tauberian theorems for weighted mean of double integrals. In the field of study triple sequences spaces, there are some recently studies that can be seen in [12, 13, 14, 15, 16, 17].

In this paper, we define the notion of weighted mean method of type (α,β,γ) determined by the functions $p(x), q(y)$ and $u(z)$. Besides, we prove that if (3) exists and $t_{\alpha,\beta,\gamma}(x,y,z)$ is bounded on $\mathbb{R}^3_+$ for some $\alpha,\beta,\gamma > -1$, the limit $\lim_{x,y,z\to\infty} t_{\alpha+i,\beta+j,\gamma+k}(x,y,z) = L$ exists for all $i,j,k > 0$. As a corollary of this result, we show that if (2) is convergent to L and the function $F(x,y,z)$ is bounded on $\mathbb{R}^3_+$. Then, $\lim_{x,y,z\to\infty} t_{111}(x,y,z) = L$. But, the converse of this implication may true under some conditions imposed on p, q, u and f. Furthermore, we give a Tauberian condition under which convergence of improper triple integrals follows from the existence of $\lim_{x,y,z\to\infty} t_{111}(x,y,z) = L$.

2 Main Results

Theorem 2.1. *If (3) exists and $t_{\alpha,\beta,\gamma}(x,y,z)$ is bounded on $\mathbb{R}^3_+$ for some $\alpha,\beta,\gamma > -1$. Then, $\lim_{x,y,z\to\infty} t_{\alpha+i,\beta+j,\gamma+k}(x,y,z) = L$ exists for all $i,j,k > 0$.*

Proof. Consider

$$\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)t_{\alpha,\beta,\gamma}(x,y,z)dtdsdv, \quad (4)$$

where

$$\psi(t,s,v;x,y,z) = \frac{1}{B(\alpha+1,i)}(\frac{p'(t)}{p(x)})^{\alpha} (1-\frac{p(t)}{p(x)})^{i-1}\frac{1}{B(\beta+1,j)}(\frac{q'(s)}{q(y)})^{\beta}(1-\frac{q(s)}{q(y)})^{j-1} \frac{1}{B(\gamma+1,k)}(\frac{u'(v)}{u(z)})^{\gamma}(1-\frac{u(v)}{u(z)})^{k-1}$$

where B denotes Beta function defined by

$$B(d,e) = \int_0^1 t^{d-1}(1-t)^{e-1}dt,\ d>0 \text{ and } e>0.$$

Letting $a = \frac{p(t)}{p(x)}$, $b = \frac{q(s)}{q(y)}$ and $c = \frac{u(v)}{u(z)}$, we have

$$\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)dtdsdv = 1. \quad (5)$$

We first prove that

$$\lim_{x,y,z\to\infty}\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)t_{\alpha,\beta,\gamma}(x,y,z)dtdsdv = L. \quad (6)$$

Since

$$\lim_{x,y,z\to\infty} t_{\alpha,\beta,\gamma}(x,y,z) = L \quad (7)$$

by hypothesis , there exist numbers $x_\varepsilon, y_\varepsilon$ and z_ε for a given $\varepsilon > 0$ such that

$$|t_{\alpha,\beta,\gamma}(x,y,z)-L|<\varepsilon,\ x\geq x_\varepsilon,\ y\geq y_\varepsilon,\ z\geq z_\varepsilon. \tag{8}$$

It follows from (5) that

$$\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)t_{\alpha,\beta,\gamma}(x,y,z)dtdsdv-L=\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)[t_{\alpha,\beta,\gamma}(x,y,z)-L]dtdsdv-L. \tag{9}$$

To prove (6), we shall show that

$$|\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)t_{\alpha,\beta,\gamma}(x,y,z)dtdsdv-L|<5\varepsilon, \tag{10}$$

provided that x,y and z are large enough. We realize that by hypothesis, the function $t_{\alpha,\beta,\gamma}(x,y,z)$ is bounded on $\mathbb{R}^3_+$. Therefore, there exists a constant K such that

