



A new synthesis procedure for TOPSIS based on AHP

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Abstract

Vega et al. [1] analyzed the influence of the attributes' dependence when ranking a set of alternatives in a multicriteria decision making problem with TOPSIS. They also proposed the use of the Mahalanobis distance to incorporate the correlations among the attributes in TOPSIS. Even in those situations for which dependence among attributes is very slight, the results obtained for the Mahalanobis distance are significantly different from those obtained with the Euclidean distance, traditionally used in TOPSIS, and also from results obtained using any other distance of the Minkowsky family. This raises serious doubts regarding the selection of the distance that should be employed in each case. To deal with the problem of the attributes' dependence and the question of the selection of the most appropriate distance measure, this paper proposes to use a new method for synthesizing the distances to the ideal and the anti-ideal in TOPSIS. The new procedure is based on the Analytic Hierarchy Process and is able to capture the relative importance of both distances in the context given by the measure that is considered; it also provides rankings, which are closer to the distances employed in TOPSIS, regardless of the dependence among the attributes. The new technique has been applied to the illustrative example employed in Vega et al. [1].

Keywords: Multicriteria Decision Making, TOPSIS, AHP, Dependence, Synthesis.

Un nuevo procedimiento de síntesis para TOPSIS basado en AHP

Resumen

Vega et al. [1] analizan la influencia que tiene la dependencia entre atributos al ordenar con TOPSIS un conjunto de alternativas en un problema de decisión multicriterio. Asimismo, estos autores proponen utilizar la distancia de Mahalanobis en TOPSIS para incorporar las correlaciones entre los atributos. El problema es que, incluso en situaciones en las que la dependencia entre atributos es muy pequeña, los resultados obtenidos utilizando la distancia de Mahalanobis difieren significativamente de los obtenidos con la distancia euclídea tradicionalmente empleada en TOPSIS, así como de los obtenidos con cualquier otra distancia de la familia de Minkowsky. Este hecho provoca serias dudas a la hora de seleccionar la distancia que debe utilizarse en cada caso. Para abordar el problema de la dependencia entre atributos y el asociado con la selección de la distancia más apropiada, este trabajo propone utilizar una nueva forma de sintetizar las distancias al ideal y anti-ideal en TOPSIS. Este nuevo procedimiento, basado en el Proceso Analítico Jerárquico (AHP), permite capturar la importancia relativa de ambas distancias en el contexto delimitado por la medida considerada y proporciona ordenaciones más cercanas que las de la síntesis tradicional para las diferentes distancias empleadas con TOPSIS, independientemente de la existencia de dependencia entre atributos. El procedimiento propuesto ha sido aplicado al ejemplo seleccionado por Vega et al. [1].

Palabras clave: Decisión Multicriterio, TOPSIS, AHP, Dependencia, Síntesis.

1. Introduction

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) is a multicriteria decision making technique used for ranking and selecting the best alternative from a

discrete group (A_i , $i=1, \dots, m$) [2,3]. This technique is based on the minimization of geometric distances from the alternatives to the ideal (A^+) and anti-ideal (A^-) solutions. In order to calculate the *relative proximity* (R_i) from A_i to A^+ and A^- , traditional TOPSIS [2] uses the Euclidean distance, which

implicitly assumes that the attributes contemplated for ordering alternatives are independent, and an indicator, or proximity index (R_i), which synthesizes (see Section 2.1.3) the information of both distances as a ratio ($R_i = d_i^+ / (d_i^+ + d_i^-)$).

Unfortunately, independence of attributes rarely occurs in the real-life cases to which the technique is applied [4]. Moreover, its study is especially complex and difficult. After analyzing the relevance of this topic, Vega et al. [1] adapted TOPSIS to the consideration of dependent attributes by means of the reformulation of Hwang and Yoon's initial proposal [3].

The modification comprised a new measurement of ideal and anti-ideal distances, based on the Mahalanobis distance [5, 6], which captures the correlation between the attributes [7] and eliminates the common problem of data normalization. This reformulation provides very different results to those obtained with the Euclidean distance even if the dependence among the attributes is very slight [1].

To deal with this conflicting point, this paper proposes a new method, based on the Analytic Hierarchy Process (AHP), for synthesizing the contribution of the two distances (d_i^+ and d_i^-) to the final ordering. The new synthesizing procedure allows the consideration of both aspects of different relative importance and without the difficulties associated with a quotient. The new *relative importance index* (W_i) for each alternative (see Section 3 for details), obtained as $W_i = w^+ w(d_i^+) + w^- w(d_i^-)$, reduces the divergence between the rankings that result from the Euclidean and Mahalanobis distances.

