Some Remarks About the Cyclic Homology of Skew $PBW$ Extensions

Algunas observaciones sobre la homología cíclica de extensiones $PBW$ torcidas

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Abstract

We study the cyclic homology for a class of noncommutative polynomial rings known as skew $PBW$ extensions. We obtain explicit computations for some important families of such extensions over fields. In particular, we consider the cyclic homology of skew $PBW$ extensions of derivation type, certain classes of Ore extensions, operator algebras, diffusion algebras, quantum algebras and 3-dimensional skew polynomial algebras.

Key words: Cyclic homology, Filtered rings, Skew $PBW$ extensions.

1. Introduction

Cyclic homology of algebras was discovered by Connes in the formulation of noncommutative differential geometry [3]. In connection with the pairing with algebraic or topological $K$-theory, cyclic homology is quite useful also for the study of $K$-theory. For instance, Connes uses cyclic cocycles to express certain characteristic classes of a foliation in connection with the topological $K$-theory of the associated foliation $C^*$-algebra. In this context, it seems to be important to compute cyclic cohomology of interesting algebras, which appear in differential topology or in algebraic geometry. Cyclic homology has been studied in a series of papers as a noncommutative generalization of de Rham cohomology (cf. [14], [23], [4], [5]) in order to interpret index theorems for non-commutative Banach algebras, via a generalization of the Chern character, where it

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was shown in [3] that cyclic homology of $C^\infty(M)$ recovers the C-coefficient de Rham homology of the compact smooth manifold $M$. Cyclic homology was also shown to be the primitive part of the Lie algebra homology of matrices by Quillen and Loday [14]. This relationship shows that cyclic homology can be considered as a Lie analogue of algebraic $K$-theory and it is sometimes referred to as non-commutative differential geometry. Following [15], the cyclic homology of an $k$-algebra $B$ ($k$ being a commutative ring) consists of abelian groups $HC_n(B), n \geq 0$. If $k$ is a field with characteristic zero, these groups are the homology groups of the quotient of the Hochschild complex by the action of the finite cyclic groups; this is the reason for the term “cyclic”. The notation $HC$ was for “Homologie de Connes”, but soon became “Homologie Cyclique”.

Since we are interested in computing the cyclic homology groups of skew PBW (PBW denotes Poincaré-Birkhoff-Witt) extensions introduced in [7], in this paper we have compiled some facts about these groups for certain examples of this kind of extensions. We consider that this study enriches the study of non-commutative differential geometry of a considerable number of noncommutative rings and quantum groups (for instance, quantum spaces whose cyclic homologies were known [4], quasi-commutative algebras [11], Q-difference operators [9], Ore extensions and some quantum algebras [10]). The techniques used here are fairly standard and follow the same path as other text on the subject. The results presented are new for skew PBW extensions and all they are similar to others existing in the literature. In this way, we continue the task of studying several properties of skew PBW extensions and its relationship with other noncommutative rings (see [1], [8], [12], [13], [17], [18], [19], [20], [22], [24] and others).

The paper is organized as follows. In Section 2 we describe the skew PBW extensions. In Section 3 we recall the higher algebraic $K$-theory of these extensions following [13], and then we present the key results of this paper concerning about cyclic homology of these rings. Finally, in Section 4 we illustrate the results about cyclic homology of Section 3 with several examples such as Ore extensions, operator algebras and 3-dimensional skew polynomial algebras.

## 2. Skew PBW Extensions

### Definition 1 ([7], Definition 1). Let $R$ and $A$ be rings. We say that $A$ is a skew PBW extension of $R$ (also called a $\sigma$-PBW extension of $R$) if the following conditions hold:

1. $R \subseteq A$.
2. There exist finite elements $x_1, \ldots, x_n \in A$ such that $A$ is a left $R$-free module with basis
   
   
   $$\text{Mon}(A) := \{x^\alpha = x_1^{a_1} \cdots x_n^{a_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

   We say also that $A$ is a left polynomial ring over $R$ with respect to the set of variables $\{x_1, \ldots, x_n\}$ and $\text{Mon}(A)$ is the set of standard monomials of $A$. In addition, $x_1^0 \cdots x_n^0 := 1 \in \text{Mon}(A)$.

3. For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i x_r = c_{i,r} x_i r + R x_1 + \cdots + R x_n$.
4. For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_i x_j = c_{i,j} x_j x_i + R x_1 + \cdots + R x_n$.

