MINIMUM BRACING STIFFNESS FOR MULTI-COLUMN SYSTEMS: THEORY

RIGIDEZ MINIMA DE ARRIOSTRAMIENTO EN SISTEMAS DE MULTI-COLUMNAS: TEORIA

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ABSTRACT: A method that determines the minimum bracing stiffness required by a multi-column elastic system to achieve non-sway buckling conditions is proposed. Equations that evaluate the required minimum stiffness of the lateral and torsional bracings and the corresponding "braced" critical buckling load for each column of the story level are derived using the modified stability functions. The following effects are included: 1) the types of end connections (rigid, semirigid, and simple); 2) the blueprint layout of the columns (i.e., the cross section orientation and location of the centroid of each column); 3) shear deformations along each column using the modified method initially proposed by Haringx in 1948; and 4) axial load distribution among the columns (i.e., load pattern). The effects of axial deformations and warping torsion are not included. The proposed method is applicable to 2D and 3D framed structures with rigid, semi-rigid, and simple connections. The formulation presented in this paper is based on a previous work presented by Aristizabal-Ochoa in 2007. It is shown that the minimum stiffness of lateral and torsional bracings required by a multi-column system depend on: 1) the blueprint layout of the columns; 2) the variation in heights and cross sectional properties among the columns; 3) the flexural and shear stiffness of each column; 4) the load pattern on the multi-column system; 5) the lack of symmetry (in the loading pattern, column layout, column sizes, and heights) that cause the combined torsion-sway buckling all of which reduce the buckling capacity of the frame as a whole; and 6) the support conditions and restraints at the top end of the columns. The proposed method is limited to multi-column systems with elastic and orthotropic columns with doubly symmetrical cross sections (i.e., with a shear center coinciding with the centroid) oriented in any direction with respect to the global axes. Four comprehensive examples are presented in detail in a companion paper that shows the effectiveness and simplicity of the proposed method.

KEY WORDS: Beam-columns, buckling, bracing, building codes, columns, construction types, frames, loads, P-Δ effects, reinforced concrete, shear deformations, seismic loads, stability

RESUMEN: Se propone un método que determina la rigidez mínima de arrostramiento requeridos por un sistema elástico de varias columnas para lograr condiciones de pandeo sin deriva entre pisos. Las ecuaciones que evalúan la rigidez mínima requerida de los arrostramientos laterales y de torsión y la carga crítica de pandeo "arriostrada" correspondiente para cada columna del nivel de piso se obtienen utilizando las funciones de estabilidad modificadas. Se incluyen los siguientes efectos: 1) tipos de conexiones (rígidas, semirrígidas y simples), 2) la distribución en planta de las columnas (es decir, la orientación de la sección transversal y la ubicación del centro de gravedad de cada columna), 3) deformaciones por cortante a lo largo de cada columna mediante el método modificado propuesto por Haringx en 1948, y 4) distribución de la carga axial entre las columnas (es decir, el patrón de carga). Los efectos de las deformaciones axial y de torsión no están incluidos. El método propuesto es aplicable a estructuras 2D y 3D con conexiones rígidas, semirígidas y simple. La formulación se presenta en este documento se basa en un trabajo previo presentado por Aristizábal-Ochoa en 2007. Se demuestra que la rigidez mínima de los arrostramientos laterales y de torsión requeridos por un sistema de varias columnas depende de: 1) la distribución en planta las columnas; 2) la variación en altura y propiedades de la sección transversal de las columnas; 3)las rigideces a flexión y a cortante de cada columna; 4) el patrón de cargas en el sistema de columnas; 5) la falta de simetría (en el patrón de carga, en la disposición de las columnas, tamaños de las columnas, y sus alturas) que hacen que el pandeo combinado de torsión y flexión lateral pueden reducir la capacidad de pandeo critico de la estructura en su conjunto; y 6) las condiciones de apoyo y restricciones en el extremo superior de las columnas. El método propuesto se limita a sistemas multi-columna con columnas elásticas y ortotrópicas con secciones transversales doblemente simétricas (es decir, con centro de corte coincidiendo con el centroide) orientadas en cualquier dirección con respecto a los ejes globales. Se presentan cuatro ejemplos en detalle en una publicación adjunta que muestran la eficacia y la simplicidad del método propuesto
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PALABRAS CLAVE: vigas-columnas, pandeo, refuerzos, códigos de construcción, columnas, tipos de construcción, marcos, cargas, efectos P-Δ, deformaciones por cortante, hormigón reforzado, cargas sísmicas, estabilidad.

1. INTRODUCTION

Bracing and how to design the proper bracing (including its strength and stiffness) in real structures are topics of major concern to structural engineers and designers since both bracing and its design are vital to their own stability performance under working loading conditions, as well as the overall integrity of the structure under extreme loadings such as those caused by severe earthquakes and strong winds or lateral loads. It is a common practice to design framed structures with lateral bracings to protect them against excessive lateral sways and deflections, to avoid buckling and premature collapses, and to utilize maximum allowable stresses in their structural members and connections. This is done to obtain safe and economical designs. Bracings are structural components or assemblies that are intended to prevent buckling or reduce the effective unsupported length of columns, towers, truss chords, and other members or structures loaded in compression. In some cases, the same bracing is also used to resist externally applied loads.

Bracing can be divided into three major categories: 1) component bracing to avoid local or individual member buckling; 2) sub-system bracing to avoid excessive distortion in vertical or horizontal assemblages such as roofs and floor diaphragms; and 3) system bracing to prevent sidesway buckling of the structure as a whole and to maintain its lateral stability, including overturning the effects of extreme drifts caused by severe lateral loads.