The structure of the remainder of this paper is as follows: Section 2 briefly describes the multicriteria decision making techniques that use the minimization of distances as methodological support; Section 3 includes the proposal of Vega et al. [1] for dealing with the dependence among the attributes; Section 4 presents the new synthesis process, based on the Analytic Hierarchy Process, and proposed for TOPSIS, the section further includes an illustrative example; finally, Section 5 briefly details the most important conclusions of the work.

2. Background

2.1. Multicriteria decision making techniques based on minimization distances

Multicriteria Decision Making can be understood as a series of models, methods and techniques that allow a more effective and realistic solution to complex problems that contemplate multiple scenarios, actors and (tangible and intangible) criteria [8,9]. A variety of multicriteria decision approaches are mentioned in the scientific literature [10].

Despite the diversity of the techniques and the many arguments and discussions that have taken place regarding the different schools and approaches, there is no general agreement that a particular technique is superior to the others [11]. Moreover, in the last decade, debates between the different schools have been replaced by attempts to take advantage of the best elements of each approach in order to

develop the most effective technique.

With respect to the multicriteria techniques based on distance minimization, the original and most commonly used is *Compromise Programming* [10,12-13]. This technique, with a priori information about the decision maker's preferences (norms and weights), simultaneously works with all the criteria and seeks a solution that minimizes the distance to the ideal point.

Let (1) be a multi-objective optimization problem where, without losing generality, it is supposed that all the q contemplated criteria are maximized:

$$\text{Max}_{x \in X} z(x) = (z_1(x), \dots, z_q(x)) \quad (1)$$

The compromise solution is obtained by resolving the optimization problem that minimizes the distance to the ideal point or vector ($z^* = (z_1^*(x), \dots, z_q^*(x))$), where that distance is usually given by a Minkowski distance expression:

$$\text{Min}_{x \in X} d(z(x), z^*, p) = \text{Min}_{x \in X} (\sum_{j=1}^q w_j^p |z_j^* - z_j(x)|^p)^{1/p} \quad (2)$$

Given that $X = \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, \dots, m\}$ and p is the norm considered for distance ($p = 1, 2, \dots, \infty$), $w_j > 0$ is the weight of j -th criterion and z^* is the ideal vector where each component z_j^* of the vector is the individualised optimum of the j -th criteria ($j = 1, \dots, q$), we have:

$$z_j^*(x) = \text{Max}_{x \in X} z_j(x) \quad (3)$$

When $p \rightarrow \infty$, the expression of the Minkowski distance is known as the Tchebycheff distance; in this case (2) it is:

$$\text{Min}_{x \in X} d(z(x), z^*, p = \infty) = \text{Min}_{x \in X} \text{Max}_j \{w_j |z_j^* - z_j(x)|\}, j = 1, \dots, q \quad (4)$$

For reasons of operational functionality, the most commonly used Minkowski norms are: $p = 1$ (Manhattan distance), $p = 2$ (Euclidean distance) and $p = \infty$ (Tchebycheff distance). In the first case, the optimization problem is linear, in the second it is quadratic and in the third case, the model can be easily transformed into a linear one.

Other well-known multicriteria techniques based on the minimization of distances that have been widely used in discrete multicriteria decision-making are: Goal Programming [14], VIKOR [15] and TOPSIS.

2.1.1. Goal programming (GP)

GP is a multicriteria technique that uses the distance minimization concept, but is more focused on the concept of satisfaction, moving away from the traditional concept of optimization. GP integrates a set of constraints that represent some resource limitations or capacities, and can be represented according to (5):

$$\text{Min}_{x \in X} \sum_{j=1}^r w_j (d_j^+ + d_j^-) \quad (5)$$

where:

$$\begin{aligned} z_j(x) + d_j^- - d_j^+ &= \hat{z}_j, \quad j = 1, \dots, r \\ g_i(x) &\leq 0, \quad j = 1, \dots, m \\ x &\geq 0, \quad d_j^- \geq 0, \quad d_j^+ \geq 0 \\ X &= \{x \in \mathbb{R}^n \mid g_i(x) \leq 0\}, \quad i = 1, \dots, m \end{aligned}$$