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

The following proposition justifies the notation for skew PBW extensions. Before, we recall the notion of skew polynomial ring. If $B$ is a ring and $\sigma$ is a ring endomorphism $\sigma : B \to B$, a $\sigma$-derivation $\delta : B \to B$ satisfies by definition $\delta(r s) = \delta(r) + \delta(s)$, and $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in B$. If $y$ is an indeterminate, and $yb = \sigma(b)y + \delta(b)$, for any $b \in B$, we denote this noncommutative ring as $B[y; \sigma, \delta]$ which is called a skew polynomial ring.

### Proposition 2 ([7], Proposition 3). Let $A$ be a skew PBW extension of $R$. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$ for each $r \in R$.

### Definition 3. Let $A$ be a skew PBW extension.

1. ([7], Definition 4.) $A$ is quasi-commutative if the conditions (iii) and (iv) in Definition 1 are replaced by
2. (iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i x_r = c_{i,r} x_i r$.
3. (iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_i x_j = c_{i,j} x_j x_i$.
4. $A$ is bijective if $\sigma_i$ is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$. 


(c) ([12], Definition 2.3.) A is a skew PBW extension of derivation type if \( \sigma = \text{id}_R \) for \( 1 \leq i \leq n \).

Definition 4. A filtered ring is a ring \( B \) with a family \( FB = \{ F_nB \mid n \in \mathbb{Z} \} \) of additive subgroups of \( B \) where we have the ascending chain \( \cdots \subset F_{n+1}B \subset F_nB \subset \cdots \) such that \( 1 \in F_0B \) and \( F_nBF_MB \subseteq F_{n+m}B \) for all \( n, m \in \mathbb{Z} \). From a filtered ring \( B \) it is possible to construct its associated graded ring \( G(B) \) taking \( G(B)_n := F_nB/F_{n-1}B \).

The next proposition computes the graduation of a general skew PBW extension of a ring \( R \). This result will very important in Proposition 12.

Proposition 5 ([13], Theorem 2.2). Let \( A \) be an arbitrary skew PBW extension of \( R \). Then, \( A \) is a filtered ring with increasing filtration given by

\[
F_mA := \begin{cases} R & \text{if } m = 0 \\ \{ f \in A \mid \deg(f) \leq m \} & \text{if } m \geq 1 \end{cases}
\]

(2.1)

and the corresponding graded ring \( G(A) \) is a quasi-commutative skew PBW extension of \( R \). Moreover, if \( A \) is bijective, then \( G(A) \) is a quasi-commutative bijective skew PBW extension of \( R \).

Remark 6. The associated graded ring \( G(A) \) is the skew PBW extension of \( R \) generated by the variables \( z_1, \ldots, z_n \) with the relations \( z_if = c_{i_1}z_{i_1} \cdots z_iz_{i_j} = c_{i_1} \cdots z_{i_j} \), for \( 1 \leq i \leq n \), where \( c_{i_1}, c_{i_j} \) are the same constants that define \( A \). See [13], Proposition 2.1 for a proof of this assertion.

Proposition 7 establishes the relation between skew PBW extensions and iterated skew polynomial rings in the sense of Proposition 2.

Proposition 7 ([13], Theorem 2.3). Let \( A \) be a quasi-commutative skew PBW extension of a ring \( R \). Then (i) \( A \) is isomorphic to an iterated skew polynomial ring, and (ii) if \( A \) is bijective, each endomorphism of the skew polynomial ring in (i) is an isomorphism.

3. Algebraic \( K \)-Theory and Lie Analogue of Higher Algebraic \( K \)-Theory

As we said in the Introduction, cyclic homology was shown to be the primitive part of the Lie algebra homology of matrices by Quillen and Loday [14]. This relationship shows that cyclic homology can be considered as a Lie analogue of algebraic \( K \)-theory and it is sometimes referred to as non-commutative differential geometry. With this in mind, in the first part of this section we recall the higher algebraic \( K \)-theory of skew PBW extensions following [13], while the second part treats with the cyclic homology of skew PBW extensions.

