Some Relations between $N$-Koszul, Artin-Schelter Regular and Calabi-Yau Algebras with Skew PBW Extensions

Algunas relaciones entre álgebras $N$-Koszul, Artin-Schelter regular y Calabi-Yau con extensiones PBW torcidas

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Abstract

Some authors have studied relations between Artin-Schelter regular algebras, $N$-Koszul algebras and Calabi-Yau algebras (resp. skew Calabi-Yau) of dimension $d$. In this paper we want to show through examples and counterexamples some relations between these classes of algebras with skew PBW extensions. In addition, we also exhibit some examples of the preservation of these properties by Ore extensions.

Key words: Skew PBW extensions, Calabi-Yau algebras, $N$-Koszul algebras, AS-regular algebras, Ore extensions.

Resumen

Algunos autores han estudiado las relaciones entre las álgebras Artin-Schelter regular, las álgebras $N$-Koszul y las álgebras Calabi-Yau (resp. skew Calabi-Yau) de dimensión $d$. En este artículo queremos mostrar a través de ejemplos y contraejemplos algunas relaciones entre estas clases de álgebras y las extensiones PBW torcidas. Además, mostraremos algunos ejemplos de preservación de estas propiedades en las extensiones de Ore.

Palabras clave: Skew PBW extensions, Calabi-Yau algebras, $N$-Koszul algebras, AS-regular algebras, Ore extensions.

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1. Introduction

Recently there have been defined some special classes of algebras such as $N$-Koszul algebras, Calabi-Yau algebras and skew PBW extensions. Koszul algebras, which in this article are called 2-Koszul algebras or Calabi-Yau algebras and skew series of examples some relationships between the Calabi-Yau algebras. Our aim is to show through a properties:

(i) $A$ has finite global dimension $d$: every graded $A$–module has projective dimension $\leq d$.

(ii) $A$ has finite Gelfand-Kirillov dimension (GKdim), i.e., $A$ has polynomial growth.

(iii) $A$ is Gorenstein, i.e., $\text{Ext}^i_A(\mathbb{K}, A) = 0$ if $q \neq d$, and $\text{Ext}^i_A(\mathbb{K}, A) \cong \mathbb{K}$.

In the current literature these algebras are called Artin-Schelter regular algebras ($AS$-regular algebras). Most of the authors do not consider the condition (ii) in the definition of $AS$-regular algebras. We say that $A$ has polynomial growth if there exist $c \in \mathbb{R}^+$ and $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\dim_{\mathbb{K}} A_n \leq cn^r$.

2. Definitions and Elementary Properties

2.1. $AS$-Regular Algebras

Regular algebras were defined by Michael Artin and William Schelter in [2]. They studied the regular algebras of global dimension three which are generated by elements of degree one and classified into thirteen types.

Definition 1 ([2]). Let $A = \mathbb{K} \oplus A_1 \oplus A_2 \oplus \cdots$ be a finitely presented graded algebra over $\mathbb{K}$. The algebra $A$ will be called regular if it has the following properties:

(i) $A$ has finite global dimension $d$: every graded $A$–module has projective dimension $\leq d$.

(ii) $A$ has finite Gelfand-Kirillov dimension (GKdim), i.e., $A$ has polynomial growth.

(iii) $A$ is Gorenstein, i.e., $\text{Ext}^i_A(\mathbb{K}, A) = 0$ if $q \neq d$, and $\text{Ext}^i_A(\mathbb{K}, A) \cong \mathbb{K}$.

In the current literature these algebras are called Artin-Schelter regular algebras ($AS$-regular algebras).

2.2. $N$-Koszul Algebras

Koszul algebras were defined by Stewart B. Priddy in [34], later in 2001, Roland Berger in [3] introduces a generalization of Koszul algebras, which are then called generalized Koszul algebras or $N$-Koszul algebras. In [17] Victor Ginzburg defined d-Calabi-Yau algebras or Calabi-Yau algebras of dimension $d$ (or simply Calabi-Yau algebras). Then in [6], Roland Berger and Rachel Taillefer introduced the definition of graded Calabi-Yau algebra. As a generalization of Calabi-Yau algebras, were also defined the skew Calabi-Yau algebras. On the other hand, the skew PBW extensions were introduced in 2011 by Oswaldo Lezama and Claudia Gallego in [16].

