A STABILITY AND SENSITIVITY ANALYSIS OF PARAMETRIC FUNCTIONS IN A SEDIMENTATION MODEL

UN ANALISIS DE ESTABILIDAD Y SENSIBILIDAD DE LAS FUNCIONES DEFINIDAS POR PARAMETROS EN UN MODELO DE SEDIMENTACION

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ABSTRACT: This paper deals with the reliable and efficient numerical identification of parameters defining the flux function and the diffusion coefficient of a strongly degenerate parabolic partial differential equation (PDE), which is the basis of a mathematical model for sedimentation-consolidation processes. A zero-flux initial-boundary value problem (IBVP) posed for this PDE describes the settling of a suspension in a column. The parameters for a given material are estimated by the repeated numerical solutions of the IBVP (direct problem) under systematic variation of the model parameters, with the aim of successively minimizing a cost functional that measures the distance between a space-dependent observation and the corresponding numerical solution. Two important features of this paper are the following. In the first place, the method proposed for the efficient and accurate numerical solution of the direct problem. We implement a well-known explicit, monotone three-point finite difference scheme enhanced by discrete mollification. The mollified scheme occupies a larger stencil but converges under a less restrictive CFL condition, which allows the use of a larger time step. The second feature is the thorough sensitivity and stability analysis of the parametric model functions that play the roles of initial guess and observation data, respectively.

Keywords: Sedimentation of suspensions, sensitivity analysis, degenerate parabolic equation, parameter estimation, discrete mollification.

RESUMEN. Este artículo se dedica a la identificación numérica confiable y eficiente de los parámetros que definen la función de flujo y el coeficiente de difusión en una ecuación diferencial parcial de tipo parabólico fuertemente degenerada que es la base de un modelo matemático para procesos de sedimentación-consolidación. Para esta ecuación, el problema de valor inicial con valores en la frontera (IBVP) en el que el flujo es nulo, describe el asentamiento de una suspensión en una columna. Los parámetros para un material dado se estiman con base en repetidas soluciones numéricas del problema directo (IBVP) con una variación sistemática de los parámetros del modelo, con el objeto de minimizar sucesivamente un funcional de costo que mide la distancia entre una observación dependiente de tiempo y la correspondiente solución numérica. En este artículo se destacan dos aspectos. El primer aspecto es que en el método propuesto para la solución numérica eficiente y acertada del problema directo, se implementa un esquema explícito monótono bien conocido basado en diferencias finitas que usan tres puntos mejorado por molificación discreta. El esquema molificado utiliza una malla de más puntos pero converge con una condición CFL menos restrictiva, lo cual permite usar pasos temporales más grandes. El segundo aspecto es el exhaustivo análisis de sensibilidad y estabilidad de las funciones definidas por parámetros en el modelo y que juegan los papeles de aproximación inicial y dato observado, respectivamente.

Palabras claves: Sedimentación de suspensiones, análisis de sensibilidad, ecuación parabólica degenerada, estimación de parámetros, molificación discreta.

1. INTRODUCTION

1.1. Scope

Our goal is the numerical identification of unknown parameters in the flux and diffusion terms for the following initial-boundary value problem (IBVP) for a strongly degenerate parabolic equation in one space dimension:

$$u_t + f(u)_x = A(u)_{xx},$$
$$(x,t) \in \Omega_T := (0,L) \times (0,T], L > 0, T > 0 \quad (1a)$$
$$u(x,0) = u_0(x),\ x \in [0,L], \quad (1b)$$
$$f(u) - A(u)_x|_{x=0} = \psi_0(t),\ t \in (0,T], \quad (1c)$$
$$f(u) - A(u)_x|_{x=L} = \psi_L(t),\ t \in (0,T], \quad (1d)$$

where A is an integrated diffusion coefficient, i.e.,

$$A(u) = \int_0^u a(s)\, ds, \quad a(u) \geq 0. \qquad (2)$$

The diffusion function a is assumed to be integrable and is allowed to vanish on u-intervals of positive length, on which (1a) turns into a first-order hyperbolic conservation law, so that (1a) is a *strongly degenerate parabolic*. On the other hand, we assume that f is piecewise smooth and Lipschitz continuous. Under suitable choices of u_0, f, a, ψ_0 and ψ_L the IBVP (1) may describe a variety of real-world applications like traffic flow [9]. We focus our attention on Equation (1) as a model of the sedimentation-consolidation process of a solid-liquid suspension [8].