Component and sub-system bracings may consist of cross-tension members where the axial stiffness of the bracings is utilized; they may be provided at concentrated locations by other members—framing transversely to the member being braced, wherein both the axial and flexural stiffnesses of the bracing members are utilized. Steel construction codes offer some guidance on these two types of bracings [1, 2, 3, 4, and 5]. However, many bracing details employed in steel construction do not prevent the twisting of column members [6] or the local buckling of the beam flanges in concentric braced frame structures, due to lack of adequate lateral support [7]. These aspects are in need of further research.

On the other hand, system bracing is intimately related to the control of interstory drifts, generally referred to as "lateral" stability. Guidance on the required stiffness and strength for story bracing for frames is not precise, but

rather indefinite. The task of designing system bracing is generally left to the structural designer. For instance, a steel-braced frame, according to AISC Chapter C of LRFD versions 1994, 2002, and 2005 [3, 4, and 5], is one in which "lateral stability is provided by diagonal bracing, shear walls or equivalent means." The vertical bracing system must be "adequate, as determined by structural analysis, to prevent buckling of the structure and maintain the lateral stability of the structure, including overturning effects of drift..." Details on diagonal bracing under tension and compression are given by the AISC-LRFD Manuals [3, 4 and, 5]. A design guide related to bracing cold-formed steel structure elements and systems is presented by Sputo and Turner [8]. This book contains design examples illustrating the bracing design for various types of cold-formed steel structures, as well as an extensive list of primary reference sources.

It is generally recognized that bracings need stiffness (to limit the deformation of the braced components or structures and to cause them to behave in an intentional manner) and strength (to provide the necessary stabilizing forces). In many situations, the stiffness and strength requirements are related to each other: reduced stiffness allows for greater deformations, which in turn results in increased force on the bracing.

Rigorous analysis for determining the required stiffness and strength of bracing systems can be very complicated. Establishing the strength requirements for stability bracing generally requires large displacement analyses on imperfect systems [9]. However, simple and approximate criteria are available for designers. For instance, in reinforced concrete buildings, "a story within a structure is nonsway if the stability index is less than or equal to 0.05..." (ACI code version 318-05, 10.11.4.2) [10] or "... if the increase in the lateral load moments from P-Δ effects does not exceed 5 percent of the first-order moments..." (ACI code version 318-05, R10.11.4) [10]. The alternative specified in the ACI 1995 Code that says: "...if bracing elements (shearwalls, shear trusses, or other types of lateral bracing) have a total stiffness at least six times the sum of the stiffnesses of all the columns within the story" has been deleted from the last three versions of the ACI Code (1998, 2005, and 2008) [10, 12, 13].

In addition, construction codes [14] still base the lateral stability design on simplified 2D analyses (i.e., models obtained by breaking the structure into vertical plane

frames) ignoring the real 3D stability behavior. The two major effects on the stability behavior of framed structures, namely, the overall torsional-flexural coupling and coupling among the columns at a story level are ignored by most codes. Except for totally symmetrical frames (i.e., symmetrical frames subjected to symmetrical axial load patterns), torsional-flexural buckling must be considered in the design of 3D framed structures since the buckling loads can be significantly below the 2D buckling flexural loads. Whereas column coupling becomes important in frames with columns of different heights or/and under different axial loads and boundary conditions as described by Aristizabal-Ochoa [15, 16, 17, 18, 19, 20 and 21].

The main objective of this article is to present a straightforward formulation for minimum stiffness of the bracings required by an elastic multi-column structure to achieve fully "braced" conditions (i.e., the required minimum stiffness of the lateral and torsional bracings). Design recommendations for strength, ductility, fatigue, structural details of connections, etc. of any particular bracing configuration, and material are beyond the scope of this paper. The proposed formulation can be applied to plane and space multi-column frames with rigid, semirigid, and simple connections, but it is limited to framed structures with doubly symmetrical columns with their principal axes oriented in any direction with respect to the global axes. The effects of shear deformations in all members are included but axial and warping torsion deformations are not. The formulation presented in this paper is based on the work presented formerly by Aristizabal-Ochoa [21].

2. STRUCTURAL MODELS

The models of an entire story of 2D and 3D framed structures are shown in Figs. 1(a) and 1(b), respectively. In both models it is assumed that the floor diaphragms are axially rigid in their own plane. This allows for the condensation of the lateral degrees of freedom (DOFs) into one DOF per floor level in 2D frames, and three DOFs per floor level (two horizontal translations and a rotation about the vertical axis at the stiffness or shear center) in 3D frames.

2D Multi-column Model. The 2D model shown in Fig. 1(a) is a linear elastic model consisting of **n** prismatic columns each one with different cross sectional properties (gross sectional area A_i, effective shear area A_{si}, and

moment of inertia I_i), height (h_i), end bending restraints (κ_{ai}, κ_{bi}), under different axial loads (P_i), all **n** columns sharing the same lateral spring restraint S and the same sidesway Δ. A typical column element A_iB_i of the multi-column system (Fig. 2a) is made up of the column itself A_iB_i and the lumped bending springs A_iA_i' and B_iB_i' at the top and bottom ends, respectively. These bending restraints have stiffness κ_{ai} and κ_{bi} (units are in Moment/Radian). It is assumed that a typical column A_iB_i is made of a homogeneous linear elastic material with: 1) a moduli of elasticity E_i and G_i; 2) a straight line centroidal axis; and 3) buckling taking place around one of the principal axis of the cross section. The ratios $R_{ai}=\kappa_{ai}/(E_iI_i/h_i)$ and $R_{bi}=\kappa_{bi}/(E_iI_i/h_i)$ will be denoted as the stiffness indices of the flexural connections of column i at ends A and B, respectively. For convenience, the following two parameters are introduced:

$$\rho_a = 1/(1+3/R_a) \qquad (1a)$$
$$\rho_b = 1/(1+3/R_b) \qquad (1b)$$

ρ_a and ρ_a are denoted as the fixity factors. For hinged connections, both the fixity factor ρ and the rigidity index R are zero; but for rigid connections, the fixity factor is equal to 1 while the rigidity index tends to infinity. Since the fixity factor can only vary from 0 to 1 (while the rigidity index R may vary form 0 to ∞), it is more convenient to use in the analysis of structures with semirigid connections [15, 16, 22, 23]. Gerstle [24] has indicated lower and upper bounds for κ_a and κ_b for plane frames. Xu and Grierson [25] used these bounds in the design of plane frames with semirigid connections.