In the current literature, it has been studied certain relations between Artin Schelter regular algebras, $N$-Koszul algebras, Calabi-Yau algebras and skew Calabi-Yau algebras. Our aim is to show through a serie of examples some relationships between the above algebras and skew PBW extensions. Unless otherwise specified, throughout this article, $\mathbb{K}$ will represent a fixed but arbitrary field.

2.3. Calabi-Yau Algebras of Dimension $d$

Calabi-Yau algebras of dimension $d$ or $d$-Calabi-Yau algebras were defined by Victor Ginzburg in [17].

Definition 3 ([17], Definition 3.2.4). A $\mathbb{K}$-algebra $A$ is called a Calabi-Yau algebra of dimension $d$ if

(i) $A$ is homologically smooth; that is, $A$ has a finite resolution of finitely generated projective $A$-bimodules;

(ii) $\text{Ext}^i_{A-Bim}(A, A \otimes A) \cong \begin{cases} A, & \text{if } i = d \\ 0, & \text{if } i \neq d \end{cases}$, as $A$-bimodules.

The space $A \otimes A$ is endowed with two $A$-bimodule structures: the outer structure defined by $a \cdot (x \otimes y) \cdot b = ax \otimes yb$, and the inner structure defined by
\( a \cdot (x \otimes y) \cdot b = xb \otimes ay \). Consequently, the Hom spaces \( \text{Hom}_{A \otimes A}(M, A \otimes A) \) of \( A \)-bimodule morphisms from \( M \) to \( A \otimes A \) endowed with the outer structure are again \( A \)-bimodules using the inner structure of \( A \otimes A \), and the same is true for the Hochschild cohomology spaces \( H^k(A, A \otimes A) \). For \( A^p = A \otimes A^{op} \), the enveloping algebra of \( A \), each \( A \)-bimodule \( M \) is a left \( A^p \)-module for the action \((a \otimes b) \cdot m = amb \) and right \( A^p \)-module for the action \( m \cdot (a \otimes b) = bma \).

Let \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) be a \( \mathbb{Z} \)-graded algebra, and \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) be a graded \( A \)-bimodule. For any integer \( l \), \( M(l) \) is a graded \( A \)-bimodule whose degree \( i \) component is \( M(l)_i = M_{i+l} \).

**Definition 4.** A graded algebra \( A \) is called a graded Calabi-Yau algebra of dimension \( d \) if

(i) \( A \) has a finite resolution of finitely generated graded projective \( A \)-bimodules, and

(ii) \( \text{Ext}^i_{A^p}(A, A \otimes A) \cong \begin{cases} 0, & \text{if } i \neq d \\ A(l), & \text{if } i = d, \end{cases} \)

as \( A \)-bimodules; for some integer \( l \).

It follows from Definition 4 that every graded Calabi-Yau algebra of dimension \( d \) is Calabi-Yau of dimension \( d \) (see [6], Proposition 4.3).

Let \( M \) be an \( A \)-bimodule, \( \nu, \mu : A \to A \) two automorphism, the skew \( A \)-bimodule \( ^\nu M^\mu \) is equal to \( M \) as a vector \( \mathbb{K} \)-space with \( a \cdot m \cdot b = \nu(a) m \mu(b) \).

**Definition 5.** Let \( A \) be a \( \mathbb{K} \)-algebra. \( A \) is called **skew Calabi-Yau** of dimension \( d \) if there exists an automorphism \( \nu \) of \( A \) such that

(i) \( A \) is homologically smooth; and

(ii) \( \text{Ext}^i_{A^p}(A, A^p) \cong 0 \) when \( i \neq d \) and \( \text{Ext}^d_{A^p}(A, A^p) \cong A^p \) as \( A^p \)-modules.

In this case, \( \nu \) is called the **Nakayama Automorphism** of \( A \). The Nakayama automorphism is unique up to an inner automorphism. A \( \nu \)-skew Calabi-Yau algebra \( A \) is Calabi-Yau in the sense of Ginzburg if and only if \( \nu \) is an inner automorphism of \( A \) (see [30], Definition 1.1). So every Calabi-Yau algebra is skew Calabi-Yau.