3.1. Higher Algebraic \( K \)-Theory of Skew PBW Extensions

Quillen [16] proposed defining the higher algebraic \( K \)-theory of a ring \( B \) to be the homotopy groups of a certain topological space which he called “\( \text{BGL}(R)^+ \)”. In fact, he provided two fundamentally different ways of defining higher algebraic \( K \)-groups, one homotopy theoretic and the other category theoretic. Following a categoric treatment, Lezama and Reyes [13] computed the higher algebraic \( K \)-theory of bijective skew PBW extensions.

Proposition 8 ([13], Theorem 5.1). Let \( R \) be a left Noetherian left regular ring. If \( A \) is a bijective skew PBW extension of \( R \), then \( K_i(A) \cong K_i(R) \) for all \( i \geq 0 \).

With this result, it is possible to obtain the Quillen’s groups \( K_i, i \geq 0 \) for several families of non-commutative rings which are examples of bijective skew PBW extensions. For instance, PBW extensions, Ore extensions of bijective type, operator algebras, diffusion algebras, some quantum algebras, and 3-dimensional skew polynomial algebras, and some localizations of skew PBW extensions (see [13] for a detailed reference of every family). A detailed list of these groups can be found in [13].

3.2. Lie Analogue of Higher Algebraic \( K \)-Theory of Skew PBW Extensions

We recall the definitions of Hochschild and cyclic homology following [6]. For more details see [4], [5], [14], [15] or [23].

Definition 9. Let \( B \) be an associative algebra over a commutative unital ring \( k \), \( k \subset B \). The Hochschild homology of \( B \), denoted by \( \text{HH}_n(B) \), is defined to be the homology of the following complex

\[
0 \leftarrow C_0(B) \leftarrow C_1(B) \leftarrow C_2(B) \leftarrow \cdots,
\]

(3.1)

where \( C_n(B) = B^\otimes_{n+1} \), and \( d : C_n(B) \to C_{n-1}(B) \) such that

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Let $R$ be a ring. If $A$ is a skew PBW extension of $R$ and $d(G(A)) < \infty$, then $HC_i(A) \cong HC_i(R)$ for all $i \geq d(G(A))$.

**Proof.** The result follows from Proposition 5 and Proposition 11.

The following example will be of great importance in the Section 4.

**Example 13.** Let $k$ be a field, $Q = (q_{i,j})_{1 \leq i, j \leq v}$ a family of elements of $k \setminus \{0\}$ verifying $q_{i,i} = 1$ and $q_{i,j}q_{j,i} = 1$ for all $i < j$, and $X = \{x_1, \ldots, x_v\}$ a set of $v$ indeterminates. The multiparametric affine space is the $k$-algebra $S_Q(X)$ generated by $x_1, \ldots, x_v$ and the relations $x_j x_i = q_{i,j} x_i x_j$, $1 \leq i < j \leq v$. Note that $S_Q(X) = \sigma(k)\langle x_1, \ldots, x_v\rangle$.

When all the coefficients $q_{i,j}$ for $i < j$ are equal to a constant $q$, the Hochschild dimension of $S_Q(X)$ is equal to $v$ if $q$ is a root of unity, and it is equal to 1 if $q$ is not a root of unity ([10], Remark 3.1.1). In the general case we have

(i) $HH_n(S_Q(X)) = 0$ for all $n > v$;

(ii) if $HH_n(S_Q(X)) \neq 0$, then $HH_m(S_Q(X)) \neq 0$ for all $m \leq n$.

(iii) The Hochschild dimension of $S_Q(X)$ is the greatest $n$ such that there exist indexes $1 \leq p_1 < \cdots < p_n \leq v$ and a family of natural members $m_{p_i}$, $1 \leq i \leq n$ verifying

$$\prod_{i=1}^n q_{p_i,p_i}^{m_{p_i}} = 1 \ \forall r \text{ with } 1 \leq r \leq n$$

(see [10], Corollary 3.1.4 for more details).

Remark 6, Proposition 12 and Example 13 guarantee the following result

**Proposition 14.** Let $A$ be a skew PBW extension of type derivation over a field $k$. Then $HC_i(A) \cong HC_i(k)$ for $i \geq d(S_Q(X))$.