3D Multi-column Model-. In the case of a 3-D multi-column system shown in Fig. 1(b), the particular floor under consideration is on the *XY* plane with the origin O located at a convenient point (generally, at the shear center of the floor). Again, This is a linear elastic model consisting of **n** prismatic columns, with the centroid of column **i** located at point (X_i,Y_i) on the global *XY*-plane, under axial load (P_i) with individual properties including: a cross area (A_i), effective shear areas (A_{sxi}, A_{syi} for transverse shear corresponding to bending around the local x_i- and y_i-axes, respectively); principal moments of inertia (I_{xi} and I_{yi} for bending around the local x_i- and y_i-axes, respectively) with its major local x_i-axis making an angle λ_i with the global X-axis; effective polar moment of inertia (J_{ei}); height (h_i); and end flexural restraints κ_{axi}, κ_{bxi}, and κ_{ayi}, κ_{byi} around the local principal x_i- and y_i-axes, respectively. All **n** columns share the same lateral spring restraints $S_{\Delta X}$, $S_{\Delta Y}$, $S_{\theta XY}$, and

interstory sideways Δ_X, Δ_Y and θ_{XY} (i.e., at each story, the top and bottom floors serve as rigid diaphragms allowing only three degrees-of-freedom per floor). All columns are assumed to have doubly symmetrical cross sections and whose shear center and centroid coincide. Only three types of overall-story buckling modes are considered: 1) pure-translational sway flexural buckling; 2) pure-torsional sway buckling; and 3) combined flexural-torsional sway buckling. Individual column flexural buckling without overall story sway is also considered; but individual column torsional buckling is not.

A typical column element A_iB_i of the 3-D multi-story system is made up of the column itself $A'_iB'_i$ and the two pairs of bending restraints $A_{xi}A'_{xi}$, $B_{xi}B'_{xi}$ and $A_{yi}A'_{yi}$, $B_{yi}B'_{yi}$ located at the top and bottom ends and around the local x_i- and y_i-axes, respectively (Fig. 2). It is assumed that a typical column $A'_iB'_i$ is made of a homogeneous linear elastic material with: 1) moduli of elasticity E_{xi}, E_{yi}, E_{zi}, G_{xi}, G_{yi} and G_{ei}; and 2) straight-line centroidal axis z_i with the external axial load P_i applied along its centroidal axis.

The bending restraints $A_{xi}A'_{xi}$ and $A_{yi}A'_{yi}$ at the top end A have stiffnesses κ_{axi} and κ_{ayi} (whose dimensions are in force-distance/radian) around the principal local x_i- and y_i-axes, respectively. The ratios $R_{axi} = \kappa_{axi}/(E_{xi}I_{xi}/h_i)$ and $R_{ayi} = \kappa_{ayi}/(E_{yi}I_{yi}/h_i)$ will be denoted as the *stiffness indices*

of the bending restraints of column i at the top end A. Similarly, the bending restraints $B_{xi}B'_{xi}$ and $B_{yi}B'_{yi}$ at the bottom end have stiffnesses κ_{bxi} and κ_{byi} and stiffness indices ratios $R_{bxi} = \kappa_{bxi}/(E_{xi}I_{xi}/h_i)$ and $R_{byi} = \kappa_{byi}/(E_{yi}I_{yi}/h_i)$. The stiffness indices vary from zero for hinged connections to infinity for fully restrained connections (i.e., perfectly clamped ends). To facilitate analysis, the following four parameters are introduced:

$$\rho_{axi} = \frac{1}{1+3/R_{axi}}; \quad \rho_{ayi} = \frac{1}{1+3/R_{ayi}};$$

$$\rho_{bxi} = \frac{1}{1+3/R_{bxi}}; \quad \rho_{byi} = \frac{1}{1+3/R_{byi}} \quad (2a\text{-}d)$$

Where ρ_{axi}, ρ_{ayi}, ρ_{bxi}, and ρ_{byi} are called the *fixity factors* at the top and bottom ends of column A_iB_i around the principal local x_i- and y_i-axes, respectively.

3.1 PROPOSED CRITERIA FOR MINIMUM STIFFNESS OF BRACING

3.1 For 2D Multi-column Systems

The minimum lateral bracing stiffness $(S_\Delta)_{min.}$ required to convert any story of a 2D multi-column system [Fig. 1] with sidesway uninhibited or partially inhibited into a fully braced story can be determined from Eq. (3).

$$\frac{(S_\Delta)_{min}}{(\eta E)_j / h_j^3} = \phi_j^2 \sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i}\left[1 - \frac{3(\rho_{ai} + \rho_b - 2\rho_{ai}\rho_b) + 9\rho_{ai}\rho_b \dfrac{tan(\phi_i/2)}{\phi_i/2}}{\eta_i\phi_i^2(1-\rho_{ai})(1-\rho_b) + 3(\rho_{ai} + \rho_b - 2\rho_{ai}\rho_b)\left(1 - \dfrac{\eta_i\phi_i}{tan\,\phi_i}\right) + 9\rho_{ai}\rho_b\left(\dfrac{tan(\phi_i/2)}{\phi_i/2} - \eta_i\right)}\right] \quad (3)$$

Fig. 1. 2D-model multi-column system with sidesway partially inhibited

$$\phi_j^2 = \frac{P_j}{(\eta E / h^2)_j}$$

Where: (4a)

$$\phi_i^2 = \frac{P_i}{(\eta E / h^2)_i} = \frac{\alpha_i \gamma_i^2 \eta_j}{\beta_i \eta_i} \phi_j^2$$ (4b)