### 2.4. Skew PBW Extensions

Skew PBW extensions or \( \sigma - \text{PBW} \) extensions were defined in 2011 by Oswaldo Lezama and Claudia Gallego in [16].

**Definition 6.** Let \( R \) and \( A \) be rings. We say that \( A \) is a **skew PBW extension** of \( R \) if the following conditions hold:

(i) \( R \subseteq A \).

(ii) There exist elements \( x_1, \ldots, x_n \) in \( A \) such that \( A \) is a left free \( R \)-module, with basis,

\[
\text{Mon}(A) := \{x_1^{a_1} \cdots x_n^{a_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.
\]

(iii) For each \( 1 \leq i \leq n \) and any \( r \in R - \{0\} \) there exists an element \( c_{i,r} \in R - \{0\} \) such that

\[
x_i r - c_{i,r} x_i r \in R.
\]

(iv) For any elements \( 1 \leq i, j \leq n \), there exists \( c_{i,j} \in R - \{0\} \) such that

\[
x_j c_{i,j} x_i r - x_i c_{i,j} x_j r \in R + Rx_1 + \cdots + Rx_n.
\]

**Proposition 1** ([16], Proposition 3). Let \( A \) be a skew PBW extension of \( R \). Then, for every \( 1 \leq i \leq n \), there exist an injective ring endomorphism \( \sigma_i : R \to R \) and a \( \sigma_i \)-derivation \( \delta_i : R \to R \) such that

\[
x_i r = \sigma_i(r) x_i + \delta_i(r),
\]

for each \( r \in R \).

In this case we write \( A := \sigma(R)(x_1, \ldots, x_n) \).

We say that \( A \) is a **bijective** if \( \sigma_i \) is bijective for each \( 1 \leq i \leq n \) and \( c_{i,j} \) is invertible for any \( 1 \leq i < j \leq n \) (see [16], Definition 4).

### 3. Relations, Examples and Counterexamples

Some authors have found some interesting relations between \( AS \)-regular algebras, \( N \)-Koszul algebras and Calabi-Yau algebras. Some examples of these relations are the following:

(i) Roland Berger and Nicolas Marconnet in Proposition 5.2 of [8] show that if \( A = T(V)/(R) \) is a connected graded \( \mathbb{K} \)-algebra such that the space \( V \) of generators is concentrated in degree 1, the space \( R \) of relations lives in degrees \( \geq 2 \), the global dimension \( d \) of \( A \) is 2 or 3, and that \( A \) is \( AS \)-regular (the polynomial growth imposed by Artin and Schelter is often removed and in fact, it is not necessary), then \( A \) is \( N \)-Koszul if \( d = 3 \), and 2-Koszul if \( d = 2 \).

(ii) Roland Berger y Rachel Taillefer in Proposition 4.3 of [6] show that if \( A \) is a connected \( \mathbb{N} \)-graded
Calabi-Yau algebra then $A$ is $AS$-regular algebra, and in Proposition 5.4 they prove that if $A$ is $AS$-regular C-algebra of global dimension 3 (with polynomial growth), then $A$ is Calabi-Yau if and only if $A$ is of type $A$ in the classification of Artin and Schelter given in [2].

(iii) Let $K$ be of characteristic zero, $V$ be an $n$-dimensional space with $n \geq 1$, $w$ be a non-zero homogeneous potential of $V$ of degree $N + 1$ with $N \geq 2$, and $A = A(w)$ be the potential algebra defined by $w$ (so that the space of generators of $A$ is $V$); Roland Berger and Andrea Solotar in Theorem 2.6 of [4] prove that if the space of relations $R$ (i.e. the subspace of $V^\otimes N$ generated by the relations $\partial_i(w)$, $x \in X$) of $A$ is $n$-dimensional, then $A$ is 3-Calabi-Yau if and only if $A$ is $N$-Koszul of global dimension 3 and $\dim R_{N+1} = 1$, where $R_{N+1} = (R \otimes V) \cap (V \otimes R) \subseteq V^{\otimes (N+1)}$.

(iv) Manuel Reyes, Daniel Rogalski and James Zhang in Lemma 1.2 of [37] show that if $A$ is a connected graded algebra, then $A$ is graded skew Calabi-Yau if and only if $A$ is $AS$-regular.