**4. Examples**

In this section we present examples of skew PBW extensions where Proposition 12 and Proposition 14 can be applied. The complete references of all examples can be found in [13].
4.1. PBW Extensions

Any PBW extension \( A = \sigma(R) \langle x_1, \ldots, x_n \rangle \) is a bijective skew PBW extension since in this case \( \sigma_i = id_R \) for every \( 1 \leq i \leq n \), and \( c_{i,j} = 1 \) for every \( 1 \leq i, j \leq n \). Thus, for PBW extensions we have \( A = id_k(R) \langle x_1, \ldots, x_n \rangle \). Some particular examples of PBW extensions are the polynomial rings, skew polynomial rings of derivation type, universal enveloping algebras, and differential operator rings.

We recall that a filtered algebra \( B \) is called quasi-commutative if its associated graded algebra is commutative. This is the case for PBW extensions above, which follows from Proposition 5. The Hochschild homology and the cyclic homology of this type of algebras was computed in [11].

Example 15.

1. ([15], p. 10, 59). For a commutative ring \( k \),
   \[ HH_0(k) = k \text{ and } HH_n(k) = 0, \quad n > 0. \]
   With respect to the cyclic homology, \( HC_{2n}(k) = k \),
   \[ HC_{2n+1}(k) = 0, \quad n > 0. \]

2. Let \( \mathcal{U}(g) \) be the universal enveloping algebra of a Lie algebra \( g \) of dimension \( n \) over a field \( k \) of characteristic 0. If \( G(\mathcal{U}(g)) \) is isomorphic to a polynomial ring \( k[x_1, \ldots, x_n] \), and \( d(\langle x_1, \ldots, x_n \rangle) = n \), then \( HC_i(\mathcal{U}(g)) \cong HC_i(k) \) for \( i \geq n \).

3. For any Lie algebra \( g \), Kassel [11] computed the Hochschild and cyclic homology groups of its enveloping algebra in terms of the canonical Lie-Poisson structure on the dual \( g^* \). For the first Weyl algebra, Kassel proves that if \( k \) is a field of characteristic zero, then
   \[
   H_i(A_1(k)) = \begin{cases} k, & i = 2, \\ 0, & \text{other case.} \end{cases}
   \]
   \[
   HC_i(A_1(k)) = \begin{cases} k, & i \text{ is even, } i \geq 2, \\ 0, & \text{other case.} \end{cases}
   \]
   (4.1)

Since \( A_1(k) \cong \sigma(k[x]) \langle y \rangle \), we have \( G(A_1(k)) = k[x,y] \) and \( d(G(A_1(k))) = 2 \). Hence, Proposition 12 implies \( HC_i(A_1(k)) = HC_i(k) \), \( i \geq 2 \), which coincides with (4.1) and [15], p. 10, 59.

4. Loday [15], p. 94, showed that for a field \( k \) containing the field of rational numbers \( \mathbb{Q} \), the Hochschild homology and the cyclic homology of the Weyl algebra \( A_n(k) \), are given by
   \[
   H_i(A_n(k)) = \begin{cases} k, & i = 2n, \\ 0, & \text{otherwise.} \end{cases}
   \]
   \[
   HC_i(A_n(k)) = \begin{cases} k, & i = 2j, j \geq n, \\ 0, & \text{otherwise,} \end{cases}
   \]
   (4.2)

respectively. Since \( A_n(k) \cong \sigma(k[x_1, \ldots, x_n]) \langle y_1, \ldots, y_n \rangle \) ([13], Section 3.1), then \( G(A_n(k)) \cong k[x_1, \ldots, x_n, y_1, \ldots, y_n] \). In this way \( d(G(A_n(k))) = 2n \), and by Proposition 12 we obtain \( HC_i(A_n(k)) = HC_i(k[x_1, \ldots, x_n]) \), \( i \geq 2n \), which coincides with (4.2).

5. With respect to universal enveloping algebras, Kassel [11] computed its Hochschild homology and cyclic homology. He obtain exact values for the groups \( H_i(\mathcal{U}(\mathfrak{so}(2, k))) \), \( HC_i(\mathcal{U}(\mathfrak{so}(2, k))) \), \( H_i(\mathcal{U}(\mathfrak{so}(4))) \), and \( HC_i(\mathcal{U}(\mathfrak{so}(4))) \), \( i \geq 0 \).