$\alpha_i = P_i/P_j$ = ratio of axial load of column i to that of representative column j;

$\beta_i = (EI)_i/(EI)_j$ = ratio of flexural stiffness of column i to that of representative column j;

$$(1 - \rho_{aj})(1 - \rho_{bj})\eta_j\phi_j^2 + 3(\rho_{aj} + \rho_{bj} - 2\rho_{aj}\rho_{bj})\left(1 - \frac{\eta_j\phi_j}{tan\,\phi_j}\right) + 9\rho_{aj}\rho_{bj}\left(\frac{tan(\phi_j/2)}{\phi_j/2} - \eta_j\right) = 0 \qquad (5)$$

The value of $(S_\Delta)_{min.}$ according to Eq. (3) can be obtained following the four steps described below:

The representative j-column is selected from the **n**-column system. The column with the lowest critical axial load under braced conditions from Eq. (5) is generally recommended.

The fixity factors ρ_{ai} and ρ_{bi} for each column must be determined for both conditions "braced" (i.e. with side sway totally inhibited) and "unbraced" (i.e. with side sway totally uninhibited).

The ϕ_j value of the representative j-column is calculated from Eq. (5) utilizing the fixity factors ρ_{aj} and ρ_{bj} for "braced" conditions. For the rest of the columns, the "braced" ϕ_i values are determined from Eq. (4b). In this step, it is important to make sure that the P_{cr} of each column for "braced" conditions is larger than that corresponding to "unbraced" conditions; otherwise, the representative j-column initially selected must changed to the one the lowest P_{cr} among the **n** columns considering that the top end of the multi-column system is "braced" against lateral sway.

The ϕ values of all columns (from step 3) and ρ_{ai} and ρ_{bi} for "unbraced" conditions (from step 2) are then substituted into Eq. (3) from which the required minimum bracing stiffness $(S_\Delta)_{min}$ can be calculated directly. The next two examples that follow show in detail the proposed procedure for single columns and plane frames.

$\gamma_i = h_i/h_j$ = ratio of height of column i to that of representative column j;

ϕ_j = ϕ-value of the representative column j calculated using Eq. (5) under braced conditions;

$\eta_i = 1/[1 + P_i/(G_iA_{si})]$ and $\eta_j = 1/[1 + P_j/(G_jA_{sj})]$.

The stability equation for the representative j-column under "braced" conditions is given by Eq. (5):

3.2 For 3D Multi-column Systems

The minimum stiffness of the lateral bracings $S_{\Delta X}$, $S_{\Delta Y}$ and torsional bracing $S_{\theta XY}$ required to convert any given story of a 3-D multi-column system with sidesway uninhibited or partially inhibited [Fig. 2] into a fully "braced" story can be determined from Eqs. (10), (11), and (12), respectively, as follows:
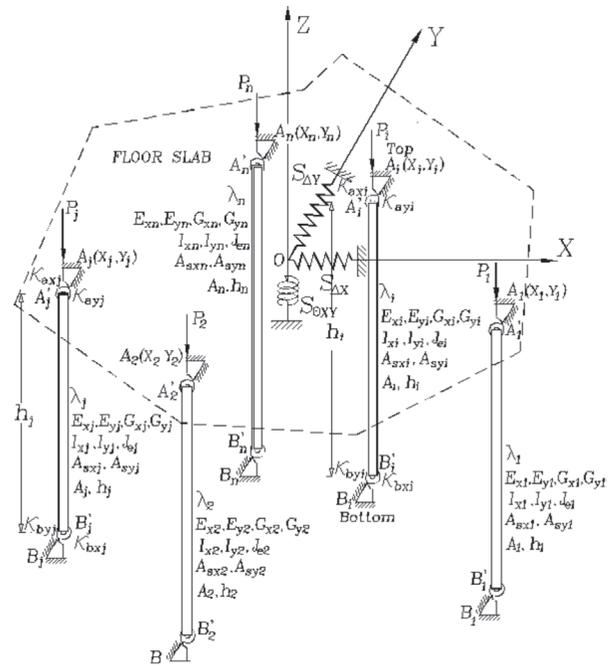
Fig. 2. 3D-model multi-column system with rotational restraints and with sidesway and torsion partially inhibited

$$(S_{\Delta X})_{min} = -\sum_{i=1}^{n}\left(S_{xi}\,cos^2\,\lambda_i + S_{yi}\,sin^2\,\lambda_i\right) \qquad (10)$$

$$(S_{\Delta Y})_{min} = -\sum_{i=1}^{n}\left(S_{xi}\,sin^2\,\lambda_i + S_{yi}\,cos^2\,\lambda_i\right) \tag{11}$$

$$(S_{\theta XY})_{min} = \sum_{i=1}^{n}\left[\left(S_{xi}sin^2\,\lambda_i + S_{yi}cos^2\,\lambda_i\right)X_i^2 + \left(S_{xi}cos^2\,\lambda_i + S_{yi}sin^2\,\lambda_i\right)Y_i^2 - 2X_iY_i\,sin\lambda_i\,cos\lambda_i\left(S_{xi}-S_{yi}\right) + G_{ei}J_{ei}/h_i\right] \tag{12}$$

Where the lateral stiffness coefficients S_{xi} and S_{yi} are given by Eqs. (13a) and (13b).

