On what there is in philosophy of mathematics

Sobre lo que hay en filosofía de las matemáticas

Ice B. Risteski
Toronto, Canada.
E-mail: ice@scientist.com

Carlos E. García D.
Universidad de Caldas, Universidad de Manizales.
E-mail: carlos.garcia_d@ucaldas.edu.co

Recibido el 30 de noviembre de 2007 y aprobado el 14 de marzo de 2008

Abstract

In this paper, we sketch roughly several moments that have a considerable influence on the development of the philosophy of mathematics. We illustrate the main problems of this discipline and discuss the import of the so called 'positiviness of mathematics'. By examining several approaches to this problem, we analyze ontological and epistemological questions which can help to clarify this complicated area.

Resumen

En este artículo presentamos un bosquejo de diversos momentos que han ejercido una influencia considerable sobre el desarrollo de la filosofía de las matemáticas. Ilustramos los principales problemas de esta disciplina y discutimos el sentido y alcance de la denominada 'positividad de la matemática'. Además, mediante el examen de diversos enfoques sobre este tópico, analizamos algunas cuestiones epistemológicas y ontológicas que pueden arrojar algo de luz sobre esta complicada área de la metafilosofía.

Key words

Philosophy of mathematics, logic, abstract mathematics, pure mathematics, application of mathematics.

Palabras clave

Filosofía de las matemáticas, lógica, matemática abstracta, matemática pura, aplicación de las matemáticas.

Philosophy of mathematics is a very complicated discipline and is full of divergent ideas. Philosophy of mathematics does the problematization of mathematics, *i.e.*, it brings the latter into question. It is a critical survey on mathematics, a survey that requires distance and abandonment of mathematical thinking. Hence it must be distinguished from the uncritical penetration, in the framework of the mathematical thinking in which mathematics itself is developed.

However, this approach can be misinterpreted by quasiphilosophers who believe that their vision of a scientist who cannot understand the essence of science is confirmed once more, *i.e.*, their vision of a mathematician who cannot understand the essence of mathematics. Such a scientist would be one who cannot leave science and understand it clearly in a philosophical way. But such a view is not maintained here. As we know, most scientists do not leave science, and most philosophers do not penetrate into science, so they cannot leave a place they have never visited. Quasiphilosophers forget that something can be walked out of only if it has been walked into beforehand. In general, only a scientist who leaves science or a philosopher who penetrates into science, maintaining a philosophical distance from it, can have a relevant philosophical view of science. Thus pure scientists have little to say in the present discussion as do the so called pure philosophers.

We discuss here a problematization of mathematics as a science, or, more precisely, as a paradigmatic example of a science, according to two directions:

1. Mathematics is a paradigmatic example of a positive science and thus it becomes a field on which any epistemology can be tested.

2. Mathematics is a paradigmatic example of a science whose positiveness must be justified by a general epistemology. Here the positiveness itself is questioned.

In the first approach the epistemology passes or fails depending of whether the positiveness of mathematics, *i.e.*, the positiveness of mathematical science can be explained or not. In the second approach it is mathematics that passes or fails depending of whether it can be epistemologically justified.

The first approach is characteristic of the philosophers who think about mathematics, while the second approach is characteristic of the mathematicians who think philosophically.

The first approach encapsulates the *old* meaning of the philosophical attitude towards the positive mathematics, while the second approach represents the *new* meaning of the philosophical attitude towards the positiveness of mathematics.

Let us examine now the following question: What is mathematically brought into question in the *old philosophy of mathematics*? Certainly not its positiveness. Whatever could contradict this positiveness, is already doubtful due to this tension. Briefly, the positiveness of mathematics in the old philosophy of mathematics is not questioned. What is brought into question here is its applicability. In order to explain this, we shall make use of some legends which should not be taken literally.

Thales created elementary mathematics. Namely, he constructed mathematics as a deductive science discovering the abstract territory in which it develops. It is known that since Thales there has existed a science which does not deal with concreteness and this science is called mathematics. This science is not subject to a change of the concrete, but it is absolutely positive in its abstractness. Mathematics proves its abstract truths. The positiveness of mathematics lies in its abstractness which enables proof.

Thales started, and Pythagoras went on developing mathematics as a positive abstract science. With Pythagoras this abstract science is not only abstract consciousness. With him mathematical concept is a structural and constitutive element of the concrete or, as Pythagoras would say, *nature is made up of numbers*. Though this phrase is often considered as mathematical mysticism, yet it seems that this is just a general estimate which says nothing and explains still less. But what is this all about? It is about the relationship between the positive abstract mathematics and the concrete nature. Here we find for the first time the problematization of this relationship, as well as the first solution of the problem noticed, which through the identification of concrete nature with abstract mathematics describes the applicability of abstract mathematics to the comprehension of concrete nature. A great number of contemporary and formerly living metallurgists, physicists and chemists who believe in the applicability of abstract mathematics, are in fact as much of mystics as Pythagoras was since their belief is based mainly on an identification of Pythagorean type. The problematization of the relationship between abstract mathematics and concrete nature or, as it was called above, the problematization of the applicability of abstract mathematics to concrete nature, in no way can be called mysticism. The point is about the main problem of the old philosophy of mathematics which does not question the positiveness of mathematics treating it as well founded in its abstractness, but it brings into question its applicability, with Pythagoras as its founder.