4.2. Ore Extensions of Bijective Type

Any skew polynomial ring \( R[x; \sigma, \delta] \) of bijective type (\( \sigma \) bijective) is a bijective skew PBW extension. In this case we have \( R[x; \sigma, \delta] \cong R \langle x \rangle \). If additionally \( \delta = 0 \), then \( R[x; \sigma] \) is quasi-commutative. In a general way, let \( R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \) be an iterated skew polynomial ring of bijective type, i.e., the following conditions hold:

- for \( 1 \leq i \leq n \), \( \sigma_i \) is bijective;
- for every \( r \in R \) and \( 1 \leq i \leq n \), \( \sigma_i(r) \delta_i(r) \in R \);
- for \( i < j \), \( \sigma_j(x_i) = c_{i,j} x_i + d_i \), with \( c, d \in R \) and \( c \) has a left inverse;
- for \( i < j \), \( \delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n \);

then, \( R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \) is a bijective skew PBW extension. Under these conditions we have \( R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \cong \sigma(R) \langle x_1, \ldots, x_n \rangle \) ([13], Section 3.2). Therefore, by Remark 6 we have \( G(\sigma(R) \langle x_1, \ldots, x_n \rangle) \cong R[z_1; \sigma_{z_1}] \cdots [z_n; \sigma_{z_n}] \), where \( \sigma(z) = c_{j,r} x_j + c_i z_i \) for \( 1 \leq i < j \leq n \).

By Proposition 12 we obtain

\[
HC_i(R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]) \cong HC_i(R),
\]
for \( i \geq d(G(\sigma(R) \langle x_1, \ldots, x_n \rangle)) \). (4.3)

Example 16. Some remarkable examples of this kind of noncommutative rings are the following:

1. Quantum plane \( \mathcal{O}_q(k^2) \). Let \( q \in k \setminus \{0\} \). The quantized coordinate ring of \( k^2 \) is a \( k \)-algebra,
denoted by $\mathcal{O}_q(k^2)$, presented by two generators $x, y$ and the relation $xy = qyx$. We have $\mathcal{O}_q(k^2) \cong \sigma(k)[x, y]$, $G(\mathcal{O}_q(k^2)) \cong k[y][x; \sigma]$. If $q$ is a root of the unity, $HC_1(\mathcal{O}_q(k^2)) \cong HC_1(k)$ for $i \geq 2$, and if $q$ is not a root of the unity, $HC_1(\mathcal{O}_q(k^2)) \cong HC_1(k)$ for $i \geq 1$.

2. The algebra of $q$-differential operators $D_{q,h}[x, y]$. Let $q, h \in k$, $\sigma$ be an extension satisfying the following conditions: if $k$ is a field, $HC(\mathcal{O}_q(k^2)) \cong HC_1(k)$ for $i \geq 2$, and if $q$ is not a root of the unity, $HC(\mathcal{O}_q(k^2)) \cong HC_1(k)$ for $i \geq 1$.

3. The mixed algebra $D_h$. It is defined by $D_h := k[x; \sigma_h, \delta_h]$, where $h \in k$ and $\sigma_h(x) := x$. Then $D_h \cong \sigma_h(\mathcal{O}_q(k^2)) \cong \sigma_h(k)[x; \sigma_h]$, again, if $q$ is a root of the unity, $HC_1(D_{q,h}) \cong HC_1(k)$ for $i \geq 3$. In other case, $HC_1(D_{q,h}) \cong HC_1(k)$ for $i \geq 1$.

Remark 17. Guccione and Guccione in [10] computed under certain conditions the Hochschild homology and the cyclic homology of Ore extensions. For instance, they proved ([10], Corollary 2.5) that if $k$ is a field, $B$ a $k$-algebra and $E = B[t; \alpha, \delta]$ an Ore extension satisfying the following conditions:

- As a $k$-module, $B$ is a direct sum $B = \bigoplus_{s \in M} B_s$, with $B_0 = k$.
- There exist $q_1, \ldots, q_s \in k \setminus \{0\}$ such that $\alpha^{-1}(a) = q_1 a \cdots q_s a$, provided that $a \in B_{(m_1, \ldots, m_s)}$.

and if $q_1 \cdots q_s = 1$ implies $m_1 = \cdots = m_s = 0$, then the Hochschild homology of $E$ with coefficients in $E$ is given by

$$HH_0(E) = HH_0(B) \oplus \bigoplus_{r \geq 1} k[t^r],$$

$$HH_1(E) = HH_1(B) \oplus \bigoplus_{r \geq 0} k[t^r] \otimes t,$$

$$HH_n(E) = HH_n(B), \quad n > 1.$$

The cyclic homology of $E$ is given by

$$HC_0(E) = HC_0(B) \oplus \bigoplus_{r \geq 1} k[t^r],$$

$$HC_n(E) = HC_n(B) \oplus \bigoplus_{r \geq 1} \langle t \rangle, \quad n > 0.$$

4.3. Operator Algebras

In this subsection we recall some important and well-known operator algebras. Some of these algebras are skew PBW extensions of fields and hence we can apply the result established in (4.3).