2.1.2. VIKOR technique

VIKOR (ViseKriterijumska Optimizacija I Kompromisno Resenje in Serbian) was originally proposed by Serafim Opricovic in his Ph. D. dissertation (1979) aimed at resolving complex decision problems with conflicting and non-commensurable criteria [15]. For the ranking of the alternatives and the selection of the best one, it uses an index that measures the proximity to the ideal solution. The distance employed for measuring the proximity belongs to the L_p -Minkowsky metric that is traditionally used in Compromise programing [12-13].

$$L_{p,i} = \left\{ \sum_{j=1}^q \left[w_j \frac{z_j^* - z_j(x^i)}{z_j^* - z_{oj}^*} \right]^p \right\}^{1/p}, \quad i = 1, \dots, m \quad (6)$$

When seeking the integration of all the attributes in the ranking process, it is necessary to express them in dimensionless scales. The normalization mode used by VIKOR is *utility normalization*:

$$n_{ij} = \frac{x_{ij} - \min x_i}{\max x_i - \min x_i}, \quad i = 1, \dots, m, \quad j = 1, \dots, n \quad (7)$$

With dimensionless measures, the next step is to estimate the measurement of *satisfaction* (S_i) and *regret* (R_i):

$$S_i = \sum_{j=1}^n \left[w_j \frac{z_j^* - z_j(x^i)}{z_j^* - z_{oj}^*} \right], \quad i = 1, \dots, m \quad (8)$$

$$R_i = \max_j \left[w_j \frac{z_j^* - z_j(x^i)}{z_j^* - z_{oj}^*} \right], \quad i = 1, \dots, m \quad (9)$$

Using these values, VIKOR estimates a proximity index for every alternative:

$$Q_i = (v) \frac{S_i - S^*}{S^- - S^*} + (1 - v) \frac{R_i - R^*}{R^- - R^*}, \quad i = 1, \dots, m \quad (10)$$

where: $S^* = \text{Min}_i S_i$, $S^- = \text{Max}_i S_i$, $R^* = \text{Min}_i R_i$, $R^- = \text{Max}_i R_i$ and v is the weight associated with the normalised difference from the best collective strategy (best solution with $p=1$) and $(1 - v)$ the weight associated with the normalised difference to the best individual rejection strategy (best solution with $p=\infty$).

Table 1.
TOPSIS decision matrix

	w_1	w_2	...	w_j	...	w_n
	c_1	c_2	...	c_j	...	c_n
A_1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}
...
A_i	x_{i1}	x_{i2}	...	x_{ij}	...	x_{in}
...
A_{m1}	x_{m1}	x_{m2}	...	x_{mj}	...	x_{mj}

Source: Authors

These weights take values in a range between 0 and 1. For example, for $v = 0.5$ indicates consensus among decision makers. If $v > 0.5$ then majority have more weight in the decision process. But if $v < 0.5$, the minority have more weight, producing a veto effect. The results for S_i , R_i and Q_i are three lists that allow the alternatives to be ranked in descending order.

2.1.3. Traditional TOPSIS technique (TOPSIS-T)

Given a discrete multicriteria decision problem with m alternatives ($A_i, i = 1, \dots, m$) evaluated with respect to n criteria ($C_j, j = 1, \dots, n$), each element x_{ij} in Table 1 represents the value associated with alternative A_i for the attribute or criterion C_j , of which the weight or importance is w_j . Traditional TOPSIS (TOPSIS-T) contemplates each alternative or object as a point or vector of n -dimensional space (see decision matrix in Table 1).

TOPSIS-T calculates the Euclidean distance between the normalized values (with the distributive mode) of the initial alternatives (A_i) and those of two special alternatives: the ideal (A^+) and the anti-ideal (A^-), on the understanding that the best alternatives are those which are closer to the ideal and further away from the anti-ideal. To apply this technique, the attributes' values should be numerical and have commensurable units.

TOPSIS implicitly assumes that the contemplated attributes are independent [16,17]. Unfortunately, this rarely occurs in the real-life cases where the technique is applied. The procedure is better described in the following steps, as suggested by Hwang and Yoon [3] in their original proposal (traditional TOPSIS):

Step 1. Calculate the normalized decision matrix

As TOPSIS allows the evaluated criteria to be expressed in different measurement units, it is necessary to convert them into normalized values. The normalization process, like the metric used to calculate the ideal and anti-ideal distances, is the Euclidean one.