3.1. Examples

In the current literature there are not explicit relations between skew PBW extensions with $AS$-regular algebras, $N$-Koszul algebras or Calabi-Yau algebras. Next we will show some examples of algebras that are $AS$-regular, or $N$-Koszul, or Calabi-Yau, or a combination of these types, that are skew PBW extensions.

3.1.1. $AS$-regular + $N$-Koszul + Calabi-Yau

Below are some examples of algebras that are $AS$-regular, $N$-Koszul and Calabi-Yau, and in addition, they are also skew PBW extensions.

1. The polynomial algebra $A = K[x,y]$ is a connected graded Noetherian algebra of global dimension 2. It follows that $A$ is $AS$-regular with $GKdim(A) = 2$ (see [40], Theorem 3.5), $A$ is 2-Koszul algebra (see [8], Proposition 5.2). Moreover, $A$ is Calabi-Yau of dimension 2 (see [28]), and $A$ is a skew PBW extension (see [16], Example 5).

2. Let $A = K[x_1, \ldots, x_n]$ be the polynomial algebra in $n$ variables. Then $A$ is a 2-Koszul algebra (see [31], Example 1.6). $A$ is a skew PBW extension (see [16], Example 5), $A$ is Calabi-Yau of dimension $n$ (see [9], page 18) and therefore, $AS$-regular (see [6], Proposition 4.3).

3. Let $A = K(x,y,z)/(yz-zy, zx-xz, xy-yx + z^2)$ which is of type $S_1$ in the classification of three-dimensional $AS$-regular algebras given in [2]. According to [8], $A$ is 3–Calabi-Yau (see [45], Example 3.6), and by Proposition 5.2 of [8] $A$ is 2-Koszul. We note that $A \cong (K[z])(x,y)$ and therefore $A$ is a skew PBW extension.

4. For any $n \geq 2$, let $A$ be a non-degenerate non-commutative quadric graded algebra in $n$ variables $x_1, \ldots, x_n$ of degree 1. Let $z$ be an extra variable of degree 1. Let $B$ be an algebra defined by a non-zero cubic potential $w$ in the variables $x_1, \ldots, x_n, z$. Assume that the graded algebra $B$ is isomorphic to a skew polynomial algebra $A[z; \sigma; \delta]$ over $A$ in the variable $z$, defined by a 0-degree homogeneous automorphism $\sigma$ of $A$ and a 1-degree homogeneous $\sigma$–derivation $\delta$ of $A$. Then $B$ is 2-Koszul and 3-Calabi-Yau (see [4], Proposition 4.1). $B$ is a skew PBW extension.

3.1.2. $AS$-Regular + $N$-Koszul

The following are some examples of $AS$-regular $N$-Koszul algebras which are skew PBW extensions. It is not clear if these algebras are Calabi-Yau or not, since we have no clear criteria for making claims in this regard.

1. The algebra $A = K(x,y,z)/(a\beta xy + a\beta yx, \alpha zx + axz, yz + a\beta zy)$ is $AS$-regular of global dimension 3 of type $S_1$ (see [2], Theorem 3.10). Moreover, $A$ is 2-Koszul (see [8], Proposition 5.2), and $A$ is a skew PBW extension.

$A$ may be or not Calabi-Yau, depends on the coefficients $a$, $\alpha$ and $\beta$ (see [6], Proposition 5.4).

2. The quantum plane $A = K(x,y)/(yx - cxy)$ ($c \neq 0$) is an $AS$-regular algebra of global dimension 2 (see [2], page 172). Moreover $A$ is a skew PBW extension as well as 2-Koszul (see [8], Proposition 5.2). For example, if $c = 1$ then the quantum plane $A$ is a 2-Calabi-Yau algebra.

3. The Jordan plane $A = K(x,y)/(yx - xy - x^2)$ is an $AS$-regular algebra of global dimension 2 (see [2], page 172). Since $A$ is a quadratic algebra and $yx - xy - x^2$ is a principal ideal, it
follows that \( A \) is 2-Koszul (see [15], page 7), \( A \cong \sigma(\mathbb{K}[x]/(y)) \) and therefore \( A \) is a skew PBW extension. The Jordan plane \( A \) is not Calabi-Yau (see [30]).