$$|t_{\alpha,\beta,\gamma}(x,y,z)-L|<K, \text{ for } 0<x,y,z<\infty.$$

Using (5) and (8), we obtain, by (9),

$$\begin{aligned}
&|\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)(t_{\alpha,\beta,\gamma}(x,y,z)-L)dtdsdv|\\
&\leq \int_0^{x_\varepsilon}\int_0^{y_\varepsilon}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)|t_{\alpha,\beta,\gamma}(x,y,z)-L|dtdsdv+\int_0^{x_\varepsilon}\int_{y_\varepsilon}^{y}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)|t_{\alpha,\beta,\gamma}(x,y,z)\\
&-L|dtdsdv+\int_{x_\varepsilon}^{x}\int_0^{y_\varepsilon}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)|t_{\alpha,\beta,\gamma}(x,y,z)-L|dtdsdv\\
&+\int_{x_\varepsilon}^{x}\int_{y_\varepsilon}^{y}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)|t_{\alpha,\beta,\gamma}(x,y,z)-L|dtdsdv+\varepsilon\int_{x_\varepsilon}^{x}\int_{y_\varepsilon}^{y}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)dtdsdv\\
&\leq K\int_0^{x_\varepsilon}\int_0^{y_\varepsilon}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv+K\int_0^{x_\varepsilon}\int_{y_\varepsilon}^{y}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv\\
&+K\int_{x_\varepsilon}^{x}\int_0^{y_\varepsilon}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv+K\int_0^{x_\varepsilon}\int_0^{y_\varepsilon}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)dtdsdv\\
&+\varepsilon\int_0^{x}\int_0^{y}\int_0^{z} \psi(t,s,v;x,y,z)dtdsdv \quad = \quad K\int_0^{x_\varepsilon}\int_0^{y_\varepsilon}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv\\
&+K\int_0^{x_\varepsilon}\int_{y_\varepsilon}^{y}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv+K\int_{x_\varepsilon}^{x}\int_0^{y_\varepsilon}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv\\
&+K\int_0^{x_\varepsilon}\int_0^{y_\varepsilon}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)dtdsdv+\varepsilon.
\end{aligned}$$

By the substitution $a=\dfrac{p(t)}{p(x)}$, $b=\dfrac{q(s)}{q(y)}$ and $c=\dfrac{u(v)}{u(z)}$, we have

$$\begin{aligned}
&\int_0^{x_\varepsilon}\int_0^{y_\varepsilon}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv\\
&=\frac{1}{B(\alpha+1,i)}\int_0^{x_\varepsilon}(\frac{p'(t)}{p(x)})^\alpha(1-\frac{p(t)}{p(x)})^{i-1}dt\\
&\times\frac{1}{B(\beta+1,j)}\int_0^{y_\varepsilon}(\frac{q'(s)}{q(y)})^\beta(1-\frac{q(s)}{q(y)})^{j-1}ds\\
&\times\frac{1}{B(\gamma+1,k)}\int_0^{z_\varepsilon}(\frac{u'(v)}{u(z)})^\gamma(1-\frac{u(v)}{u(z)})^{k-1}dv\\
&=\frac{1}{B(\alpha+1,i)}\int_0^{p(x_\varepsilon)/p(x)} a^\alpha(1-a)^{i-1}da\\
&\times\frac{1}{B(\beta+1,j)}\int_0^{q(y_\varepsilon)/q(y)} b^\beta(1-b)^{j-1}db\\
&\times\frac{1}{B(\gamma+1,k)}\int_0^{u(z_\varepsilon)/u(z)} c^\gamma(1-c)^{k-1}dc,
\end{aligned}$$

which tends to zero when $x,y,z\to\infty$ for any fixed $x_\varepsilon, y_\varepsilon$ and z_ε. Therefore, there exist some $x^1_\varepsilon, y^1_\varepsilon$ and z^1_ε such that

$$K\int_0^{x_\varepsilon}\int_0^{y_\varepsilon}\int_0^{z_\varepsilon} \psi(t,s,v;x,y,z)dtdsdv<\varepsilon,\ x\geq x^1_\varepsilon, y\geq y^1_\varepsilon, z\geq z^1_\varepsilon.$$

By the substitution $a=\dfrac{p(t)}{p(x)}$, $b=\dfrac{q(s)}{q(y)}$ and $c=\dfrac{u(v)}{u(z)}$, we have