It is well known that solutions of (1a) are, in general, discontinuous even if u_0 is smooth, and need to be defined as weak solutions along with an entropy condition to select the physically relevant solution, the *entropy solution*. For the definition, existence and uniqueness of entropy solutions of (1) we refer to [7, 8, 10].

In the present work we are interested in a stability and sensitivity analysis of the parametric model functions. In order to perform the tests, we first proceed with a numerical estimation procedure based on repeated numerical solutions of the direct problem (1) under successive variation of parameters appearing in the coefficient functions f and a. In this phase the main components are the efficient and stable solver of the direct problem and the optimization procedure based on the Nelder- Mead Simplex Method. Our goal is the stability and sensitivity analysis of the resulting inverse problem. Theoretical aspects related to identifiability are not our concern in this paper (but cf., e.g., [11]). By sensitivity analysis we mean an intensive set of tests for the numerical identification of parameters with or without noisy observation data. Our approach follows the methodology of [4] but we acknowledge the existence of other ways to perform a sensitivity analysis, for instance [19].

1.2. Related work and outline of the paper

The discrete mollification method is a convolution-based filtering procedure suitable for the regularization of ill-posed problems and for the stabilization of explicit schemes for the solution of PDEs. For the numerical identification of diffusion coefficients by discrete mollification, see [16] and its references.

Inverse problems for strongly degenerate parabolic equations are of particular interest in the context of the sedimentation-consolidation model. In fact, in applications such as wastewater treatment and mineral processing, the reliable extraction of material-specific parameters appearing in the model functions f and a from laboratory experiments allows the operation and control of continuous clarifier-thickeners handling the same material to be simulated [10, 21]. For the special case $A \equiv 0$, i.e., when effects of sediment compressibility are absent or negligible, (1a) reduces to a first-order nonlinear conservation law, portions of the function f can be identified by comparing observed space-time trajectories of concentration discontinuities, with trajectories appearing in closed-form solutions for piecewise constant initial concentrations [6, 14]. In the presence of sediment compressibility, closed-form solutions are not available and one has to resort to numerical techniques to solve the parameter identification problem [5, 11].

The paper is organized as follows. Section 2 presents the sedimentation-consolidation model along with details on the schemes for the solution of the direct problem, including a brief description of the mollification method. Section 3 deals with the parameter identification problem, the proposed algorithm, the sensitivity analysis and the effect of noisy observation data. This section ends with some conclusions.

2. THE APPLICATION OF THE MATHEMATICAL MODEL

2.1. Sedimentation model

According to [8, 10] and the references cited in these works, (1) can be understood as a model for the settling of a flocculated suspension of small solid particles dispersed in a viscous fluid, where $u = u(x,t)$ is the local solid concentration as a function of height x and time t. For batch settling in a closed column of height L we set $\psi_0 = 0$ and $\psi_L = 0$; the function u_0 denotes the initial solid concentration. The material specific function f describes the effect of hindered settling.

We employ here the following typical parametric expression:

$$f(u) = \begin{cases} v_\infty u\left(1-\dfrac{u}{u_{max}}\right)^C & 0 \le u \le u_{max}, \\ 0 & otherwise, \end{cases} \quad (3)$$

where $v_\infty < 0$ is the settling velocity of a single particle in an unbounded fluid, C>1 is a dimensionless exponent that quantifies how rapidly the settling velocity decreases (as an absolute value) with increasing solids concentration, and $0 < u_{\max} \le 1$ is a (nominal) maximal solids concentration. The function A is given by (2), where we define

$$a(u) = -\frac{f(u)\sigma_e'(u)}{(\rho_s-\rho_f)gu}, \quad (4)$$

where ρ_s and ρ_f are the solid and fluid densities, respectively, g is the acceleration of gravity and $\sigma_e'(u) = \frac{d\sigma_e}{du}$ is the derivative of the material specific solid stress function σ_e.