3.1.3. Skew Calabi-Yau algebras

The following is an example of skew Calabi-Yau algebra that is skew PBW extension. Multi-parameter quantum affine \( n \)-spaces \( O_q(\mathbb{K}^n) \) can be obtained by iterated Ore extensions. Let \( n \geq 1 \) and \( q \) be a matrix \( (q_{ij})_{n \times n} \) with entries in a field \( \mathbb{K} \) where \( q_{ii} = 1 \) and \( q_{ij}q_{ji} = 1 \) for all \( 1 \leq i, j \leq n \). Then quantum affine \( n \)-space \( O_q(\mathbb{K}^n) \) is defined to be \( \mathbb{K} \)-algebra generated by \( x_1, \ldots, x_n \) with the relations \( x_jx_i = q_{ij}x_ix_j \) for all \( 1 \leq i, j \leq n \). The \( \mathbb{K} \)-algebra \( O_q(\mathbb{K}^n) \) is skew Calabi-Yau when the Nakayama automorphism \( \nu \) such that \( \nu(x_i) = (\prod_{j=1}^n q_{ji})x_i \) (see [30], Proposition 4.1). This \( \mathbb{K} \)-algebra is a skew PBW extension (see [29]).

The Jordan plane \( A = \mathbb{K}(x,y)/(xy - xy - x^2) \) is skew Calabi-Yau, but not Calabi-Yau (see [30]).

3.1.4. The universal enveloping algebra and the Sridharan enveloping algebra of Lie algebra

Let \( \mathcal{G} \) be a finite dimensional Lie algebra over \( \mathbb{K} \) with basis \( \{x_1, \ldots, x_n\} \). The universal enveloping algebra of \( \mathcal{G} \), denoted \( \mathcal{U}(\mathcal{G}) \), is a PBW extension of \( \mathbb{K} \) since \( x_i r - r x_i = 0 \), \( x_i x_j - x_j x_i = [x_i, x_j] \in \mathcal{G} = \mathbb{K} + \mathbb{K} x_1 + \cdots + \mathbb{K} x_n \), \( r_i \in \mathbb{K} \), for \( 1 \leq i, j \leq n \). Ji-Wei He, Fred Van Oystaeyen and Yinhao Zhang showed that for the 3-dimensional Lie algebra \( \mathcal{G} \) with basis \( \{x,y,z\} \), \( \mathcal{U}(\mathcal{G}) \) is a Calabi-Yau algebra if and only if the Lie bracket is given by \( [x,y] = ax + by + wz, [x,z] = cx + vy - bz, [y,z] = ux - cy + az \), where \( a, b, c, u, v, w \in \mathbb{K} \); and if \( \mathcal{G} \) is a finite dimensional Lie algebra, \( \mathcal{U}(\mathcal{G}) \) is Calabi-Yau of dimension 3 if and only if \( \mathcal{G} \) is isomorphic to one of the following Lie algebras (see [22], Proposition 4.5 and Proposition 4.6):

(i) The 3-dimensional simple Lie algebra \( sl(2, \mathbb{K}) \);

(ii) \( \mathcal{G} \) has a basis \( \{x,y,z\} \) such that \( [x,y] = y, [x,z] = -z \) and \( [y,z] = 0 \);

(iii) The Heisenberg algebra, that is; \( \mathcal{G} \) has a basis \( \{x,y,z\} \) such that \( [x,y] = z \) and \( [x,z] = y, [y,z] = 0 \);

(iv) The 3-dimensional abelian Lie algebra.

We note that if \( \mathcal{G} \) is a finite dimensional Lie algebra over a field \( \mathbb{K} \) and \( \mathcal{U}(\mathcal{G}) \) is the universal enveloping algebra of \( \mathcal{G} \), then \( \mathcal{U}(\mathcal{G}) \) is a skew PBW extension (see [16]); in particular, universal enveloping Calabi-Yau algebra \( \mathcal{U}(\mathcal{G}) \) of dimension 3 is a skew PBW extension.