Here an additional explanation is necessary. The above mentioned identification by Pythagoras takes place in two steps. In the first step, the geometrical abstractions are considered as constructive elements of the concrete nature, and in the second step as a final reduction of these abstractions the number is discovered. The second step of these identifications is shattered by the discovery of the possible incommensurable lengths. It also shatters the specific Pythagorean reduction of the concrete nature of the number but not the Pythagorean view that the abstract foundation of mathematics brings into question its applicability to concreteness, *i.e.*, the view which represents a fundamental problem of the old philosophy of mathematics.

Here we shall underline that this problem really and basically remains a problem of the old philosophy of mathematics.

To many people it is difficult to understand the genesis of Democritus' mathematical atomism [1]. It can be easily understood if we see in it a concordance of the abstract mathematics with the concrete atomistically comprehended nature, a concordance which should justify the applicability of mathematics.

Similar problems are encountered by many investigators of Zeno's aporia. While some see in these just an idle sophistical talk, others find mathematical errors in Zeno's arguments. The well known sophisms cannot confuse those who look primitively and concretely at the motion of the arrow and Achilles's victory over the tortoise. They cannot be deceived and led along a wrong way by abstract arguments. On the other hand, the isolated abstract argument is based on the erroneous mathematical premise that the sum of infinitely many finite quantities must be infinitely great. Is not Zeno's argument just a sophistical game which brings into question the elementary understanding of the concrete nature by using an abstract mathematical argument which remains silent about a false mathematical premise? Maybe, but only under the condition that the applicability of the abstract mathematical argument to the concrete Achilles's situation is considered self-evident, only if we identify the concrete path of Achilles with an abstract mathematical length, only if we think that the infinite divisibility of the abstract mathematical length is the same thing as the infinite divisibility of the concrete path of Achilles. Thus Zeno's effect can be considered a hollow sophism only if the applicability of the abstract mathematical argument to the concrete Achilles's situation is considered self-evident. But is not just this self-evidence of the applicability of abstract mathematics brought into question by Zeno's argument? Is not the point here about the fundamental problem of the old philosophy of mathematics? Of course, this is exactly the point.[1]

In the modern philosophy of mathematics we can trace the shift from the objective normative relationship of the abstract to the concrete towards the thought about the abstract which standardizes the description of the concrete. Kant stands on the top of this thought when he says that mathematical sense is positive being a sense of the *a priori* shapes of consciousness, space and time. This sense is applicable since the *a priori* shapes of consciousness first enable the meeting with reality, *i.e.*, these are constituents of reality. The opposition abstract - concrete is replaced by the opposition transcendental - immanent, and applicability should not be understood as applicability of the abstract in the comprehension of the transcendental since the transcendental is not concrete. Both the concrete and the abstract are immanent, and that is how they are related to each other.

The old philosophy of mathematics does not end with Kant. Its fundamental problem is still one of the basic problems of philosophy of mathematics. The terms *old* and *new* philosophy should not be understood literally.

The *new philosophy* of mathematics brings into question the positiveness of abstract mathematics. Solving this problem does not touch at all the question of applicability. Its problem is to explain the positiveness of pure mathematics, that is, the positiveness which in the old philosophy of mathematics is self-evident. The new philosophy of mathematics is so preoccupied with this problem, that it does not consider the fundamental problem of the old philosophy, that is, *the possibility of applying mathematics*. Sometimes the old and new philosophies of

[1] Plato gives up the simple identification of the abstract with the concrete. He solves this relationship by replacing the abstract idea by a concrete one. The abstract idea is an objective norm of the concreteness.

mathematics are thought to be in some conflict. In fact, these put into the focus of their interests two different but equally essential problems.

Where does the positiveness of pure mathematics lie? On what is it grounded and on what is it based? In the 20th century three answers are clearly outlined. These are the answers of logicism, intuitionism and formalism. Now we shall say something more about each of them.

A sapid, and to a certain extent popular, presentation of the logicistic, intuitionistic and formalistic foundations of mathematics can be found in the works of Carnap [2], Heyting [3] and von Neumann [4] which were written in 1930. The difficulties and the mature logicism were then already known, and on their basis Carnap gave his review. Intuitionism is at that time a young and revolutionary teaching which is fighting for its place under the sun, so Heyting's self-confidence, sometimes exclusivism, must be regarded in this context. The formalistic theory of proof is in an upsurge, Gödel's results about the incompleteness of formal systems are still unknown, so one can explain von Neumann's relative optimism. The logicists deduce the positiveness of mathematics from the self-evidence of logic, thus they deduce mathematics from logic. Mathematics is an extension of logic and as such it acquires its inexorable positiveness. To such understanding of the relationship between logic and mathematics, which results from the creation of mathematical logic, one can come from the following opposition.