The nth Weyl algebra $A_n(k)$ over $k$ coincides with the $k$-algebra of linear partial differential operators with polynomial coefficients $k[t_1, \ldots, t_n]$. As we have seen, the generators of $A_n(k)$ satisfy the following relations $t_i t_j = t_j t_i$, $\partial_i \partial_j = \partial_j \partial_i$, for $1 \leq i < j \leq n$, and $\partial_i t_j = t_j \partial_j + \delta_{ij}$, for $1 \leq i, j \leq n$, where $\delta_{ij}$ is the Kronecker symbol. Therefore $\sigma(k)[t_1, \ldots, t_n; \partial_1, \ldots, \partial_n]$, its associated graded ring is isomorphic to $k[t_1, \ldots, t_n; \partial_1, \ldots, \partial_n]$ and $HC_1(\sigma(k)[t_1, \ldots, t_n; \partial_1, \ldots, \partial_n]) \cong HC_1(k)$, for $i \geq 2n$.


For a fixed $q \in k \setminus \{0\}$, this is the $k$-algebra $k[t_1, \ldots, t_n; D_1^{(q_1)}, \ldots, D_m^{(q_m)}], n \geq m$, subject to the relations:

$$t_i t_j = t_j t_i, \quad 1 \leq i < j \leq n,$$

$$D_j^{(q_1)} t_i = q_i D_j^{(q_1)} + 1, \quad 1 \leq i \leq m,$$

$$D_j^{(q_1)} t_i = t_j D_j^{(q_1)}, \quad i \neq j,$$

$$D_j^{(q_1)} D_j^{(q_2)} = D_j^{(q_1)} D_j^{(q_2)}, \quad 1 \leq i < j \leq m.$$

If $m = n$, this operator algebra coincides with the additive analogue $A_n(q_1, \ldots, q_n)$ of the Weyl algebra $A_n(q)$ (Section 4.5, Example (a)). This algebra can be expressed as the skew PBW extension $\sigma(k)[t_1, \ldots, t_n; D_1^{(q_1)}, \ldots, D_m^{(q_m)}]$, and hence $HC_1(\sigma(k)[t_1, \ldots, t_n; D_1^{(q_1)}, \ldots, D_m^{(q_m)}]) \cong HC_1(k)$, $i \geq n + m$.

3. Operator differential rings. Let $R$ be an algebra over a commutative ring $k$ and let $\Lambda = \{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations of $R$. Let $T = R[t_1, \ldots, t_n; \theta_1, \ldots, \theta_n]$ be the operator differential ring. The elements of $T$ can be written in a unique way as left $R$-linear combinations with the ordered monomials in $\theta_1, \ldots, \theta_n$. The product on $T$ is defined extending the product from $R$ subject to the relation $\theta_i r - r \theta_i = \delta_i(r), \quad r \in R, \quad i = 1, \ldots, n$, and $\theta_i \theta_j - \theta_j \theta_i = 0, \quad i, j = 1, \ldots, n$, and $T = \sigma(R)[\theta_1, \ldots, \theta_n]$. Then $HC_1(T) = HC_1(R)$ for $i \geq 1$.  

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4.4. Diffusion Algebras

Diffusion algebras arose in physics as a possible way to understand a large class of 1-dimensional stochastic processes. A diffusion algebra $\mathcal{A}$ with parameters $a_{ij} \in \mathbb{C} \setminus \{0\}$, $1 \leq i, j \leq n$ is a $\mathbb{C}$-algebra generated by indeterminates $x_1, \ldots, x_n$ subject to relations $a_{ij} x_j x_i - b_{ij} x_i x_j = r_{ij} x_j - r_{ji} x_i$, whenever $i < j$, $b_{ij}, r_i \in \mathbb{C}$ for all $i < j$. Therefore $\mathcal{A}$ admits a $\text{PBW}$-basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$, that is, $\mathcal{A}$ is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for $\mathcal{A}$. From Definition 1, (iii) and (iv), it is clear that the family of skew $\text{PBW}$ extensions are more general than diffusion algebras.