Among several proposed semi-empirical approaches for σ_e we chose the power law type function

$$\sigma_e(u) = \begin{cases} 0 & \text{for } 0 \le u \le u_c, \\ \sigma_0\left[(u/u_c)^\beta - 1\right] & u_c < u, \end{cases}$$

with material-dependent parameters $\sigma_0 > 0$ and $\beta > 1$. The values of β, σ_0 and u_c characterize the compressibility of the sediment formed by a given material.

Values of the primitive $A(u)$ usually have to be determined by numerical quadrature. However, if f and a are given by (3)–(4) and β is an integer, then $A(u)$ can be evaluated in closed form by $A(u) = 0$ for $0 \le u \le u_c$ (equation (1a) is strongly degenerate) and $A(u) = \boldsymbol{A}(u) - \boldsymbol{A}(u_c)$ for $u > u_c$, where the function $\boldsymbol{A}$ is defined by

$$\boldsymbol{A}(u) := \frac{v_\infty \sigma_0}{\Delta_\rho g u_c^\beta u_{\max}^C} \times \sum_{k=1}^{\beta}\left(\prod_{l=1}^{k}\frac{\beta+1-l}{C+l}\right)(u_{\max}-u)^{C+k}u^{\beta-k}.$$

2.2. Discrete mollification

The discrete mollification method [17, 18] consists in replacing a set of data $y = \{y_j\}_{j\in\mathbb{Z}}$ by its mollified version $J_\eta y$, where J_η is the *discrete mollification operator* defined by

$$[J_\eta y]_j := \sum_{i=-\eta}^{\eta}\omega_i y_{j-i}, \ j \in \mathbb{Z}.$$

The support parameter $\eta \in \mathbb{N}$ indicates the width of the mollification stencil, and the weights ω_i satisfy $\omega_i = \omega_{-i}$ and $0 \le \omega_i \le \omega_{i-1}$ for $i = 1,\dots,\eta$ along with $\omega_{-\eta} + \cdots + \omega_{\eta-1} + \omega_\eta = 1$. The weights ω_i are obtained by numerical integration of a suitable truncated Gaussian kernel. Details can be found in [1, 2, 3, 16, 20].

2.3. Discretization of the direct problem

The domain Ω_T is discretized by a standard Cartesian grid by setting $x_j := j\Delta x, j = 0,\dots,\mathcal{N}$, where $\mathcal{N}\Delta x = L$ and $t_n := n\Delta t, n = 0,\dots,\mathcal{M}$, where $\mathcal{M}\Delta t = T$.

We denote by u_j^n an approximate value of the cell average of $u = u(x, t)$ over the cell $[x_j, x_{j+1}]$ at time $t = t_n$ and correspondingly set

$$u_j^0 = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0(x)\,dx, \ j = 0,\dots,\mathcal{N}-1.$$

We solve (1) numerically using two convergent finite difference methods. The first one [8, 13] has the following form, where $\lambda := \Delta t/\Delta x$ and $\mu := \Delta t/\Delta x^2$:

$$u_j^{n+1} = u_j^n - \lambda\Delta_+ F^{EO}(u_{j-1}^n, u_j^n) + \mu\left(A(u_{j+1}^n) - 2A(u_j^n) + A(u_{j-1}^n)\right). \quad (5)$$

Here F^{EO} stands for the well-known Engquist-Osher numerical flux [12], and Δ_+ denotes the standard forward difference operator. Scheme (5) is monotone and convergent under the CFL condition

$$\lambda\|f'\|_\infty + 2\mu\|a\|_\infty \le 1. \quad (6)$$

The second finite difference method is the mollified scheme [2], which is also monotone and convergent and takes the form

$$u_j^{n+1} = u_j^n - \lambda\Delta_+ F^{EO}(u_{j-1}^n, u_j^n) + 2\mu C_\eta\left([J_\eta A(u_j^n)]_j - A(u_j^n)\right), \quad (7)$$

with

$$C_\eta := \left[\sum_{j=-\eta}^{\eta} j^2 \omega_{-j}\right]^{-1}.$$

This is an explicit method and has the convenient CFL condition

$$\lambda \|f'\|_\infty + 2\mu\varepsilon_\eta \|a\|_\infty \leq 1, \quad (8)$$

where $\varepsilon_\eta < 1$. (For the particular mollification weights considered herein, we obtain $\varepsilon_3 = 0.7130, \varepsilon_5 = 0.3969$ *and* $\varepsilon_8 = 0.1960$.) Clearly, condition (8) is more favorable than (6) since it shows that for a given value of Δx, mollified schemes may proceed by larger time steps. See [2] for more details on this scheme.