$$\begin{aligned}
&\int_0^{x_\varepsilon}\int_{y_\varepsilon}^{y}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)dtdsdv\\
&=\frac{1}{B(\alpha+1,i)}\int_0^{x_\varepsilon}(\frac{p'(t)}{p(x)})^\alpha(1-\frac{p(t)}{p(x)})^{i-1}dt\\
&\times\frac{1}{B(\beta+1,j)}\int_{y_\varepsilon}^{y}(\frac{q'(s)}{q(y)})^\beta(1-\frac{q(s)}{q(y)})^{j-1}ds\\
&\times\frac{1}{B(\gamma+1,k)}\int_{z_\varepsilon}^{z}(\frac{u'(v)}{u(z)})^\gamma(1-\frac{u(v)}{u(z)})^{k-1}dv\\
&=\frac{1}{B(\alpha+1,i)}\int_0^{p(x_\varepsilon)/p(x)} a^\alpha(1-a)^{i-1}da\\
&\times\frac{1}{B(\beta+1,j)}\int_{q(y_\varepsilon)/q(y)}^{1} b^\beta(1-b)^{j-1}db\\
&\times\frac{1}{B(\gamma+1,k)}\int_{u(z_\varepsilon)/u(z)}^{1} c^\gamma(1-c)^{k-1}dc,
\end{aligned}$$

which tends to zero when $x,y,z\to\infty$ for any fixed $x_\varepsilon, y_\varepsilon$ and z_ε (see that $\dfrac{1}{B(\beta+1,j)}\displaystyle\int_{q(y_\varepsilon)/q(y)}^{1} b^\beta(1-b)^{j-1}db$ and $\dfrac{1}{B(\gamma+1,k)}\displaystyle\int_{u(z_\varepsilon)/u(z)}^{1} c^\gamma(1-c)^{k-1}dc$ tends to 1 as $y,z\to\infty$). Hence, there exist some $x^2_\varepsilon, y^2_\varepsilon$ and z^2_ε such that

$$K\int_0^{x_\varepsilon}\int_{y_\varepsilon}^{y}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)dtdsdv<\varepsilon,\ x\geq x^2_\varepsilon, y\geq y^2_\varepsilon, z\geq z^2_\varepsilon.$$

Similarly, the integral

$$\int_{x_\varepsilon}^{x}\int_0^{y_\varepsilon}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)dtdsdv$$

tends to zero when $x,y,z\to\infty$ for any fixed $x_\varepsilon, y_\varepsilon$ and z_ε (see that $\dfrac{1}{B(\alpha+1,i)}\displaystyle\int_{p(x_\varepsilon)/p(y)}^{1} a^\alpha(1-a)^{i-1}da$ and $\dfrac{1}{B(\gamma+1,k)}\displaystyle\int_{u(z_\varepsilon)/u(z)}^{1} c^\gamma(1-c)^{k-1}dc$ tends to 1 as $x,z\to\infty$). Hence, there exist some $x^3_\varepsilon, y^3_\varepsilon$ and z^3_ε such that

$$K\int_{x_\varepsilon}^{x}\int_0^{y_\varepsilon}\int_{z_\varepsilon}^{z} \psi(t,s,v;x,y,z)dtdsdv<\varepsilon,\ x\geq x^3_\varepsilon, y\geq y^3_\varepsilon, z\geq z^3_\varepsilon.$$

Similarly, the integral

$$\int_{x_\varepsilon}^{x}\int_{y_\varepsilon}^{y}\int_0^{z} \psi(t,s,v;x,y,z)dtdsdv$$

tends to zero when $x,y,z\to\infty$ for any fixed $x_\varepsilon, y_\varepsilon$ and z_ε (see that $\dfrac{1}{B(\alpha+1,i)}\displaystyle\int_{p(x_\varepsilon)/p(y)}^{1} a^\alpha(1-a)^{i-1}da$ and $\dfrac{1}{B(\beta+1,j)}\displaystyle\int_{q(y_\varepsilon)/q(y)}^{1} b^\beta(1-b)^{j-1}db$ tends to 1 as $y,z\to\infty$). Hence, there exist some $x^4_\varepsilon, y^4_\varepsilon$ and z^4_ε such that

$$K\int_{x_\varepsilon}^{x}\int_{y_\varepsilon}^{y}\int_0^{z} \psi(t,s,v;x,y,z)dtdsdv<\varepsilon,\ x\geq x^4_\varepsilon, y\geq y^4_\varepsilon, z\geq z^4_\varepsilon.$$