In the applications to physics the parameters $a_{ij}$ are strictly positive reals and the parameters $b_{ij}$ are positive reals as they are unnormalised measures of probability. We will denote $q_{ij} := \frac{b_{ij}}{a_{ij}}$. The parameter $q_{ij}$ is a root of unity if and only if $i$ is equal to 1. It is therefore reasonable that we will sometimes assume these parameters not to be a root of unity other than 1. If all coefficients $q_{ij}$ are nonzero, then the corresponding diffusion algebra have a $\text{PBW}$-basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$ and hence these algebras are skew $\text{PBW}$ extensions, but not Ore extensions (see [13]). Therefore, $HC_i(\mathcal{A}) = HC_i(\mathbb{C})$ for $i \geq d(G(\mathcal{A}))$.

4.5. Quantum Algebras

The term quantum group is usually used, not only for the $q$-analogue of the coordinate ring of a semisimple algebraic group but also for the $q$-analogue of the universal enveloping algebra of a semisimple Lie algebra. A quantum group is a Hopf algebra but in this article we will only consider with the algebra structure and so the coalgebra structure will not be considered. We remark that the exact value of the Hochschild dimension of the associated graded ring of every skew $\text{PBW}$ extension $A$, that is, $d(G(A))$ can be found it using Example 13.

Example 18.

1. Additive analogue of the Weyl algebra. The $\mathbb{k}$-algebra $A_\nu(q_1, \ldots, q_n)$ is generated by the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the relations:

$$ x_j x_i = x_i x_j, \quad 1 \leq i, j \leq n, $$
$$ y_j y_i = y_i y_j, \quad 1 \leq i, j \leq n, $$
$$ y_i x_j = x_j y_i, \quad i \neq j, $$
$$ y_i x_i = q_i x_i y_i + 1, \quad 1 \leq i \leq n, $$

where $q_i \in k \setminus \{0\}$. In this way we have $A_\nu(q_1, \ldots, q_n) \cong \sigma(k)[x_1, \ldots, x_n, y_1, \ldots, y_n]$, that is, $A_\nu(q_1, \ldots, q_n)$ is a skew $\text{PBW}$ extension of the field $k$.

2. Quantum algebra $\mathcal{W}(\mathfrak{so}(3, k))$. This algebra is the $q$-analogue of the universal enveloping algebra $\mathfrak{so}(3, k)$. By definition it is the $k$-algebra generated by the variables $l_1, l_2, l_3$ subject to relations $l_2 l_1 - q l_1 l_2 = -q^{-1/2} I_3$, $l_3 l_1 - q^{-1} I_1 l_3 = q^{-1/2} I_3$, $l_2 l_3 - q l_3 l_2 = -q^{-1} l_1$, where $q \in k \setminus \{0\}$. We have $\mathcal{W}(\mathfrak{so}(3, k)) \cong \sigma(k)(l_1, l_2, l_3)$.

3. Dispnic algebra $\mathcal{W}(\mathfrak{osp}(1, 2))$. This algebra is generated by $x, y, z$ over the commutative ring $k$ satisfying the relations $yz = zy = z$, $xz = y$, $xy = yx = x$. Thus, $\mathcal{W}(\mathfrak{osp}(1, 2)) \cong \sigma(k)(x, y, z)$.

4. Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2, k))$. This algebra is generated by $x, y, z$ subject to the relations $xz - v^4 zx = (1 + v^2)x$, $xy - v^2 yx = vz$, $zy - v^4 yz = (1 + v^2)y$, where $\nu \in k \setminus \{0\}$ is not a root of unity. Under certain conditions on $\nu$ we have the isomorphism $\mathcal{W}_\nu(\mathfrak{sl}(2, k)) \cong \sigma(k)(x, y, z)$.