In the thirties and forties of the 19th century, formal logic develops mainly in the English elaboration of the scholastic syllogistics due to Hamilton and De Morgan. There are characteristic attempts in the study of logic as a science which tests the laws in the field of quality to apply the mathematical methods of analysis which prove to be fruitful and positive in the study of the laws of quantity. The final goal would be the elaboration of the corresponding symbolics and the confirmation of the law of its manipulation taking pattern from mathematics, more precisely, arithmetic. The first real success in such attempts was achieved in 1847 by Boole's work *Mathematical Analysis of Logic*. Boole has shown that the basic operations with concepts can be represented by the arithmetic operations + and $\cdot$, and the basic notions of something and nothing–by the numbers 1 and 0. These operations are subject precisely to the laws of arithmetic, *i.e.*, to the laws for operations with quantities, with one additional law $x^2 = x$ called *the principle of tautology* which is characteristic and distinctive for the operations with qualities. In this sense, the logic of concepts becomes a special arithmetic and Boole himself understood it this way.

Premises are equalities from which by arithmetic operations one can obtain other equalities which are consequences of the arithmetized conclusions. By this analysis it was considered that the logic of concepts acquired simplicity, positiveness and generality.

No interpreted equalities that appear in the arithmetized proofs disturb the pure logical understanding of the basic laws, so Boole, Jevons, Venn, Schröder and other prominent logicians began with a *purge of arithmetic*.

The final result was Boole-Schröder's algebra which represents a logic of concepts, but at the same time a logic of propositional functions as well as a logic of propositions. This algebra thus appears as an abstract mathematical system with more than one possible interpretation. Such system is not defined with its interpretation, *i.e.*, with the object of study, but independently of its possible interpretations it is founded as a deductive axiomatic system whose axioms, and thus the system itself, can admit different interpretations.

The setting and study of such systems is characteristic for pure mathematics which is no longer defined with its object of study as arithmetic or geometry were, but develops independently of this object constructing pure no interpreted axiomatic systems.

Thus *pure mathematics* was born.[2] By the way, Boole-Schröder's algebra which should represent a pure, *purged* from arithmetic, foundation of logic, is in some sense paradoxical as a foundation of logic. Namely, the mathematical deductions, which are introduced there as in any other purely mathematical system, are subject to logical laws. However, if in this Boole-Schröder's algebra we want to found logic itself, then we should not use in it the laws of the very logic, which on the other hand is not possible since mathematics presupposes logic. Naturally, this belief in the priority of logic is not a belief of Boole and his time. In that period, formal logic is considered a special case of pure mathematics. Mathematical logic understood as mathematics of logic is in concordance with an abstract system of pure mathematics, *i.e.*, with

[2] Russell must be understood in this sense when he says that Boole was the father of pure mathematics.

Boole-Schröder's algebra. This system, which is mathematics of logic, is not a pure mathematical logic since it is not founded independently of its logical interpretation. Boole is the father of pure mathematics which presupposes logic, but not the father of pure mathematical logic. Pure mathematical logical arises from the contradiction of the Boolean mathematics of logic which passes into logic of mathematics. Now the natural question arises: How does one come to that logicistic contradiction?

The criticism of differential and integral calculus and the mathematicians' return to the problem of its foundation in the middle of the 19th century gave a clear notice of the need of foundation of the arithmetic of real numbers. This was a period of critical motion in mathematics, a period when Weierstrass achieved a supreme mathematical authority, a period when with the informal logical analysis the intuitions of space and time as foundations of arithmetic were rejected. In that period Dedekind arithmetized the continuum of real numbers, to set afterwards even the problem of the genesis of the very natural numbers with the words [5]: "*If we call arithmetic, algebra or analysis just one branch of logic, by this we already mean that we consider the concept of number quite independent of the notion of space and time; on the contrary, we consider it an immediate product of the pure laws of thought*." That was a period when there was no speech about the Boolean logic seeking for its positiveness in mathematics. On the contrary, in that period mathematics would seek for its positiveness in logic. However, this radical contradiction was developing gradually, with the adoption and development of one more aspect of the critical motion in mathematics. In the minds of the leading mathematicians of the end of the 19th century the ideal of the mathematical theory, which is deduced from a small number of mathematical premises in accordance with the logical principles, was clearly drawn, so the ideal of the theory rejected intuition even as means of demonstration. It is obvious that the Boolean period in which pure mathematics was elaborated, essentially contributed to the creation of that ideal. Radical thinkers, such as Peano, saw that a deductive science so understood, that required postulation and definition of the basic entities but also a radical purge of the deductive procedure from the propensity to intuition, could come true only in the framework of a symbolic language free of the intuitive contents of the natural languages. To a great extent, that language had been already developed by mathematics. Meanwhile, Peano in concordance with the new conceptions of deductive science considered necessary to do what had not been done in mathematics till then, namely, to formalize and symbolically describe only the mathematical argumentation. So he constructed the symbolic logic which he used in the formalization of mathematics. By means of that logic, the deduction of the conclusions from the premises was replaced by a formal generation of the respective symbolic expressions from other such expressions by a quasi-algebraic process.

Such a modern, but hardly simpler and more readable presentation of mathematical logic of first order was given by Beth [6]. He also proved its completeness by comprising all logical principles of first order. This was Gödel's result which became still more interesting in the context of his later results of incompleteness of the formal mathematics of first order. An alternative system of logic, the so called natural system, was proposed by Prawitz [7], which can be used in the formalization of mathematics, more concretely, of elementary arithmetic.