3. SENSITIVITY AND STABILITY ANALYSIS

3.1. Parameter identification problem

The inverse problem can be formulated as follows: given observation data $u^{obs}(x)$ at the final time $T > 0$ and functions u_0, ψ_0 and ψ_L, find the flux f and the diffusion function a such that the entropy solution $u(x, T)$ of problem (1) is as close as possible to $u^{\text{obs}}(x)$ in some suitable norm. The inverse problem is solved by minimizing the cost function

$$J(u(.,T)) = \frac{1}{2}\int_0^L \left|u(x,T) - u^{\text{obs}}(x)\right|^2 dx. \quad (9)$$

Since the functions f and a depend on a vector of parameters, the inverse problem corresponds to the following parameter identification problem:

$$Minimize\ J(\boldsymbol{p})\ w.r.t. parameter\ vector\ \boldsymbol{p}. \quad (PI)$$

The functions f and a are associated to the current parameter vector $\boldsymbol{p}$.

We define the piecewise constant function u^Δ by $u^\Delta(x,t) = u_j^n$ for $x \in [x_j, \ x_{j+1})$ and $t \in [t_n, t_{n+1})$ for $j = 0,\ldots, \mathcal{N} - 1$ and $n = 0,\ldots, \ \mathcal{M} - 1$ and replace u^{obs} by a piecewise constant function $u^{\text{obs},\Delta}$ formed by cell averages as follows:

$$u^{\text{obs},\Delta}(x) = u_j^{\text{obs}} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u^{\text{obs}}(x)\,\mathrm{d}x \quad \text{for } x\epsilon[x_j, x_{j+1}),$$

where $j = 0,\ldots, \mathcal{N} -1$. The parameter dependent cost function is

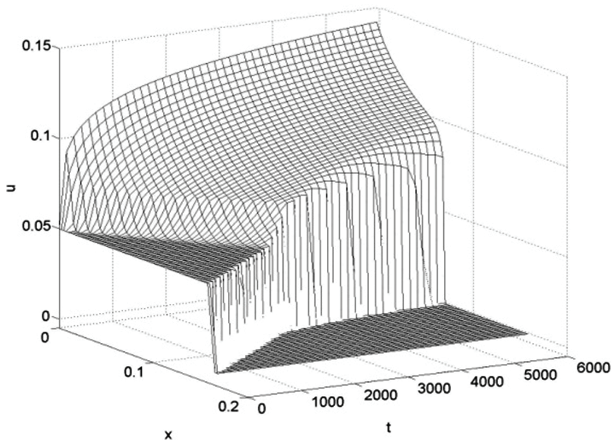


Figure 1. Reference solution

$$J^\Delta(\boldsymbol{p}) = \frac{1}{2}\int_0^L \left|u^\Delta(x,T) - u^{\text{obs},\Delta}(x)\right|^2 \mathrm{d}x = \frac{\Delta x}{2}\sum_{j=0}^{\mathcal{N}-1}\left|u_j^{\mathcal{M}} - u_j^{\text{obs}}\right|^2. \quad (10)$$

This yields a discrete version of (PI) given by

$$\begin{cases} minimize\ J^\Delta(\boldsymbol{p})\ w.r.t.\ \boldsymbol{p} \\ u^\Delta, numerical\ solution\ of\ (1) \\ f\ and\ a\ associated\ to\ current\ \boldsymbol{p}. \end{cases} \quad (PI^\Delta)$$

There are many options for the numerical implementation of the optimization procedure. We selected a globalized bounded Nelder-Mead Method with restarts (MATLAB function *fminsearch,* see [15] for details), which is a major improvement over the basic simplex method. The strategy is described by the following algorithm. Suppose $\boldsymbol{p}_j = (p_j^1, \ldots, p_j^K)$, that is, K different parameters are sought.