$$S_{xi} = \left[\frac{3(\rho_{axi}+\rho_{bxi}-2\rho_{axi}\rho_{bxi})+9\rho_{axi}\rho_{bxi}\dfrac{tan(\phi_{xi}/2)}{\phi_{xi}/2}}{\eta_{xi}\phi_{xi}^2(1-\rho_{axi})(1-\rho_{bxi})+3(\rho_{axi}+\rho_{bxi}-2\rho_{axi}\rho_{bxi})\left(1-\dfrac{\eta_{xi}\phi_{xi}}{tan\,\phi_{xi}}\right)+9\rho_{axi}\rho_{bxi}\left(\dfrac{tan(\phi_{xi}/2)}{\phi_{xi}/2}-\eta_{xi}\right)}-1\right]\frac{\alpha_i}{\gamma_i}\frac{\eta_{xj}\phi_{xj}^2E_jI_{xj}}{h_j^3} \tag{13a}$$

$$S_{yi} = \left[\frac{3(\rho_{ayi}+\rho_{byi}-2\rho_{ayi}\rho_{byi})+9\rho_{ayi}\rho_{byi}\dfrac{tan(\phi_{yi}/2)}{\phi_{yi}/2}}{\eta_{yi}\phi_{xi}^2(1-\rho_{ayi})(1-\rho_{byi})+3(\rho_{ayi}+\rho_{byi}-2\rho_{ayi}\rho_{byi})\left(1-\dfrac{\eta_{yi}\phi_{yi}}{tan\,\phi_{yi}}\right)+9\rho_{ayi}\rho_{byi}\left(\dfrac{tan(\phi_{yi}/2)}{\phi_{yi}/2}-\eta_{yi}\right)}-1\right]\frac{\alpha_i}{\gamma_i}\frac{\eta_{yj}\phi_{yj}^2E_jI_{yj}}{h_j^3} \tag{13b}$$

Where:

$$\phi_{xj}^2 = \frac{P_j}{(\eta EI/h^2)_{xj}} \tag{14a}$$

$$\phi_{xi}^2 = \frac{P_i}{(\eta EI/h^2)_{xi}} = \frac{\eta_{xj}}{\eta_{xi}}\frac{\alpha_i}{\beta_{xi}}\phi_{xj}^2\gamma_i^2 \tag{14b}$$

$$\phi_{yj}^2 = \frac{P_j}{(EI/h^2)_{yj}} \tag{14c}$$

$$\phi_{yi}^2 = \frac{P_i}{(\eta EI/h^2)_{yi}} = \frac{\eta_{yj}}{\eta_{yi}}\frac{\alpha_i}{\beta_{yi}}\phi_{yj}^2\gamma_i^2 \tag{14d}$$

$\alpha_i = P_i/P_j =$ ratio of axial load of column i to that of representative j-column;

$\beta_{xi} = (EI)_{xi}/(EI)_{xj} =$ ratio of flexural stiffness around the local x-axis of column i to that of representative j-column;

$\beta_{yi} = (EI)_{yi}/(EI)_{yj} =$ ratio of flexural stiffness around the local y-axis of column i to that of representative j-column;

$\gamma_i = h_i/h_j =$ ratio of height of column i to that of representative column j;

ϕ_{xj} and $\phi_{xj} =$ stability factors of the representative j-column corresponding to buckling along the local x_j- and y_j-axes calculated from Eq. (5) under "braced" conditions, respectively;

$\eta_{xi} = 1/[1+P_i/(G_{xi}A_{sxi})]$; and

$\eta_{yi} = 1/[1+P_i/(G_{yi}A_{syj})]$.

The stability equations for "braced" conditions along the X- and Y- directions are given by Eqs. (15a-b).

$$(1-\rho_{axj})(1-\rho_{bxj})\phi_{xj}^2 + 3(\rho_{axj}+\rho_{bxj}-2\rho_{axi}\rho_{bxj})\left(1-\frac{\eta_{xj}\phi_{xj}}{tan\,\phi_{xj}}\right)+9\rho_{axj}\rho_{bxj}\left(\frac{tan(\phi_{xj}/2)}{\phi_{xj}/2}-\eta_{xy}\right)=0 \tag{15a}$$

$$(1-\rho_{ayj})(1-\rho_{byj})\phi_{yj}^2 + 3(\rho_{ayj}+\rho_{byj}-2\rho_{ayi}\rho_{byj})\left(1-\frac{\eta_{yj}\phi_{yj}}{tan\,\phi_{yj}}\right)+9\rho_{ayj}\rho_{byj}\left(\frac{tan(\phi_{yj}/2)}{\phi_{yj}/2}-\eta_{yj}\right)=0 \tag{15b}$$

The stiffnesses $(S_{\Delta X})_{min.}$, $(S_{\Delta Y})_{min.}$ and $(S_{\theta XY})_{min.}$ can be obtained following the 4 steps described below:

1) Similar to 2-D frames, a representative j-column is selected from the **n** columns within the story level under consideration [Fig. 2]. The column with the

lowest critical axial load under braced conditions is generally recommended.

2) The fixity factors ρ_{ai} and ρ_{bi} of each column must be determined in the local x_i- and y_i-directions for both conditions, "braced" and "unbraced";

3) The "braced" ϕ_{xj} and ϕ_{yj} factors of the representative column j are calculated from Eqs. (15a) and (15b) utilizing the fixity factors ρ_{axj}, ρ_{ayj}, ρ_{bxj}, and ρ_{by} for the "braced" case. For the rest of the columns the "braced" factors ϕ_{xj} and ϕ_{yj} can be determined from Eqs. (14b) and (14d). At this step, it is important to check that P_{cr} of each column for "braced" conditions is larger than that corresponding to "unbraced" conditions; otherwise, the j-column initially selected must be changed to the column with the smallest P_{cr} among the n columns considering that story level "braced".

4) The "braced" ϕ-factors (from step 2), the fixity factors ρ_{ai} and ρ_{bi} in x- and y-directions for "unbraced" conditions (from step 1), the XY coordinates of each column, and the individual column torsional stiffness $(G_{ei}J_{ei}/h_i)$ are then substituted into Eqs. (10), (11) and (12) from which the required minimum stiffness of the bracings $(S_{\Delta X})_{min.}$, $(S_{\Delta Y})_{min.}$ and $(S_{\theta XY})_{min.}$ be calculated directly. The two examples that follow show the proposed procedure for 3-D framed structures.