In this way mathematics is freed from intuition, being transformed into a set of propositions of the form p **implies** q, where p represents the conjunction of the postulated mathematical assertions and q are their quasi-algebraically deduced consequences. The connection of this aspect of the critical motion with that presented by Dedekind, who claimed that the main constituents of the mathematical proportions could be logically defined, showed the possibility that p and q were purely logical proportions, *i.e.*, that mathematics freed from intuition became a set of purely logical propositions of the form p **implies** q. In other words, mathematics became a branch of logic. The connection of these two aspects of the critical motion in mathematics and the consequent adoption of that attitude were accomplished by the logicists Frege and Russell. They occupied themselves with the proof that pure mathematics was a branch of logic, the branch called mathematical logic, being the logic of mathematics, *i.e.*, the logic from which mathematics started.

The logic which is necessary for the logicistic reconstruction of the pure mathematics extensively overlaps with Cantor's set theory whose positiveness at the beginning of the 20^{th} century was brought into question by the discovery of its paradoxicality. Logicism reduced mathematics to logic, but not to inexorable positive logic but to logic which is perhaps less positive than mathematics itself. The intuitionists, or more generally the constructivists as Brouwer and others go along the way of repeated acquisition of the positiveness of mathematics in mathematics itself [8].

The contradiction between the logicistic and intuitionistic foundations of mathematics can be more clearly seen in the opinions concerning the existence of mathematical entities. A brilliant presentation of this contradiction, in which the consideration of the formalistic opinion is also included, as well as the connection with the traditional philosophical contradiction, can be found in Quine's philosophy of mathematics[3] [9]. Some logicists, like Russell, do not give up their original opinion and must treat that situation as a catastrophe of logicism. Others such as Ramsey and Quine consider the logicistic program fulfilled with the very adoption of Platonism. We must underline that the Platonistic opinion concerning the existence of mathematical entities is the *official* opinion of modern mathematics. This in fact is confirmed by Gödel [10, 11] through his still unsurpassed presentation of Platonism in mathematics. The constructivistic opinions retreat from this main trend. Such an opinion is also present in Goodstein's theory [12]. We shall show where the constructivists find positiveness in mathematics by means of a text on the definition of mathematics.

What is the subject of mathematics? Does mathematics describe the entities and what is the mode of existence of these entities? What is spoken about in mathematics? When one speaks of the most fundamental, *i.e.*, of the first subject of mathematics, the natural number, our comprehension encounters a lot of troubles. We shall try to explain where these unknowns come from, and also the uncertainty characteristic for the encounters with these questions.

The first description of mathematics which deserves a serious consideration is Russell's very often quoted aphorism: "*Mathematics can be defined as a subject about which we never know what is it we are talking about, nor whether what we say is true*."[4]

This definition contains four great distinctions.

1. It is rejected by common sense. In this point common sense is in conflict with what the usual *common sense* does not do. One of the main services that mathematics has done to mankind is putting common

[3] Here an explanation is needed. The original logicistic opinion, for instance Russell's opinion concerning the existence of mathematical entities, is not realism or what it is often called nowadays Platonism. Carnap's philosophical approach [2] also testifies for this. By the way, as we already said, the logic which is *necessary for the logicistic reconstruction of mathematics* overlaps itself with Cantor's set theory and this is in fact Platonistic.

[4] RUSSELL, B. (1901) "Recent work on the Principles of Mathematics." *Int. Monthly 4*, pp. 83-101.

sense where it belongs, on the highest shelf next to the dusty tin box with the label *thrown out nonsense*.

2. Russell's determination fixes an entirely abstract character of mathematics.

3. It gives the possibility of realizing in just a few words one of the basic thoughts of mathematics since 1890, *i.e.*, the entire mathematics and the mature scientific disciplines should be reduced to a postulation form, so that mathematicians, philosophers, scientists and people with normal common sense could see precisely that what each one of them thinks when he speaks about it.

4. Russell's determination of mathematics sends a loud farewell to the shattered tradition which still respects the composers of dictionaries, according to whom, mathematics is a science of number, quantity and measure. These subjects comprise a great part of the matter to which mathematics was applied. But they are in no case higher mathematics, just as the paints in the master's tubes are not the masterpiece that he paints. They are for mathematics what linseed oil and blue paint are for the painter's skill.

What the above aphorism said seems to be true, as the examples below will testify. We do not know what we are talking about in mathematics, but there is another side of the story which distinguishes mathematics from the unclear way of reasoning of some philosophers and people who occupy themselves with speculative sciences. We must be certain we have not changed the subject of discussion in the course of the intricate and delicate mathematical reasoning, or that our initial premises really contain what we think while speaking. Once again the mathematicians had to destroy the erected structure of their own knowledge since they, like all other human beings, are prone to errors not taking notice of some trivial defects in the foundations of mathematics.[5]

[5] Before abandoning Russell's definition, let us compare with these two definitions. According to Peirce, "*mathematics is a science that deduces the indispensable conclusions*." And again Russell pronounces a similar thought: "*Pure mathematics consists exclusively of assertions of the form: if a statement is true for some entity, then some other statement is also true. The essence is not to discuss whether the first statement is actually true and not to mention what is this which is presupposed to be true*." And again: "*pure mathematics is a class of all statements of the form p implies q, where p and q are statements . . .* "
All quotations by Russell come from: RUSSELL, B. (1901) "Recent work on the Principles of Mathematics." *Int. Monthly 4,* pp. 83-101

The evolution of this too abstract view of mathematics was slow and in its mature form it represents a characteristic product of the mathematical activity in the 20th century. Not all mathematicians agreed with the determinations of that type. Many gave priority to something more concrete. Only a few accepted the dogma that skill in manipulating postulates, in order to come to what Kant called analytical judgment, is sufficient for the creation or understanding of mathematics. In mathematics usually something more than impeccable logic is sought. A skillful logician cannot be considered as mathematician due to his ingenuity in logic.