Let \( \mathcal{G} \) be a finite dimensional Lie algebra, and let \( f \in \mathbb{Z}^3(\mathcal{G}, \mathbb{K}) \) be an arbitrary 2-cocycle, that is,

\[
\begin{aligned}
\mathcal{G} \times \mathcal{G} &\to \mathbb{K} \\
(x, y) &\mapsto f(x, y) + f(y, x)
\end{aligned}
\]

for all \( x, y \in \mathcal{G} \).

The Sridharan enveloping algebra of \( \mathcal{G} \) is defined to be the associative algebra \( \mathcal{U}_f(\mathcal{G}) = T(\mathcal{G})/I \), where \( I \) is the two-sided ideal of \( T(\mathcal{G}) \) generated by the elements

\[
(x \otimes y - (y \otimes x) - [x,y] - f(x,y)), \text{ for all } x, y \in \mathcal{G}.
\]

For \( x \in \mathcal{G} \), we still denote by \( x \) its image in \( \mathcal{U}_f(\mathcal{G}) \). \( \mathcal{U}_f(\mathcal{G}) \) is a filtered algebra with the associated graded algebra \( \text{gr}(\mathcal{U}_f(\mathcal{G})) \) being a polynomial algebra.

Let \( \mathbb{K} \) be a field and algebraically closed with characteristic zero. If \( \mathcal{G} \) is a Lie \( \mathbb{K} \)-algebra of dimension three then, the Sridharan enveloping algebra \( \mathcal{U}_f(\mathcal{G}) \), for \( f \in \mathbb{Z}^3(\mathcal{G}, \mathbb{K}) \), is isomorphic to one of 10 following associative \( \mathbb{K} \)-algebras, defined by three generator \( x, y, z \) and the following commutation relations (see [32], Theorem 1.3):

<table>
<thead>
<tr>
<th>Type</th>
<th>( [x,y] )</th>
<th>( [y,z] )</th>
<th>( [z,x] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( x )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( x )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>( ay )</td>
<td>( -x )</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>( y )</td>
<td>( -(x+y) )</td>
</tr>
<tr>
<td>6</td>
<td>( z )</td>
<td>( -2y )</td>
<td>( -2x )</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>( x )</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>( x )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>( y )</td>
<td>( x )</td>
</tr>
</tbody>
</table>

where \( a \in \mathbb{K} - \{0\} \). Therefore the Sridharan enveloping algebra \( \mathcal{U}_f(\mathcal{G}) \) is a skew PBW extension.

Let \( \mathcal{G} \) be a finite dimensional Lie algebra. Then for any 2-cocycle \( f \in \mathbb{Z}^3(\mathcal{G}, \mathbb{K}) \), the following statements are equivalent (see [22], Theorem 5.3).
(i) The Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension $d$.

(ii) The universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is Calabi-Yau of dimension $d$.

Let $\mathcal{U}_f(\mathcal{G})$ be a Sridharan enveloping algebra of a finite dimensional Lie algebra $\mathcal{G}$. Then $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension 3 if and only if $\mathcal{U}_f(\mathcal{G})$ is isomorphic to $\mathbb{K}(x,y,z)/\langle R \rangle$ with the commuting relations $R$ listed in the following table (see [22], Theorem 5.5):

<table>
<thead>
<tr>
<th>Case</th>
<th>${x,y}$</th>
<th>${x,z}$</th>
<th>${y,z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z$</td>
<td>$-2x$</td>
<td>$2y$</td>
</tr>
<tr>
<td>2</td>
<td>$y$</td>
<td>$-z$</td>
<td>$0$</td>
</tr>
<tr>
<td>3</td>
<td>$z$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>4</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>5</td>
<td>$y$</td>
<td>$-z$</td>
<td>$1$</td>
</tr>
<tr>
<td>6</td>
<td>$z$</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>7</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

where $\{x,y\} = xy - yx$.

From the above discussion we have the following result.

**Proposition 2.** Let $\mathcal{U}_f(\mathcal{G})$ be a Sridharan enveloping algebra of a finite dimensional Lie algebra $\mathcal{G}$. If $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension 3 then $\mathcal{U}(\mathcal{G})$ is a skew PBW extension.