Note that similar to what happens in 2D multi-column system when the values of both fixity factors ρ_a and ρ_b for "braced" and "unbraced" conditions in any column of a 3D multi-column system remain unchanged in any of its local x- and y-directions. Thus $(S_{\Delta X})_{min}$ or/and $(S_{\Delta Y})_{min}$ and $(S_{\theta XY})_{min.}$ become infinity except for leaning columns. This comes from the fact that denominator of Eqs. (13a-b) which is identical to expressions (15a-b) become equal to zero, making $(S_{\Delta X})_{min}$ or/and $(S_{\Delta Y})_{min}$ and $(S_{\theta XY})_{min.}$ equal to infinity, with the shear factors η having no effect.

4. SUMMARY AND CONCLUSIONS

A method that determines the minimum stiffness of the lateral and torsional bracings required by a multi-column elastic system, with the goal of achieving non-sway buckling conditions is proposed. Equations that evaluate the required minimum stiffness of the lateral and torsional bracings and the corresponding "braced"

critical buckling load for each column of the story level are derived using the modified stability functions. The following effects are included: 1) the plan layout of the columns (i.e., the orientation and location of the centroid of the cross section of all columns); and 2) shear deformations along each column using the modified method proposed by Timoshenko and Gere [26]. The proposed method is applicable to 2D and 3D multi-column systems with rigid, semi-rigid, and simple connections. It is shown that the minimum stiffness of the bracings required by a multi-column system depends on: 1) the blueprint layout and the layout of the columns; 2) the variation in height and cross sectional properties among the columns; 3) the flexural and shear stiffness of each column; 4) the applied axial load pattern on the columns; 5) the lack of symmetry in the loading pattern, column layout, column sizes, and heights that cause the combined torsion-sway buckling which reduces the buckling capacity of the frame as a whole; and 6) the support conditions and end restraints of the columns. The proposed method is limited to elastic framed structures with columns with a doubly-symmetrical cross section oriented in any direction with respect to the global axes. Three examples are presented in detail to show the effectiveness and simplicity of the proposed method and the effects of the orientation of the cross section and shear deformations along the height of the columns.

Definite criterion for minimum stiffness of bracing for 2-D and 3-D elastic framed structures is presented, and the corresponding equations are derived using the modified method proposed by Timoshenko and Gere [26]. A condensed approach that determines the minimum stiffness of story bracing required by plane and space framed structures to achieve non-sway buckling conditions is proposed. The proposed approach and corresponding equations are applicable to multi-column frames with rigid, semirigid, and simple connections. The proposed method is only applicable to elastic framed structures with the following limitations: 1) the floor diaphragms including the ground floor of the framed structure are assumed to be axially rigid; and 2) all columns are assumed to be doubly-symmetrical with their principal cross-sectional axes parallel to the XY global axes (i.e., columns whose shear center and centroid coincide). The effects of axial deformations are also neglected. As a consequence, overall story flexural buckling occurring along the X- or/and Y-axes and overall flexural-torsional buckling occurring in the XY plane and about the Z axis are considered in

3-D frames. Overall story flexural buckling occurring in the plane of the frame is only considered in the bracing analysis of 2-D frames. Pure torsional buckling in a single column is not considered herein.

The proposed method finds great applications in systems made of materials with relatively low shear stiffness such as orthotropic composite materials (FRP) and multilayer elastomeric bearings used for seismic isolation of buildings. It allows the analyst to investigate directly not only the combined effects of shear and bending deformations along each column, but also the topology of the columns (i.e., their layout and orientation on the XY-plane), load patterns, semirigid connections, flexural hinges, interstory bracing, and the properties of the members (span, cross sectional geometry, and elastic moduli) on the minimum bracing stiffness requirements for plane and space framed structures. The proposed approach is more accurate and general than any other method available in the technical literature.

5. APPENDIX I. DERIVATION OF BRACING STIFFNESS EQUATIONS

5.1 Determination of Lateral Stiffness Coefficients S_{xi} and S_{yi}-.

The stability analysis of a prismatic column including bending and shear deformations (Figs. 3b-c) is formulated using the modified approach proposed by Timoshenko and Gere [26, page 134]. This approach has been utilized by Aristizabal-Ochoa [21] in the stability analysis of columns and elastomeric isolation bearings. The governing equations are as follow:

$$\eta EI\, u''(x) + P\,u(x) = -M_a - (M_a + M_b + P\Delta)\frac{x}{h} \quad (16a)$$

$$\eta EI\, \psi''(x) + P\,\psi(x) = \frac{M_a + M_b + P\Delta}{h} \qquad (16b)$$

Where $u(x)$ = lateral deflection of the column center line; and $\psi(x)$ = rotation of the cross section, as shown by Fig. 2c. The solutions for the second-order linear differential Eqs. (16a)-(16b) are as follows:

$$u(x) = A\cos(\frac{x}{h}\phi) + B\sin(\frac{x}{h}\phi) + \frac{x}{h}\left(\frac{M_a + M_b}{P} + \Delta\right) - \frac{M_a}{P} \qquad (17a)$$

$$\psi(x) = C\cos(\frac{x}{h}\phi) + D\sin(\frac{x}{h}\phi) + \frac{1}{h}\left(\frac{M_a + M_b}{P} + \Delta\right) \qquad (17b)$$

Where: $\phi^2 = P/(\eta EI/h^2)$.

Fig. 3. Model of column with sidesway partially inhibited and with rotational and lateral end restraints: (a) Structural model; b) end moments, forces, rotations and deflections; c) differential element including bending and shear deformations; and d) column orientation of cross-section local axis

The unknown coefficients A, B, C, and D can be obtained from the following boundary conditions:

At x = 0: $u = 0$, and $\psi = \psi_a$

At x = h: $u = \Delta$ and $\psi = \psi_b$

Where:

ψ_a and ψ_b = rotations of cross sections at A and B due to bending, respectively;

Δ_a and Δ_b = lateral sway at A and B, respectively.