These conclusions can be supported to a great extent by the thought of Klein, a leading German mathematician in the last quarter of the 19th century, who said that *mathematics in general is a fundamental science about the entities which are self-evident*. Though this is profoundly true, can be misunderstood.

First of all, the modern critical development taught most of the mathematicians to be extremely mistrustful with respect to *entities which are self-evident*. According to this, a mathematician can be misled if he accepts that the complicated chains of the strict reasoning are easy, or that they can be avoided from the very beginning. If mathematics were actually a science about self-evident entities, then mathematicians would be a fantastic mob of fools who spoil tons of high-quality paper proving that fact. Mathematics (according to this philosophy of mathematics) is abstract and difficult, and any assertion of simplicity is true only in a strictly professional sense. Only the premises mathematics comes from are simple.

Each of the attempts to define mathematics contributes to the elucidation of particular details of the picture. These attempts, as well as others that were not mentioned here, show the hopelessness of the attempt to paint the bright sunrise in one color. The attempt to condense the free spirit of modern mathematics within a few inches in some dictionary is vain, just as the endeavor to compress into a small phial a cloud full of electricity which constantly tends to expansion [13].

Let us make this clear. For the Greeks, mathematics was first of all geometry, and if geometry is studied in the traditional way, then a flood of philosophical questions arises from the very beginning. Euclid defined the point as *something that has no parts*, but how must this be understood? Indeed, is it possible that something without parts exists? Even if such things could exist, could we ever see them, or know anything about them? People often consider Euclidean geometry a description of the physical world, but it seems hard to believe that the world can be constructed of points, since if the points are not expanded, then, even an infinite number of points are not sufficient to obtain some volume. Are the points just ideas of our mind? Or are they real things that cannot be observed? Yet how can engineers apply the principles of geometry? Here we find several related questions: What kind of meaning do geometrical terms have? Can the principles of geometry be true? How in general is knowledge acquired from geometry and what for is geometry applied?

The creation of non-Euclidean geometries adds new fuel to that fire. If those geometries that contain laws logically incompatible with the Euclidean geometry are mathematically exact, then what happens with the concept of mathematical truth? When a law is incompatible with another one, then both cannot be true at the same time. Have mathematicians ceased dealing with truth? It is hard to imagine a study of geometry which has any sense without presupposing the seeking for truth in space.

Concerning arithmetic, there are also a number of similar questions which refer to the meaning of the terms used, the possibilities of confirming the truth, in fact, whether truth is also sought in this field of mathematics. Important questions are raised about what kind of knowledge do we speak when we use such term, or why the laws of numbers can be applied to reality. In relation to the mathematics which studies the numbers another problem arises, namely the problem of mathematical existence. The principles of geometry can be understood as hypothetical principles which do not claim the existence of anything, for instance: *if there exists a triangle, then the sum of its angles equals two right angles*. Geometry cannot be conceived as a theory which makes existential claims like *there exists a triangle*. On the other hand, in the mathematics which studies the numbers there are a number of laws which seem to claim the existence of some things, for instance: *there exists a unique number y such that x multiplied by y yields x, no matter what number x is*. Laws of this form, no doubt, seem to claim the existence of something, *i.e.*, the number 1, in such a way that the law cannot be easily understood in a hypothetical sense, as it was the case with the laws of geometry. But what kind of existence are we talking about? What kind of reality does that field of mathematics refer to? Should the statements of existence be understood literally, or quite figuratively?

All these are philosophical problems since they refer to very general and fundamental questions about meaning, truth, reality and knowledge. The hard-working mathematicians who make efforts to expand their subject, usually devote little attention to such problems. Someone could say: *rightly so, since the above mentioned problems are just mixed pseudoproblems, such a type of philosophical speculation about mathematics is meaningless*.[6]

The mathematics of the numbers, or arithmetic, shows by means of examples that a science about the numbers is possible without ontologyzing the number. The problem of arithmetic is not the numbers as an object of some ontology, but the role of the numbers, *i.e.*, in the addition, subtraction, multiplication, division, etc. The presence of this role in arithmetic is realized by the carriers of the role, *i.e.*, with the symbol of the number, and this presence enables the construction of arithmetic. The role of these carriers is determined by: their enumeration in the sense of a potential realization of these numbers, their addition in the sense of a potential realization of these numbers, etc. This potential realization is usually encoded in the form of rules:

enumeration of numbers
$\Rightarrow 0$
$m \Rightarrow m|$,
addition of numbers
$\Rightarrow m + 0 = m$
$m + n = p \Rightarrow m + n| = p|$, etc.

If we understand the so determined roles as transformation rules of the symbols of the numbers, then we can agree (though not in any disparaging formal sense) with the often repeated assertion that arithmetic is not a study of the numbers themselves, but of the transformation rules potentially realized with the symbols of the numbers. Though in arithmetic we speak about numbers, this implicitly shows that the speech is free of ontological accent.