The Sridharan enveloping algebra of an $n$-dimensional abelian Lie algebra is $n$-Calabi-Yau; in particular the Weyl algebra $A_n$ is $2n$-Calabi-Yau (see [9], Theorem 6.5) as well as a skew PBW extension (see [16], Example 5).

### 3.2. Counterexamples

Next we will show some examples of algebras that are $AS$-regular, or $N$-Koszul, or Calabi-Yau, but are not skew PBW extensions.

1. $A = \mathbb{K}(x,y,z)/\langle xy - yx - x^2, yz - z^2, zx - xz - y^2 \rangle$ is $AS$-regular of global dimension 3 of type $A_2$ (see [2], Theorem 3.10), $A$ is 3-Koszul (see [8], Proposition 5.2) and Calabi-Yau of dimension 3 (see [6], Proposition 5.4).

2. $A = \mathbb{K}(x,y)/\langle x^3 + xy^2 + y^2x + xy + y^3 \rangle$ is $AS$-regular of global dimension 3 of $A_3$ (see [2], Theorem 3.10), $A$ is 3-Koszul, and Calabi-Yau of dimension 3 (see [6], Proposition 5.4).

3. $A = \mathbb{K}(x,y)/\langle xy \rangle$ is not $AS$-regular algebra. $A$ is the only graded algebra of global dimension 2 and $GK$-dimension 2 which is not Noetherian (see [2], page 172). $A$ is $2$-Koszul (see [15], page 7). $A$ is not $2$-Calabi-Yau (see [6], Proposition 4.3).

4. The exterior algebra $A = \mathbb{K}(x_1, \ldots, x_n)/\langle x_i^2 \rangle$ in $n$ variables is a 2-Koszul algebra (see [31], Example 1.6).

5. If $A = \mathbb{K}(x_1, \ldots, x_n)/I$ is an quadratic algebra and $I$ is principal, then $A = 2$-Koszul (see [15], page 7). It depends on the ideal $I$ whether $A$ is Calabi-Yau or not.

6. Consider $V$ of dimension 1, $V = \mathbb{K}x$ and $w = x^{N+1}$. Then, $\text{dim} R = \text{dim} R_{N+1} = 1$, $A(w)$ is $N$-Koszul (since the global dimension of $A(w)$ is infinite, and $A(w)$ is not 3-Calabi-Yau (see [4], Example 2.12).

### 4. Some Properties Preserved by Ore Extensions

Suppose $\sigma : A \to A$ is a graded algebra automorphism and $\delta : A(-1) \to A$ is a graded $\sigma$-derivation. If $B := [z, \sigma, \delta]$ is the associated Ore extension, then $B$ is a skew PBW extension. In this case we have $B = A[z, \sigma, \delta] = \sigma(A)(x)$ (see [16], Example 5).

Below we list some properties that are preserved by Ore extensions:

1. If $A$ is a connected graded algebra then $B$ is a connected graded algebra.

2. If $A$ is homologically smooth, then so is $B$ (see [30], Proposition 3.1).

3. $B$ is 2-Koszul if and only if $A$ is 2-Koszul (see [33], Corollary 1.3).

4. Let $A = \mathbb{K}(x_1, \ldots, x_n)/\langle f \rangle$ where $f = (x_1, \ldots, x_n)^T M (x_1, \ldots, x_n)$ and $M$ is an $n \times n$ matrix. Then $A$ is Calabi-Yau of dimension 2 if and only if $M$ is invertible and anti-symmetric (see [24], Corollary 1).

Let $\delta$ be a graded derivation of the free algebra $\mathbb{K}(x_1, \ldots, x_n)$ of degree 1. If $\delta(f) = 0$, then $\delta$...
induces a graded derivation $\tilde{\delta}$ on $A$. Let $B = A[z; \tilde{\delta}]$ be the Ore extension of $A$ defined by the graded derivation $\tilde{\delta}$. Then $B$ is a graded Calabi-Yau algebra of dimension 3 (see [21], Proposition 1.3).

5. If $A$ is $\nu$-skew Calabi-Yau projective $\mathbb{K}$-algebra of dimension $d$, then $B$ is skew Calabi-Yau of dimension $d + 1$ and the Nakayama automorphism $\nu'$ of $B$ satisfies that $\nu'_p = \sigma^{-1}\nu$ and $\nu'(z) = uz + b$, with $u, b \in A$ and $u$ invertible (see [30], Theorem 3.3).