$\Delta = \Delta_a - \Delta_b$ = relative sidesway of column end A with respect to its bottom end B.

$$\psi_a = \frac{M_a}{EI/h}\frac{\sin\phi - \eta\phi\cos\phi}{\eta\phi^2 \sin\phi} + \frac{M_b}{EI/h}\frac{\sin\phi - \eta\phi}{\eta\phi^2 \sin\phi} + \frac{\Delta}{h} = -\frac{M_a}{\kappa_a} \tag{18a}$$

$$\psi_b = \frac{M_a}{EI/h}\frac{\sin\phi - \eta\phi}{\eta\phi^2 \sin\phi} + \frac{M_b}{EI/h}\frac{\sin\phi - \eta\phi\cos\phi}{\eta\phi^2 \sin\phi} + \frac{\Delta}{h} = -\frac{M_b}{\kappa_b} \tag{18b}$$

Where: $\eta = 1/[1+P/(A_s G)]$; and A_s = effective shear area of the column. Notice that, in this approach V includes the component $P\psi$, as suggested by Haringx [27]. This component was not included in the paper

Therefore: $A = \dfrac{M_a}{P}$; $B = \dfrac{M_a}{P}\tan(\phi/2) - \dfrac{M_a + M_b}{P\sin\phi}$; $C = $

$$\psi_a - \frac{1}{h}\left(\frac{M_a + M_b}{P} + \Delta\right); \text{ and}$$

$$D = \frac{\psi_a - \psi_b \cos\phi}{\sin\phi} - \left(\frac{M_a + M_b}{P} + \Delta\right)\tan(\phi/2)$$

Since $u' = \psi + V/(A_s G)$ and $V = P\psi - \dfrac{M_a + M_b + P\Delta}{h}$ the following expressions for ψ_a and ψ_b can be obtained:

by Aristizabal-Ochoa [20] on the effects of shear deformations on the stability of single beam-columns.

Eqs. (18) can be represented in matrix form as follows:

$$\frac{h}{EI}\begin{bmatrix} \dfrac{\sin\phi - \eta\phi\cos\phi}{\eta\phi^2 \sin\phi} + \dfrac{EI/h}{\kappa_a} & \dfrac{\sin\phi - \eta\phi}{\eta\phi^2 \sin\phi} \\ \dfrac{\sin\phi - \eta\phi}{\eta\phi^2 \sin\phi} & \dfrac{\sin\phi - \eta\phi\cos\phi}{\eta\phi^2 \sin\phi} + \dfrac{EI/h}{\kappa_b} \end{bmatrix}\begin{Bmatrix} M_a \\ M_b \end{Bmatrix} = -\frac{\Delta}{h}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{19}$$

Inverting the 2×2 matrix in Eq. (19), M_a and M_b can be expressed in term of the interstory drift Δ/h as follows:

$$M_a = -\frac{\dfrac{1}{R_b} + \dfrac{1 - \tan(\phi/2)}{\phi}}{\dfrac{\eta\phi^2}{R_a R_b} + \left(\dfrac{1}{R_a} + \dfrac{1}{R_b}\right)\left(1 - \dfrac{\eta\phi}{\tan\phi}\right) + \dfrac{\tan(\phi/2)}{\phi/2} - \eta}P\Delta \tag{20a}$$

$$M_a = -\frac{\dfrac{1}{R_a} + \dfrac{\tan(\phi/2)}{\phi}}{\dfrac{\eta\phi^2}{R_a R_b} + \left(\dfrac{1}{R_a} + \dfrac{1}{R_b}\right)\left(1 - \dfrac{\eta\phi}{\tan\phi}\right) + \dfrac{\tan(\phi/2)}{\phi/2} - \eta}P\Delta \tag{20b}$$

Since three unknowns (M_{ai}, M_{bi}, and Δ) are involved, one more equation is required at the element level. This equation is obtained applying rotational equilibrium of column AB in Fig. 2b:

$$M_a + M_b + P\Delta + F h = 0 \tag{21}$$

Substituting Eqs. (20a-b) into Eq. (21)

$$\left[\frac{\dfrac{1}{R_a}+\dfrac{1}{R_b}+\dfrac{tan(\phi/2)}{\phi/2}}{\dfrac{\eta\phi^2}{R_a R_{bi}}+\left(\dfrac{1}{R_{ai}}+\dfrac{1}{R_{bi}}\right)\left(1-\dfrac{\eta\phi}{tan\,\phi}\right)+\dfrac{tan(\phi/2)}{\phi/2}-\eta}-1\right]\eta\phi^2\,\frac{EI}{h^3}=\frac{F}{\Delta} \tag{22}$$

The left term of Eq. (22) represents the lateral stiffness (including the effects of shear deformations) provided by column AB to the floor system in the direction of buckling. Eq. (22) can be expressed in terms of the fixity factors [ρ_{ai} and ρ_{bi} from Eqs. (1a-b)] and ratios α_i and γ_i, as follows:

$$\left\{\frac{3(\rho_{ai}+\rho_{bi}-2\rho_{ai}\rho_{bi})+9\rho_{ai}\rho_{bi}\dfrac{tan(\phi_i/2)}{\phi_i/2}}{\eta_i\phi_i^2(1-\rho_{ai})(1-\rho_{bi})+3(\rho_{ai}+\rho_{bi}-2\rho_{ai}\rho_{bi})\left(1-\dfrac{\eta_i\phi_i}{tan\phi_i}\right)+9\rho_{ai}\rho_{bi}\left(\dfrac{tan(\phi_i/2)}{\phi_i/2}-\eta_i\right)}-1\right\}\frac{\alpha_i}{\gamma_i}\phi_j^2\,\frac{\eta_j E_j I_j}{h_j^3}=\frac{F_i}{\Delta} \tag{23}$$

The left term in Eq. (23), when applied in the local x_i- and y_i-principal directions, becomes the stiffness coefficients S_{xi} and S_{yi} given by Eqs. (13a) and (13b) previously utilized in Eqs. (10)-(13).