[6] By the way, this remark is too rough. Perhaps most of the difficulties experienced by philosophers who consider mathematics arise from the misunderstanding of this kind of problems, yet these questions represent serious intellectual problems because the misunderstandings they arise from are significant and convincing but not ridiculous nor easy to eliminate. These problems, which are not easily discarded, deserve being tested and solved. Those who cut the Gordian knot instead of untying it, show that the knot bothers them for a long time.

As for the existence of the symbols of the numbers and the transformation rules which determine their roles, let us add the following. The transformation rules have a normative but not descriptive character. To those rules it corresponds a norm of potential realization of the numbers, their addition, etc. On the other hand, the symbol of the number opposes the ontological accent by its arbitrarity. Anything can be the symbol of a number. But this ontological arbitrarity of the number is the basis for the construction of arithmetic since the possibility of realization of a new number lies in it, which enables the statement of the potential realization in the form of the previously mentioned transformation rules.

In the normative character of the pure mathematics and the ontological arbitrarity which enables its construction lies its positiveness or, as it is sometimes with pride and more often with incomprehension called, its eternal truth.

This positiviness is confronted still being unaware of its source, and then under ontological pressure we want to discover and ontologically found the subject of mathematics, and to understand it as a description of that somewhere existing subject. It is clear that in the ontological involvement and the descriptive character of the so understood mathematics the positiveness which we confronted and whose source is important for us is lost. While seeking the very source of positiveness we become uncertain and, confused, we begin to whirl in the set of unknowns.

This is why we must be conscious that the ontological foundation of mathematics as a descriptive science through the foundation of its subject cannot have that character of positiveness which pure mathematics has. In view of the character of the pure mathematics and its foundation we shall say a few words about the apodictical truth of the mathematical assertions.

The previously presented reasoning of the pure mathematics about the numbers led us to understand these, in a clearly defined sense, as a study of the transformation rules of the symbols of the numbers. This study requires a speech which is assertive in nature.

A simple arithmetic assertion is the claim of the potential realization of some calculation, for instance, addition, multiplication, etc. By such an assertion we claim the potential realization of the triple $m + n = p$, by the application of the rule of addition. This assertion of the potential realization or constructibility can be given the form $\vdash m + n = p$.

The apodictical truth of such an assertion lies exactly in the potential realization of normatively directed calculations. This is guaranteed by the realization of the prescribed construction, this means by the realization of the synthesis which is not descriptive but is normatively directed, *i.e.*, according to the old vocabulary, it is *a priori*. We can say that by the meaning of the assertion (in our case this is the potential realization of a construction) its truth is determined.

Meanwhile, we do not abide by the words *simple proposition*. We combine the proposition by means of connectives, negate it, particularize it, etc. In what lies and what is the truth value of the individual propositions of such a rich assertive speech? What are the meanings of these propositions?

While in our speech we are satisfied with a simple combination of the propositions by means of the so called propositional connectives (including the negation), the meanings and the criteria of truth of the so obtained propositions as a whole are understood with the two-valued classical concepts of true and false as predicates which are assigned to the proposition in concordance with the so called truth tables of the values of the individual propositions.

This comprehension is in concordance with the comprehension of the simple propositions, so that it can be extended in an appropriate way to the universal propositions as shown by an example of the recursive arithmetic elaborated in 1923 by Skolem [14]. He has shown that in recursive arithmetic as a whole it is possible to simulate speech by propositional connectives of combined propositions through purely calculated speech of simple propositions. In this way a purely arithmetic logic, which is deprived of the direct use of the existential and universal quantifiers, is founded.

The following questions remain open: What is the meaning of the universal and particular propositions? In what lies and what is the truth value of these propositions? What do the propositions 'at least one odd number is perfect', or 'each odd number is imperfect' mean? If by an ontological accent we understand these propositions as a description of the actual set of the numbers, then the meanings of these propositions are descriptions of states of the objects in this set. The condition of truth determined by this meaning corresponds to the description of the state of the objects. The uncertainty which arises from this understanding is already designated. Moreover, this condition of truth of some proposition does not make simultaneously possible a criterion for its truth.

If the positiveness of the pure mathematics is important for us here, we already know where its source lies. Some work is being done on the foundation of a mathematics with assertive speech, but free of ontological accent. This mathematics, *i.e.*, mathematical logic, in some sense already precedes the one already described, freely speaking, in the same sense in which speech precedes enumeration.

Thus we see how constructivism determines pure mathematics and where in concordance with this determination finds its positiveness. We also see that the constructivistically understood pure mathematics does not presuppose a logic. Yet logic itself is a branch of the pure mathematics. It is a pure *mathematics with assertive speech free of ontological accent*.

But the constructivistic reconstruction of the entire classical mathematics is not possible. Constructivism finds positiveness in the pure mathematics which does not overlap with the standard classical mathematics. To many people this seems to be a too big sacrifice.

The term *formalism* is used in two senses. Quite often it means a nominalistic attitude towards the existence of the mathematical objects. Here we will use it in another sense. For us (and for others like von Neumann [4]) formalism will be a synonym for Hilbert's mathematical foundation of mathematics since this foundation requires in one of its steps a methodical comprehension of the nominalistic formalism. We shall explain what foundation we are talking about.