Step 1 Input $\boldsymbol{p}_0, \epsilon$

Step 2 for $j = 1\ to\ M$

$$\boldsymbol{p}_j = \texttt{fminsearch}(J^\Delta, \boldsymbol{p}_{j-1}).$$

If $\max_{1 \leq k \leq K}\left|\frac{p_j^k - p_{j-1}^k}{p_{j-1}^k}\right| \leq \epsilon$ then break, end

Step 3 End.

3.2. Numerical examples

The reference solution is generated by the corresponding numerical scheme (5) or (7) on a very fine grid. For examples 1, 2 and 3 we consider batch settling in a column of height $L = 0.16$ m and parameter values

$u_{\max} = 0.5, g = 9.81\mathrm{m/s}^2, \rho_{\mathrm{s}} - \rho_{\mathrm{f}} = 1660\ \mathrm{kg/m}^3, v_\infty = -2.7 \times 10^{-4}\ \mathrm{m/s}$, $C = 21.5$, $\beta = 5, u_{\mathrm{c}} = 0.07$ and $\sigma_0 = 1.2\mathrm{Pa}$.

The objective is to obtain an accurate identification of the parameters u_{c} σ_0 and C in eight different instances described in Table 1. Our experiments include clean and noisy observation data. Data at the instant $T = 800$ s will play the role of u^{obs}. Figures 1 and 2 (a) show the reference solution over the whole computational domain and the profile at $T = 800$ s, respectively. The restarting parameter and the tolerance parameter for the optimization are $M = 10$ and $\epsilon = 10^{-4}$, respectively.

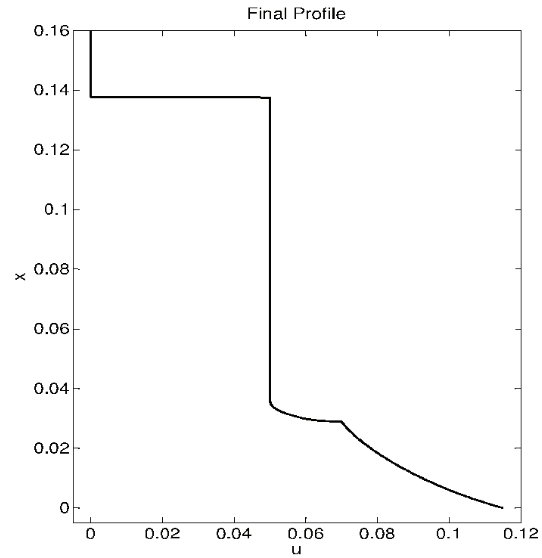
Example 1: Sensitivity to mollification parameters. Clean observation data (no noise added) and $\Delta x = L/256$. The results are summarized in Table 2. Here, j denotes the number of calls to the `fminsearch` algorithm, $\boldsymbol{p_j}$ is the vector of parameter values found, E_J is the required number of computed solutions of the direct problem, e_∞ is the maximum relative error in the result for each parameter (usually due to σ_0), and CPU denotes the total CPU time of each run.

Example 2: Sensitivity to initial guess. We randomly generate 100 initial guesses and carry out the identification task. Each initial guess $\boldsymbol{p_0} = (u_{\mathrm{c}}^0, \sigma_0^0, C^0)^T$ is generated in the form

$$u_c^0 = (1 + 0.3\xi_1)u_c, \sigma_0^0 = (1 + 0.3\xi_2)\sigma_0,$$
$$C^0 = (1 + 0.3\xi_3)C,$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3$ is a uniformly distributed random vectorial variable whose components are between −1 and 1. The results are indicated in Table 3. Here, the average $\bar{e}_\infty$ of e_∞ and its standard deviation σ are included. Additionally, column "# restarts" stands for the number of calls of `fminsearch` and E_J for the number of solutions of the direct problem.