5. $q$-Heisenberg algebra. The $k$-algebra $H_n(q)$ is generated by the set of variables $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$ subject to the relations:

$$ x_j x_i = x_i x_j, z_j z_i = z_i z_j, y_j y_i = y_i y_j, \quad 1 \leq i, j \leq n, $$
$$ z_j y_i = y_i z_j, z_j x_i = x_i z_j, y_j x_i = x_j y_i, \quad i \neq j, $$
$$ z_i y_i = q y_i z_i, z_i x_i = q^{-1} x_i z_i + y_i, y_i x_i = q x_i y_i, \quad 1 \leq i \leq n, \text{ with } q \in k \setminus \{0\}. \quad \text{Then } H_n(q) \cong \sigma(k)(x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n)$.

4.6. 3-Dimensional Skew Polynomial Algebras

The universal enveloping algebra $\mathcal{W}(\mathfrak{sl}(2, k))$ of the Lie algebra $\mathfrak{sl}(2, k)$, the Woronowicz’s algebra $\mathcal{W}_\nu(\mathfrak{sl}(2, k))$ and the dispnic algebra $\mathcal{W}(\mathfrak{osp}(1, 2))$, are examples of algebras classified by Smith and Bell which are known as 3-dimensional skew polynomial algebras. Next we recall the definition of this algebras and we will see that they are particular examples of skew $\text{PBW}$ extensions.
**Definition 19.** A 3-dimensional skew polynomial algebra $\mathcal{A}$ is the $k$-algebra generated by the variables $x, y, z$ restricted to relations
\[
yz - \alpha zy = \lambda, \quad zx - \beta xz = \mu, \quad xy - \gamma xy = \nu,
\]
such that
\[
1. \lambda, \mu, \nu \in k + kx + ky + kz, \quad \text{and} \quad \alpha, \beta, \gamma \in k^*; \\
2. \text{Standard monomials } \{x^iy^jz^k | i, j, k \geq 0\} \text{ are a } k\text{-basis of the algebra.}
\]

It is clear from Definition 4.4 that 3-dimensional skew polynomial ring are skew PBW extensions of the field $k$. Hence, $HC_i(\mathcal{A}) = HC_i(k)$ for $i \geq d(G(\mathcal{A}))$.

Finally, we recall Proposition 20 which establishes a classification of 3-dimensional skew polynomial algebras.

**Proposition 20** ([21], Theorem 4.3.1, p. 101). Let $\mathcal{A}$ be a 3-dimensional skew polynomial algebra. Then $\mathcal{A}$ is one of the following algebras:

(a) If $|\{\alpha, \beta, \gamma\}| = 3$, then $\mathcal{A}$ is defined by
\[
yz - \alpha zy = 0, \quad zx - \beta xz = 0, \quad xy - \gamma xy = 0.
\]
(b) If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, $\mathcal{A}$ is one of the following algebras:

(i) $yz - zy = z, \quad zx - \beta xz = \gamma$, \quad $xy - yx = x$;
(ii) $yz - zy = z, \quad zx - \beta xz = y$, \quad $xy - yx = x$;
(iii) $yz - zy = 0, \quad zx - \beta xz = y$, \quad $xy - yx = 0$;
(iv) $yz - zy = 0, \quad zx - \beta xz = y$, \quad $xy - yz = 0$;
(v) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = x$;
(vi) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = 0$.

Here $\alpha, \beta$ are any elements of $k$. All nonzero values of $b$ give isomorphic algebras.

(c) If $|\{\alpha, \beta, \gamma\}| = 2$ and $\alpha \neq \beta = \gamma \neq 1$, then $\mathcal{A}$ is one of the following algebras:

(i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b$, \quad $xy - \alpha xy = 0$;
(ii) $yz - \alpha zy = 0, \quad zx - \beta xz = y$, \quad $xy - \alpha xy = 0$.

In this case $b$ is an arbitrary element of $k$. Again, any nonzero values of $b$ give isomorphic algebras.

(d) If $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is the algebra
\[
yz - \alpha zy = a_1 x + b_1, \quad zx - \alpha xz = a_2 y + b_2, \quad xy - \alpha xy = a_3 z + b_3.
\]

If $a_i = 0, i = 1, 2, 3$, all nonzero values of $b_i$ give isomorphic algebras.

(e) If $\alpha = \beta = \gamma = 1$, $\mathcal{A}$ is isomorphic to one of the following algebras

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z$;
(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z$;
(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b$;
(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0$;
(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0$.

Parameters $a, b \in k$ are arbitrary and all nonzero values of $b$ generates isomorphic algebras.

References