6. Let $A$ be a 2-Koszul $AS$-regular algebra of global dimension $d$ with the Nakayama automorphism $\xi$. Then $B = A[z, \xi]$ is a Calabi-Yau algebra of dimension $d + 1$ (see [25], Theorem 3.3).

7. Let $A$ be a $\nu$-skew Calabi-Yau algebra of dimension $d$ and $\sigma \in Aut(A)$, then $A[x; \sigma]$ and $A[x^{\pm 1}; \sigma]$ are Calabi-Yau algebras of dimension $d + 1$ (see [18], Theorema 1.1). Furthermore, if $A[x; \sigma]$ is Calabi-Yau, then $A[x^{\pm 1}; \sigma]$ is Calabi-Yau.

8. Now we present an example of skew Calabi-Yau algebra that is not Calabi-Yau (see [30]), and then, we consider the corresponding Ore extension. Let $A = \mathbb{K}(x, y)/(yx - xy - x^2)$ be the Jordan plane, $A$ is $AS$-regular algebra of dimension 2 and therefore $A$ is 2-Koszul, $A = \mathbb{K}[x][y, \delta_1]$ with $\delta_1(x) = x^2$. It follows that $A$ is skew Calabi-Yau but not Calabi-Yau. $A$ has Nakayama automorphism given by $\nu(x) = x$ and $\nu(y) = 2x + y$. $B = A[z; \nu]$ is an Ore extension of Jordan plane. Then $B$ is skew Calabi-Yau with the Nakayama automorphism $\nu'$ such that $\nu'(x) = x$ and $\nu'(y) = y$. $B = \mathbb{K}[x, z][y; \delta]$ where $\delta$ is given by $\delta(x) = x^2$ and $\delta(z) = -2xz$. So, $\nu'(z) = z$. It follows that $B$ is Calabi-Yau, which was already proved by Berger and Pichereau in [5].

9. In [44], $AS$-regular algebras of dimension 5 generated by two generators of degree 1 with three generating relations of degree 4 are classified under some generic condition. There are nine types such $AS$-regular algebras in this classification list. Among them, the algebras $D$ and $G$ are given by iterated Ore extensions (see [44], Section 5.2).

The algebra $D$ is skew Calabi-Yau with the Nakayama automorphism $\nu$ given by $\nu(x) = p^{-3}q^4x$; $\nu(y) = p^3q^{-3}y$. $D$ is Calabi-Yau if and only if that $p, q$ satisfy the system of equations (see [30], Theorem 4.3):

\[
\begin{cases}
    p^3 = q^4, \\
    2p^3 - p^2q + q^2 = 0.
\end{cases}
\]

The algebra $G$ is skew Calabi-Yau with the Nakayama automorphism $\nu$ given by $\nu(x) = gx$; $\nu(y) = g^{-1}y$. $D$ is Calabi-Yau if and only if $g = 1$.

They study and classification of $AS$-regular algebras of dimension five with two generators under an additional $\mathbb{Z}^2$-grading uses Gröbner basis computations (see [48]).

10. Let $\mathbb{K}$ be a field, let $n$ be an even natural number $\geq 2$, and let $A$ be the associative $\mathbb{K}$-algebra defined by generators $x_1, \ldots, x_n$ subject to the single relation

\[
\sum_{1 \leq i \leq \frac{n}{2}} [x_i, x_i + \frac{n}{2}] = \nu + \lambda,
\]

where the bracket stands for the commutator, $\nu$ is a linear combination of the $x_i$’s, and $\lambda \in \mathbb{K}$. Then the filtered algebra $A$ is 2-Koszul. Furthermore $A$ is 2-Calabi-Yau if and only if $\nu = 0$ (see [9], Theorem 6.4). So, if $\sigma_2 = i_{\mathbb{K}[x_1]}$ and $\delta_2(\mathbb{K}[x_1]) \subseteq \mathbb{K}$, then the skew PBW extension $\sigma(\mathbb{K})/(x_1, x_2) \cong \mathbb{K}[x_1][x_2; \sigma_2, \delta_2]$ is 2-Calabi-Yau.

References