(a)



(b)

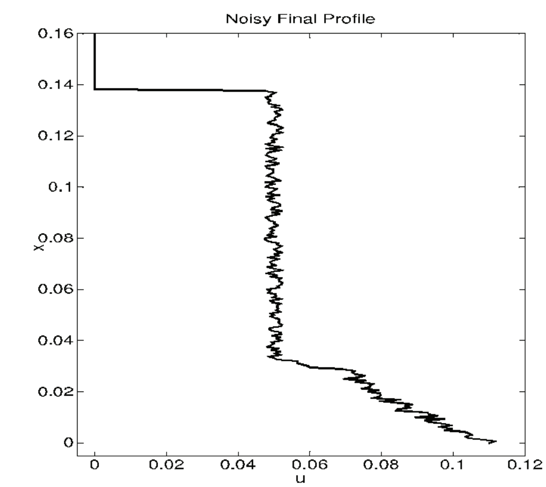


Figure 2. Profiles u^{obs} for Examples 1 and 2 (clean data) and Example 3 (noisy data $\epsilon = 0.05$) respectively

Example 3: Effect of noisy observation data. We randomly generate 100 final profiles and associate them to the previously generated initial guesses. The corrupted profile is generated as follows:

$$u_j^\varepsilon = \left(1 + \varepsilon\varphi_j\right)u^{\mathrm{obs}}\left(x_j\right),$$
$$j = 0, \ldots, \mathcal{N} - 1,$$

where $\varepsilon = 0.01, 0.03$ and 0.05, and φ_j is a uniformly distributed random variable assuming values between −1 and 1. The results are in Table 4.