Hilbert understands the constructivistic criticism of the classical mathematics as non-positive mathematics. Still what he considers positive mathematics is just a part of the constructivistically understood pure mathematics, the so called *finitist* part. But, unlike the constructivists or intuitionists, he does not want to reject that part of the classical mathematics which by the constructivistic reconstruction is not justified as a positive science. His program is a program of justification of the constructivistically rejected non-positive mathematics by the so called method of *ideal elements*.[7]

7 The explanation of the basic ideas of Hilbert's program can be found with more details in Prawitz [7].

The first problem of that program is the formalization of the standard classical mathematics. The main results of that problem have been already obtained by the logicists transforming mathematics into a system of propositions of the form *p* **implies** *q*, where *p* represents a conjunction of postulated mathematical assertions, and *q* are their quasialgebraically deduced consequences. But while logicism seeks a logical interpretation of the propositions of that system, Hilbert accepts only a finite interpretation. In simpler words, Hilbert gives a meaning only to these propositions of the systems which have a simple computational sense, for instance, the propositions of the form $m + m = p$. All other propositions which cannot be simply explained, are not interpreted at all nor given any meaning by Hilbert. Strictly speaking, for him and his school, they have no sense at all. They are symbols which mean nothing, *i.e.*, they are understood as nominalistic signs. The uninterpreted propositions of the systems are ideal elements which can be formally logical consequences of the interpreted propositions, from which, again, other uninterpreted propositions can be deduced formally logically and, at last, also interpreted ones. Formally these are member of equal rights of the deductive system. The deductive system as a whole is justified by a mathematical proof of consistency which must provide the following: *each interpreted proposition which has a simple finitist meaning and which can be proved in the framework of the system, even by using ideal elements, according to its meaning is true*.

The ideal elements which have no sense and which should be given no meaning serve to prove more quickly and more simply the finitist propositions.

In this program a moment which distinguishes it from the nominalistically understood formalism should be noted. This is the moment of choice, or better to say construction of the system which is justified. On the one hand, for that justification to be at the same time a desired justification of the classical mathematics, the system itself should be a formalization of the classical mathematics, and this can be the case only if the classical mathematics is its interpretation. Thus at the very moment of formalization the system must be interpreted if the justification of the classical mathematics is important and not the formal result about the formal system. On the other hand, the very justification is found by a methodical forgetting of that interpretation. The final Hilbert's conclusion should be the following: *We understand the classical mathematics but are not positive about some of its parts. Meanwhile, if we wholly reject* *the very possibility for understanding these non-finitist parts and retain as comprehensible and positive a part of the classical mathematics, namely, its finitist part, then we can show the following: the assertions which we do not understand and which have no positive sense can be understood as auxiliary means which enable the simpler and easier proof of finitely comprehensible assertions.*

We can often read that the unrealizability of Hilbert's program is proved by Gödel's incompleteness theorems. By his first theorem, Gödel has proved that a complete formalization of any mathematical theory which contains arithmetic is not possible. In any formalization of a mathematical theory of the type mentioned one can find a formal proposition which can be neither proved nor rejected in the formal system. Still the interpretation of that proposition is a true mathematical assertion.

From this follows a second theorem which claims that the consistency of a formalized mathematical theory which contains arithmetic cannot be proved in the formalized theory itself. These theorems are explained and proved in Kleene's logic [15]. Let us move to the consequences of Gödel's theorems about Hilbert's program.

1. It is completely clear that by Hilbert's program the entire mathematics cannot be justified since by the first Gödel's theorem it cannot be formalized, and the formalization of the theory which is justified is the first step in Hilbert's program.

2. The proof of the consistency of the formal system does not ensure that each interpreted and ideally proved proposition of the system will be a true proposition. Hilbert's claim that the real truth of the ideally proved and interpreted propositions of some formal system follows from the proof of the consistency of that system and is based on the erroneous premise of complete formalization. The proof of the real truth of the ideally proved propositions must be direct, and in view of this the proofs of consistency lose the significance they used to have in Hilbert's program. This direct proof, though possible, is much more complicated that the pure proof of consistency. The possibility of achieving Hilbert's program is thus not destroyed by Gödel's theorems. The use of the program becomes complicated and partial. On the other hand, the destructive criticism of the postulation of a formalistic mathematics (which independently of Hilbert's program has begun to live a life of

its own as an illegitimate child of logicism and formalism repudiated by both parents) can be found in Kreisel's philosophy [16] which can be also understood as a subtle justification of Platonism in mathematics by the very fact of its evident presence.

Now we shall propose a general framework in which we can more clearly see the entire mathematics and by means of which we can perhaps more easily understand the above views, attitudes and determinations.

Mathematics, in its most elementary form, is already an abstract construction which enables an encounter with reality. On the other hand, the consciousness of the pure forms of this reality is also called mathematics if it is achieved by the following procedure: *some intuitively observed spatial or temporal principles are postulated and included into the language as so called axioms, and then from this axiomatic kernel the further properties are deduced by means of the formal linguistic rules of the classical logic*. The intuitively observed principles are an object of constant doubt [17]. Objects of doubt are, for instance, the spatial intuition and analysis. To eliminate this means to achieve positiveness in mathematics. The arithmetization of analysis is an example of elimination of spatial intuition. Here three possibilities are realized, two radical and one pragmatic.