Hence, we have (10) for $x\geq\max\{x_\varepsilon,x^1_\varepsilon,x^2_\varepsilon,x^3_\varepsilon,x^4_\varepsilon\}$, $y\geq\max\{y_\varepsilon,y^1_\varepsilon,y^2_\varepsilon,y^3_\varepsilon,y^4_\varepsilon\}$, $z\geq\max\{z_\varepsilon,z^1_\varepsilon,z^2_\varepsilon,z^3_\varepsilon,z^4_\varepsilon\}$ and this proves (6). Therefore, we obtain

$$\begin{aligned}
&\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)t_{\alpha,\beta,\gamma}(t,s,v)dtdsdv\\
&=\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)(\int_0^t\int_0^s\int_0^v(1-\frac{p(a)}{p(t)})^\alpha(1-\frac{q(b)}{p(s)})^\beta(1-\frac{u(c)}{u(v)})^\gamma f(a,b,c)dadbdc)dtdsdv\\
&=\int_0^x\int_0^y\int_0^z f(a,b,c)(\int_a^x\int_b^y\int_c^z \psi(t,s,v;x,y,z)(1-\frac{p(a)}{p(t)})^\alpha(1-\frac{q(b)}{p(s)})^\beta(1-\frac{u(c)}{u(v)})^\gamma dtdsdv)dadbdc\\
&=\int_0^x\int_0^y\int_0^z f(a,b,c)I(a,b,c;x,y,z)dadbdc,
\end{aligned}$$

where

$$I(a,b,c;x,y,z) = \int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)(1-\frac{p(a)}{p(t)})^{\alpha}(1-\frac{q(b)}{p(s)})^{\beta}(1-\frac{u(c)}{u(v)})^{\gamma}dtdsdv.$$

Now, we write $I(a,b,c;x,y,z)$ as

$$\begin{aligned}
I(a,b,c;x,y,z) &= \int_a^x\int_b^y\int_c^z \psi(t,s,v;x,y,z)(1-\frac{p(a)}{p(t)})^{\alpha}(1-\frac{q(b)}{p(s)})^{\beta}(1-\frac{u(c)}{u(v)})^{\gamma}dtdsdv\\
&= (\frac{1}{B(\alpha+1,i)}\int_a^x \frac{p'(x)}{p(x)}(\frac{p(t)}{p(x)})^{\alpha}(1-\frac{p(t)}{p(x)})^{i-1}(1-\frac{p(a)}{p(t)})^{\alpha}dt)\\
&\times(\frac{1}{B(\beta+1,j)}\int_b^y \frac{q'(y)}{q(y)}(\frac{q(s)}{q(y)})^{\beta}(1-\frac{q(s)}{q(y)})^{j-1}(1-\frac{q(b)}{q(s)})^{\beta}ds)\\
&\times(\frac{1}{B(\gamma+1,k)}\int_c^z \frac{u'(z)}{u(z)}(\frac{u(v)}{u(z)})^{\gamma}(1-\frac{u(v)}{u(z)})^{k-1}(1-\frac{u(c)}{u(v)})^{\gamma}dv)\\
&= (\frac{1}{B(\alpha+1,i)}(\frac{1}{p(x)})^{\alpha+1}\int_a^x p'(t)(1-\frac{p(t)}{p(x)})^{i-1}(p(t)-p(a))^{\alpha}dt)\\
&\times(\frac{1}{B(\beta+1,j)}(\frac{1}{q(y)})^{\beta+1}\int_b^y q'(s)(1-\frac{q(s)}{q(y)})^{j-1}(q(s)-q(b))^{\beta}ds)\\
&\times(\frac{1}{B(\gamma+1,k)}(\frac{1}{u(z)})^{\gamma+1}\int_c^z u'(v)(1-\frac{u(v)}{u(z)})^{k-1}(u(v)-u(c))^{\gamma}dv)\\
&= I_1(a,x)I_2(b,y)I_3(c,z),
\end{aligned}$$

where

$$I_1(a,x) = \frac{1}{B(\alpha+1,i)}(\frac{1}{p(x)})^{\alpha+1}\int_a^x p'(t)(1-\frac{p(t)}{p(x)})^{i-1}(p(t)-p(a))^{\alpha}dt,$$

$$I_2(b,y) = \frac{1}{B(\beta+1,j)}(\frac{1}{q(y)})^{\beta+1}\int_b^y q'(s)(1-\frac{q(s)}{q(y)})^{j-1}(q(s)-q(b))^{\beta}ds$$

and

$$I_3(c,z) = \frac{1}{B(\gamma+1,k)}(\frac{1}{u(z)})^{\gamma+1}\int_c^z u'(v)(1-\frac{u(v)}{u(z)})^{k-1}(u(v)-u(c))^{\gamma}dv.$$

Substituting $p(t) = p(x)-(p(x)-p(a))x$ in $I_1(a,x)$, we have

$$I_1(a,x) = \frac{1}{B(\alpha+1,i)}(1-\frac{p(a)}{p(x)})^{i-1}(1-\frac{p(a)}{p(x)})^{\alpha+1}\int_0^1 x^{i-1}(1-x)^{\alpha}dx = (1-\frac{p(a)}{p(x)})^{\alpha+i}.$$