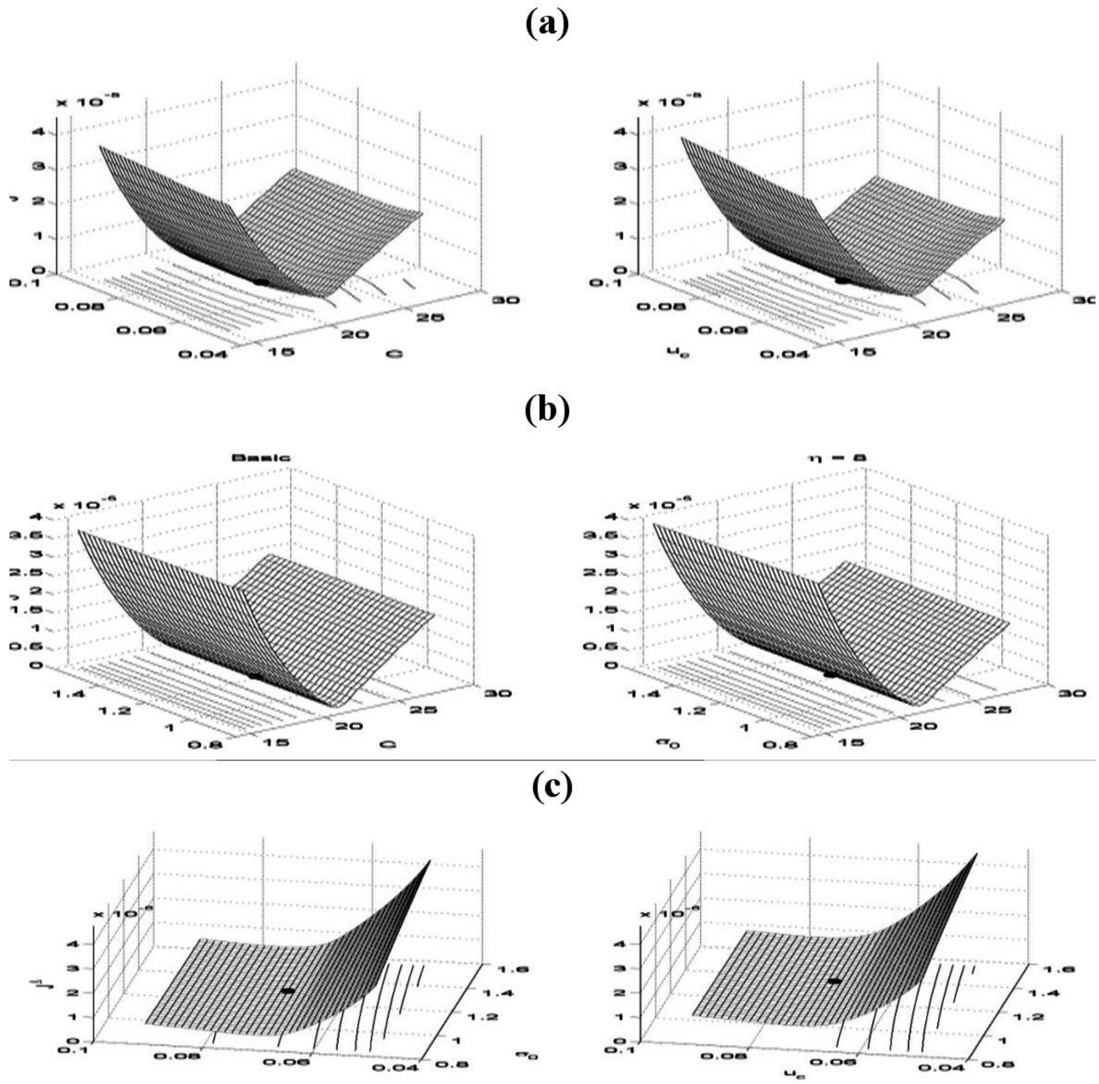


Figure 3. Two parameter cost functionals for parameter sets (a) (C, u_c), (b) (C, σ_0) and (c) (u_c, σ_0).

Table 1. Example 1: initial guesses used for identification experiments.

initial guess	parameter values	initial guess	parameter values
A	$(0.7u_c, 0.7\sigma_0, 0.7C)$	E	$(1.3u_c, 0.7\sigma_0, 0.7C)$
B	$(0.7u_c, 0.7\sigma_0, 1.3C)$	F	$(1.3u_c, 0.7\sigma_0, 1.3C)$
C	$(0.7u_c, 1.3\sigma_0, 0.7C)$	G	$(1.3u_c, 1.3\sigma_0, 0.7C)$
D	$(0.7u_c, 1.3\sigma_0, 1.3C)$	H	$(1.3u_c, 1.3\sigma_0, 1.3C)$

Table 2. Example 1: Results for the basic scheme (5) and the mollified scheme (7) with η = 3, 5 and 8.

	IG	**j**	p_j	E_j	e_∞	**CPU [s]**
Basic scheme (5)	A	2	(0.0697, 1.1219, 21.4706)	290	0.0651	79.863
	B	4	(0.0696, 1.1111, 21.4700)	517	0.0741	90.310
	C	5	(0.0697, 1.1324, 21.4700)	616	0.0564	163.74
	D	3	(0.0696, 1.1252, 21.4699)	669	0.0623	127.96

Continuation Table 2.

	IG	j	p_j	E_j	e_∞	CPU [s]
	E	3	(0.0696, 1.1179, 21.4700)	367	0.0684	59.662
	F	6	(0.0696, 1.1117, 21.4700)	674	0.0736	108.77
	G	3	(0.0696, 1.1114, 21.4700)	421	0.0738	71.391
	H	2	(0.0696, 1.1180, 21.4700)	391	0.0684	64.747
Mollified scheme (7)	A	2	(0.0695, 1.1104, 21.5067)	356	0.0746	89.43
with η = 3	B	4	(0.0695, 1.1105, 21.5067)	490	0.0746	89.77
	C	2	(0.0695, 1.1104, 21.5067)	296	0.0747	103.1
	D	5	(0.0695, 1.1103, 21.5067)	797	0.0747	155.8
	E	3	(0.0696, 1.1259, 21.5075)	332	0.0617	58.19
	F	3	(0.0696, 1.1294, 21.5066)	358	0.0588	61.25
	G	3	(0.0695, 1.1104, 21.5068)	402	0.0747	73.08
	H	5	(0.0695, 1.1104, 21.5065)	676	0.0747	119.7
Mollified scheme (7)	A	5	(0.0500, 0.1486, 21.5456)	685	0.8762	152.8
with η = 5	B	5	(0.0697, 1.1651, 21.5465)	664	0.0291	105.8
	C	3	(0.0695, 1.1124, 21.5466)	486	0.0730	103.6
	D	3	(0.0696, 1.1301, 21.5465)	639	0.0582	107.1
	E	3	(0.0696, 1.1301, 21.5466)	445	0.0583	69.26
	F	6	(0.0695, 1.0945, 21.5466)	697	0.0879	107.7
	G	3	(0.0696, 1.1299, 21.5466)	485	0.0584	78.23
	H	4	(0.0696, 1.1472, 21.5465)	807	0.0440	128.4
Mollified scheme (7)	A	2	(0.0696, 1.1174, 21.5777)	293	0.0688	49.008
with η = 8	B	3	(0.0697, 1.1530, 21.5776)	461	0.0391	64.792
	C	4	(0.0697, 1.1531, 21.5776)	649	0.0391	107.35
	D	2	(0.0697, 1.1530, 21.5777)	418	0.0391	60.759
	E	3	(0.0697, 1.1531, 21.5776)	535	0.0391	74.026
	F	3	(0.0696, 1.1174, 21.5777)	543	0.0688	73.858
	G	4	(0.0697, 1.1531, 21.5776)	576	0.0391	80.604
	H	2	(0.0697, 1.1531, 21.5776)	640	0.0391	89.225