The stiffness matrix $\begin{bmatrix} S_{xi} & 0 \\ 0 & S_{yi} \end{bmatrix}$ of column i in the principal $x_i y_i$ local system (Fig. 2c) is then transformed into the global system XY as follows:

$$[K]_i=\begin{bmatrix} cos\,\lambda_i & -sin\,\lambda_i \\ sin\,\lambda_i & cos\,\lambda_i \end{bmatrix}\begin{bmatrix} S_{xi} & 0 \\ 0 & S_{yi} \end{bmatrix}\begin{bmatrix} cos\,\lambda_i & sin\,\lambda_i \\ -sin\,\lambda_i & cos\,\lambda_i \end{bmatrix}=$$
$$\begin{bmatrix} S_{xi}\,cos^2\,\lambda_i+S_{yi}\,sin^2\,\lambda_i & (S_{xi}-S_{yi})\,sin\,\lambda_i\,cos\,\lambda_i \\ (S_{xi}-S_{yi})\,sin\,\lambda_i\,cos\,\lambda_i & S_{xi}\,sin^2\,\lambda_i+S_{yi}\,cos^2\,\lambda_i \end{bmatrix} \tag{24}$$

Using Eq. (24) for each column i (with its centroid located at X_i and Y_i with respect to the global XY-system), and applying static equilibrium of the top floor of the model shown in Fig. 2 along the X-axes, and assuming that the floor is braced along the Y-axes and restrained to rotate about the Z-axis for 3D frames, then:

$$\left[\left(\sum_{i=1}^{n}S_{xi}\,cos^2\,\lambda_i+S_{yi}\,sin^2\,\lambda_i\right)+S_{\Delta X}\right]\Delta_X=0 \tag{25}$$

Now, considering static equilibrium along the Y-axis and around the Z-axis independently (Fig. 1b) in a similar fashion as it was done along the X-axis, the following two equations can be obtained:

$$\left[\left(\sum_{i=1}^{n}S_{xi}\,sin^2\,\lambda_i+S_{yi}\,cos^2\,\lambda_i\right)+S_{\Delta Y}\right]\Delta_Y=0 \tag{26}$$

$$\sum_{i=1}^{n}\begin{bmatrix}\left(S_{xi}\,sin^2\,\lambda_i+S_{yi}\,cos^2\,\lambda_i\right)X_i^2+\\ \left(S_{xi}\,cos^2\,\lambda_i+S_{yi}\,sin^2\,\lambda_i\right)Y_i^2-\\ 2X_iY_i\,sin\,\lambda_i\,cos\,\lambda_i\left(S_{xi}-S_{yi}\right)+G_iJ_{ei}/h_i\end{bmatrix}\theta_{XY}=0 \tag{27}$$

Where the coefficients within the square matrix are given by Eqs. (10)-(12). The stability analysis, on the other hand, can be carried out by making the determinant of Eq. (28) equal to zero. The three modes of buckling of a story in 3D multi-column systems are related to the types of symmetry about the X- and Y-axes and they are fully described by Aristizabal-Ochoa [21].

ACKNOWLEDGEMENTS

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NOTATION

The following symbols are used in this paper:
A_s = effective shear area of the column.
E = Elastic modulus of the material;
G = Shear modulus of the material;
h_i = height of column i;
i = subscript indicating column-i of the 3-D multi-column system;

j = subscript indicating the representative column-j of the multi-column system;

I_g = girder moment of inertia;

I_i or I_c = moment of the inertia of column i;

κ_{ai} and κ_{bi} = the flexural stiffness of the end connections at Ai and Bi, respectively;

L_g = girder span;

$(P_{cr})_j$ = buckling load of representative column j;

$(P_{cr})_i$ = buckling load of column i [$=\alpha_i (P_{cr})_j$];

$\mathbf{n}$ = total number of columns in the story system;

$S_{\Delta x}$ = interstory lateral stiffness or bracing stiffness provided to the system along the X-axis;

$S_{\Delta y}$ = interstory lateral stiffness or bracing stiffness provided to the system along the Y-axis;

$S_{\theta xy}$ = stiffness of the interstory torsional bracing provided to the system about the Z-axis;

R_{ai} = stiffness index of the flexural connection at A_i [$= \kappa_{ai}/(EI_i/h_i)$];

R_{bi} = stiffness index of the flexural connection at B_i [$=\kappa_{bi}/(EI_i/h_i)$];

X_i and Y_i = X-and Y-coordinates of column i with respect to origin O;

x and y = subscripts that indicate that the calculation is in the global X- and Y-directions, respectively;

α_i = ratio of axial load of column i to that of representative column j [$= P_i/P_j$];

β_i = ratio of flexural stiffness of column i to that of representative column j [$= (E_i I_i)/(E_j I_j)$];

η = [$1+P/(A_s G)$];

γ_I = ratio of height of column i to that of representative column j [$= h_i/h_j$];

Δ = interstory drift;

ρ_{ai} and ρ_{bi} = fixity factors at A_i and B_i of column i, respectively;

$$\phi_i = \sqrt{\frac{P_i}{\eta_i (E_i I_i / h_i^2)}} = \text{Stability factor of column } i;$$

Γ_{ai} and T_{bi} = ratios $\sum(EI/h)_c/(EI/L)_g$ at ends A_i and B_i of column I, respectively;

θ = interstory angle of twist of the story floor;

θ'_{ai} and θ'_{bi} = rotations of column i at A'_i and B'_i with respect to column's cord, respectively.