Similarly, we have

$$I_2(b,y) = (1-\frac{q(b)}{q(y)})^{\beta+j},$$

and similarly we have

$$I_3(c,z) = (1-\frac{u(c)}{u(z)})^{\gamma+k}.$$

These show that

$$\begin{aligned}
&\int_0^x\int_0^y\int_0^z \psi(t,s,v;x,y,z)t_{\alpha,\beta,\gamma}(t,s,v)dtdsdv\\
&= \int_0^x\int_0^y\int_0^z (1-\frac{p(a)}{p(x)})^{\alpha+i}(1-\frac{q(b)}{q(y)})^{\beta+j}(1-\frac{u(c)}{u(z)})^{\gamma+k}f(a,b,c)dadbdc\\
&= t_{\alpha+i,\beta+j,\gamma+k}(x,y,z).
\end{aligned}$$

□

Corollary 2.2. *If $\lim_{x,y,z\to\infty} F(x,y,z) = L$ exists and $F(x,y,z)$ is bounded on $\mathbb{R}^3_+$. Then, $\lim_{x,y,z\to\infty} t_{111}(x,y,z) = L$ exists.*

Proof. Take $\alpha = \beta = \gamma = 0$ and $i,j,k = 1$ in Theorem 2.1. □

The proofs of theorems 2.3, 2.4 and 2.5 can be obtained by the similar techniques and steps as in the proof of Theorem 3 in [4], and so we omit them.

Theorem 2.3. *If (2) is integrable to L by the weighted mean method of type $(1,0,0)$ determined by the function $p(x)$ and*

$$\frac{p(x)}{p'(x)}\int_0^y\int_0^z f(x,s,v)dsdv = O(1) \tag{11}$$

holds. Then, (2) converges to L.

Theorem 2.4. *If (2) is integrable to L by the weighted mean method of type $(0,1,0)$ determined by the function $q(y)$ and*

$$\frac{q(y)}{q'(y)}\int_0^x\int_0^z f(t,y,v)dtdv = O(1) \tag{12}$$

holds. Then, (2) converges to L.

Theorem 2.5. *If (2) is integrable to L by the weighted mean method of type $(0,0,1)$ determined by the function $u(z)$ and*

$$\frac{u(z)}{u'(z)}\int_0^x\int_0^y f(t,s,z)dtds = O(1) \tag{13}$$

holds. Then, (2) converges to L.

Theorem 2.6. *If (2) is integrable to L by the weighted mean method of type $(1,1,1)$ determined by the function $p(x), q(y)$ and $u(z)$ and*

$$\frac{p(x)}{p'(x)}\int_0^y\int_0^z f(x,s,v)dsdv = O(1), \tag{14}$$

$$\frac{q(y)}{q'(y)}\int_0^x\int_0^z f(t,y,v)dtdv = O(1) \tag{15}$$

and

$$\frac{u(z)}{u'(z)}\int_0^x\int_0^y f(t,s,z)dtds = O(1). \tag{16}$$

Then, (2) converges to L.

Proof. Consider that (2) is integrable to L by the weighted mean of method of type $(1,1,1)$ determined by the functions $p(x), q(y)$ and $u(z)$, that is

$$G(x,y,z) = \int_0^x\int_0^y\int_0^z (1-\frac{p(t)}{p(x)})(1-\frac{q(s)}{q(y)})(1-\frac{u(v)}{u(z)})f(t,s,v)dtdsdv \to L;\ x,y,z\to\infty. \tag{17}$$

We rewrite $G(x,y,z)$ as

$$G(x,y,z) = \int_0^z (1-\frac{u(v)}{u(z)})\frac{\partial}{\partial v}G_1(x,y,v)dv, \tag{18}$$

where

$$G_1(x,y,z) = \int_0^x\int_0^y (1-\frac{p(t)}{p(x)})(1-\frac{q(s)}{q(y)})f(t,s,v)dtdsdv. \tag{19}$$

It follows from (17),(18) and (19) that $\frac{\partial}{\partial z}G_1(x,y,z)$ is integrable to L by the weighted mean method of type $(0,0,1)$ determined by the function $u(z)$.

Then, we have

$$G(x,y,z) = G_1(x,y,z) - H_1(x,y,z), \tag{20}$$

where

$$H_1(x,y,z) = \frac{1}{u(z)} \int_0^z u(v) \frac{\partial}{\partial v} G_1(x,y,v) dv. \tag{21}$$

Now, we have to show that $H_1(x,y,z) \to 0$ as $x,y,z \to \infty$.