Table 3. Example 2: Results for the basic scheme (5) and the mollified scheme (7) for different values of Δx and η.

$\Delta x/L$	Scheme	# restarts	E_j	$e_\infty \pm \sigma$	CPU[s]
1/128	(5)	387	53693	0.1400 ±0.0329	49.17
	(7), $\eta = 3$	340	49274	0.1295 ±0.0338	53.92
	(7), $\eta = 5$	333	50659	0.0909 ±0.0315	50.91
	(7), $\eta = 8$	276	41263	0.0534 ±3.72e-05	39.36
1/256	(5)	388	48842	0.0696 ±0.0096	136.03
	(7), $\eta = 3$	398	53513	0.0739 ±0.0081	160.74
	(7), $\eta = 5$	355	48280	0.0587 ±0.0174	125.46
	(7), $\eta = 8$	326	48392	0.0431 ±0.0190	114.35
1/512	(5)	334	38879	0.0312 ±0.0038	440.91
	(7), $\eta = 3$	352	40416	0.0332 ±0.0049	439.55
	(7), $\eta = 5$	346	41128	0.0278 ±0.0057	352.47
	(7), $\eta = 8$	374	47250	0.0195 ±0.0072	335.24

Table 4. Example 3, $\Delta x = L/256$: results for the basic scheme (5) and the mollified scheme (7) for different values of ε and η.

ε	Scheme	# restarts	EJ	$e_{\infty} \pm\sigma$	CPU[s]
0.01	(5)	385	50310	0.0661 ± 0.0264	140.63
	(7), $\eta = 3$	367	50489	0.0674 ± 0.0238	152.60
	(7), $\eta = 5$	368	52214	0.0533 ± 0.0256	135.85
	(7), $\eta = 8$	331	47153	0.0401 ± 0.0260	110.56
0.03	(5)	376	49025	0.0801 ± 0.0554	137.65
	(7), $\eta = 3$	364	48447	0.0807 ± 0.0533	146.72
	(7), $\eta = 5$	374	50622	0.0760 ± 0.0528	132.10
	(7), $\eta = 8$	353	49136	0.0710 ± 0.0524	115.31
0.05	(5)	378	49716	0.1163 ± 0.0913	140.31
	(7), $\eta = 3$	376	51246	0.1149 ± 0.0897	156.50
	(7), $\eta = 5$	359	50344	0.1129 ± 0.0902	131.10
	(7), $\eta = 8$	332	49139	0.1084 ± 0.0915	115.54

3.3. Conclusions

According to Table 2, most of the identifications are successful and $\eta = 8$ seems to be the best choice. For the initial guess *A* the method for $\eta = 5$ does not converge, but it does converge when started with initial guesses close to A. The results in Table 3, corresponding to Example 2, illustrate how by improving the spatial resolution (i.e., reducing Δx) the quality of the identification is increased.

Table 4 indicates that the level of noise influences the quality of the recovery but stability is never lost.

Summarizing, this parameter identification procedure yields good results for both the basic scheme and its mollified versions but the mollified approach returned advantages not only in CPU time (in s), but also in the error level, the sensitivity to the initial guess and the effect of noise in the data. This well-posed behavior was already suggested by the convex-shape of the cost functional (Figure 3.)

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