The radical ones reject postulation. The one possibility, regarding mathematics as a construction independent of linguistic-logical methods builds up the so called constructive mathematics. The other possibility, regarding mathematics as a logical-linguistic activity, tries to derive all of its concepts and results from that field. Pragmatic mathematics does not reject postulation but decreases its significance. Anything can be postulated, and the validity of the mathematics so obtained will be shown by its applications.

For constructive mathematics it is hard to achieve the richness of the structures accessible for pragmatic mathematics, but when it directly collides with some of its results, then it seriously brings them into question. The logically oriented mathematics in fact cannot pass over the postulation, but it reduces it to just a few absent postulates, for instance, the axiom of infinity and reducibility with Russell. Pragmatic mathematics which in fact is also everyday mathematics, *i.e.*, standard mathematics, by its applicability enters into experimental sciences, so it can be eventually confirmed or refuted through them. Still an inversion is also possible, by which it can again achieve mathematical positiveness. Namely, the science in which mathematics is applied can be also some positive part of it, say, finite or constructive. Now the *experimental* verification is to see that this application will not lead to finitely or constructively imprecise results. Perhaps this is possible to prove finitely or constructively.

This framework seems to be made by the standards of logicism, intuitionism and formalism. For instance, in this framework mathematical empirism can be also put as denoted. Still in this framework we see how close in its method Hilbert's mathematics is to mathematical empirism.

It is clear that a mathematical theory would have no sense if by deduction it could be derived from it that an assertion is valid and not valid. In this sense it is stated that the mathematical theory is contradictory, and the non-contradictoriness (consistency) of the formalized mathematical theories is put as a basic requirement. To decide whether a mathematical theory is non-contradictory one could attempt to test all of its assertions, but this is impossible in the general case. For instance, the number of assertions in the Euclidean geometry is infinite. To this end two other procedures are applied, one of which is relative and the other absolute. The first consists in finding a model or a realization of the tested mathematical theory in another mathematical theory which is known, *i.e.*, a one-to-one correspondence between the objects of the two theories is established so that the tested axioms be assertions in the known mathematical theory. If the known mathematical theory is non-contradictory, then the tested theory will be noncontradictory or contradictory according to whether the assertions in the known theory which correspond to the axioms of the tested theory are non-contradictory or contradictory. The methods of the analytical geometry show, for instance, that Euclidean geometry is non-contradictory if arithmetic is non-contradictory, *i.e.*, finally if the number theory is non-contradictory with respect to set theory. But it is time to mention that Klein has shown that Lobachevskian and Riemannian geometries are non-contradictory if Euclidean geometry is [18].

References

[1] BOYER, C. (1959). *The History of Calculus.* New York: Dover Pub. Inc.

[2] CARNAP, R. (1931). "Die logizistische Grundlegund der Mathematik". En: *Erkenntnis 2*, pp. 91-106.

[3] HEYTING, A. (1931). "Die intuitionistische Grundlegung der Mathematik". En: *Erkenntnis 2*, pp. 106-116.

[4] VON NEUMAN, J. (1931). "Die formaliistische Grundlegung der Mathematik". En: *Erkenntnis 2*, pp. 116-122.

[5] DEDEKIND, R. (1888). *Was sind und was sollen die Zahlen.* Brunswick.

[6] BETH, E. W. (1955). "Semantik entialment and formal derivability". Mededeling van de Koninklijke Nederlandse Akademie van Watenschappen, *Afdeling Letterkunde, N. R. 18*, 309-342.

[7] PRAWITZ, D. (1971). *Ideas and results in proof theory*. Proc. 2nd Scand. Logic Symp., North-Holland, pp. 235-309.

[8] COMBÈS, M. (1971). *Fondements des Mathématiques.* Paris: Presses Univ.

[9] QUINE, W. V. (1961). *From a Logical Point of View.* Cambridge: Harvard University Press.

[10] GÖDEL, K. (1947). "What is Cantor's continuum problem?" En: *Amer. Math. Monthly 54*, pp. 515-527.

[11] BENACERRAF, B. & PUTNEM, H. (1964). *Philosophy of Mathematics.* Oxford.

[12] GOODSTEIN, R. L. (1968). *Existence in Mathematics.* Amsterdam.

[13] BELL, E. T. (1951). *Mathematics Queen and Servant of Science.* New York: McGraw-Hill Book Co. Inc.

[14] SKOLEM, T. (1923). "Begründung der elementaren Arithmetik". *Skripter utgit av det Norske Videnskaps-Akademie, I No. 6b*, Oslo

[15] KLEENE, S. C. (1967). *Mathematical Logic.* London: John Wiley & Sons.

[16] LAKATOS, I. (1972). *Problems in Philosophy of Mathematics.* North-Holland.

[17] TARSKI, A. (1995). *Introduction to Logic and to the Methodology of Deductive Sciences.* New York: Dover Pub. Inc.

[18] MAMUZIĆ,Z. (1963). *Klasična – moderna – konstruktivna matematika.* Uvodjenje mladih u naučni rad III, Matematička biblioteka No. 20, Zavod za izdavanje udjbenika SR Srbije, Beograd pp. 9-37.