By (18), we find

$$\frac{\partial}{\partial z}G(x,y,z) = \frac{u^{'}(z)}{u(z)^2} \int_0^z u(v) \frac{\partial}{\partial v} G_1(x,y,v) dv = \frac{u^{'}(z)}{u(z)} H_1(x,y,z). \tag{22}$$

Besides, we have

$$\begin{aligned}\int_{z_1}^{z_2} \frac{\partial}{\partial z} G(x,y,z) dz &= G(x,y,z_1) - G(x,y,z_2) \\ &= \int_{z_1}^{z_2} \frac{u^{'}(z)}{u(z)} H_1(x,y,z) dz \\ &= \int_{\log u(z_2)}^{\log u(z_1)} H_1(x,y,u^{-1}(e^w)) dw \\ &= \int_{\log u(z_2)}^{\log u(z_1)} R(x,y,w) dw.\end{aligned}$$

Now, we used the substitution $u(z) = e^w$ and $R(x,y,w) = H_1(t,s,u^{-1}(e^w))$. So, we need to show that $\lim_{w\to\infty} R(x,y,w) = 0$. By the simple calculation, we have

$$\frac{\partial}{\partial w} R(x,y,w) = \frac{e^w}{u^{'}(u^{-1}e^w)} \frac{\partial}{\partial w} H_1(x,y,u^{-1}(e^w)) = \frac{u(z)}{u^{'}(z)} \frac{\partial}{\partial z} H_1(x,y,z). \tag{23}$$

By (21), we have

$$u(z)H_1(x,y,z) = \int_0^z u(v) \frac{\partial}{\partial v} G_1(x,y,v) dv. \tag{24}$$

Differentiating the both sides of (24) with respect to z gives

$$H_1(x,y,z) + \frac{u(z)}{u^{'}(z)} \frac{\partial}{\partial z} H_1(x,y,z) = \frac{u(z)}{u^{'}(z)} \frac{\partial}{\partial z} G_1(x,y,z) \tag{25}$$

Taking the weighted mean of type $(1,0,0)$ and $(0,1,0)$ of the both sides of (15) (16), we have

$$\frac{u(z)}{u^{'}(z)} \frac{\partial}{\partial z} G_1(x,y,z) = O(1), \tag{26}$$

which implies that

$$\frac{u(z)}{u^{'}(z)} \frac{\partial}{\partial z} H_1(x,y,z) = O(1), \tag{27}$$

by (23). Hence, by (23), we attain that $\frac{\partial}{\partial z}R(x,y,z)$ is bounded. Since $G(x,y,z)$ is convergent, given any $\varepsilon > 0$ there exists a z_ε such that

$$|G(x,y,z_1) - G(x,y,z_2)| < \varepsilon$$

when $z_1, z_2 > y_\varepsilon$.

Now, suppose $\phi > lnu(z_\varepsilon)$ and $R(x,y,\phi) > 0$. Then, $R(x,y,z) > 0$ for $\phi - \xi < z < \phi$ and $\phi < z < \phi + \xi$ where $\xi = \frac{R(x,y,\phi)}{K}$. If we integrate $R(x,y,z)$ between $\phi - \xi$ and $\phi + \xi$, we have

$$\int_{\phi-\xi}^{\phi+\xi} R(x,y,z) dz > \frac{2R^2(x,y,\phi)}{K}. \tag{28}$$

Moreover, by (28) we have

$$\frac{2R^2(x,y,\phi)}{K} < \int_{\phi-\xi}^{\phi+\xi} R(x,y,z) dz = G(x,y,u^{-1}(e^{\phi+\xi})) - G(x,y,u^{-1}(e^{\phi-\xi})) < \varepsilon. \tag{29}$$

Therefore,

$$R(x,y,\phi) < \sqrt{\frac{K\varepsilon}{2}} \tag{30}$$

which shows that $H_1(x,y,z) \to 0$ as $x,y,z \to \infty$. It follows from (17) and (20) that $G_1(x,y,z) \to L$ as $x,y,z \to \infty$.

Since $G_1(x,y,z) \to L$ as $x,y,z \to \infty$ and the conditions (14) and (15), we conclude $\lim_{x,y,z\to\infty} F(x,y,z) = L$ by Theorems 2.3 and 2.4.

□

Conflict of interest

The authors declare that they have no conflict of interest.

Availability of data and materials

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Authors' contributions

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