In this case, the normalization of element x_{ij} of the decision matrix (Euclidean normalisation mode, $\sum_{i,j} n_{ij}^2 = 1$) is calculated as:

$$n_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^m x_{ij}^2}}, \quad i = 1, \dots, m, \quad j = 1, \dots, n \quad (11)$$

Step 2. Calculate the weighted normalized decision matrix

The weighted normalized value v_{ij} is calculated as:

$$v_{ij} = w_j \cdot n_{ij}, i = 1, \dots, m, \quad j = 1, \dots, n$$

with $\sum_{j=1}^n w_j = 1$. The weights w_j can be obtained by means of different procedures [10]: direct assignation, AHP, etc.

Step 3. Determine the “positive ideal” and “negative ideal” alternatives

Without losing generality and supposing that all the criteria are maximized, the ideal positive solution is given by $A^+ = \{v_1^+, \dots, v_n^+\}$, where $v_j^+ = \text{Max}_i(v_{ij}), i = 1, \dots, m, j = 1, \dots, n$, and the ideal negative or anti-ideal solution is given by $A^- = \{v_1^-, \dots, v_n^-\}$ where $v_j^- = \text{Min}_i(v_{ij}), i = 1, \dots, m, j = 1, \dots, n$.

Step 4. Calculate the distances

The separation of each alternative A_i from the ideal solution A^+ is calculated as $d_i^+ = \left(\sum_{j=1}^n |v_j^+ - v_{ij}|^2\right)^{1/2}$, $i = 1, \dots, m$. In a similar way, the separation of each alternative A_i from the anti-ideal solution A^- is calculated as $d_i^- = \left(\sum_{j=1}^n |v_j^- - v_{ij}|^2\right)^{1/2}$, $i = 1, \dots, m$.

Step 5. Calculate the relative proximity to the ideal solution

The relative proximity from A_i to A^+ and A^- is given by:

$$R_i = \frac{d_i^+}{d_i^+ + d_i^-}, i = 1, \dots, m \quad (12)$$

where R_i is named the *proximity index* and low values are better.

Step 6. Preference order

Finally, R_i is used to rank the alternatives; the nearest the value of the proximity index R_i is to 0, the greater its proximity to the ideal, the higher its priority. In short, $(A_i > A_j \Leftrightarrow R_i < R_j)$.

2.2. The Analytic Hierarchy Process (AHP)

The *Analytic Hierarchy Process* (AHP) is a multicriteria decision making methodology created by the mathematician Thomas Saaty in the 1970s. It deals with the multicriteria ranking and selection of a discrete set of alternatives in a context with multiple scenarios, actors and criteria (tangible and intangible).

The AHP methodology [18-19] comprises the following stages:

(i) Modeling the problem: the construction of a hierarchy, the identification of the mission, the relevant criteria to its execution, the sub-criteria for each criterion, the actors and the alternatives.

(ii) Valuation: the incorporation of the actors’ preferences by means of pairwise comparisons between the elements of the hierarchy that hang from the same node; this process uses judgments from Saaty’s fundamental scale (see Table 2) and the result is a square, reciprocal and positive pairwise comparison matrix that reflects the relative importance of

Table 2.
Saaty’s Fundamental Scale [18]

Numerical Scale of Importance	Verbal Definition	Explanation
1	Equal importance	Both elements meet criteria by contributing equally to the objective.
3	Moderate importance: one element a little more important than the other	Judgment and experience favor one of the elements.
5	Strong importance	Judgment and experience strongly favor one element over the other
7	Very strong importance	One element is much more important and its dominance is demonstrated in practice.
9	Extreme importance	The evidence favoring one element over another is absolute

Intermediate numeric values (2, 4, 6 and 8) reflect intermediate categories
Source: Saaty (1980) [18]

two elements with respect to the common element in the higher level of the hierarchy.

(iii) Prioritization and synthesis: the determination of the local and global priorities for the elements of the hierarchy and the total priorities for the alternatives of the problem. Saaty’s Eigen Vector method (EGV) and the Row Geometric Mean method (RGM) are the two most common prioritization procedures [18].

One of the characteristics which differentiates this methodology from other multicriteria approaches is that AHP measures the inconsistency of the actors when eliciting the judgments of the pairwise comparison matrices in a formal, elegant and intrinsic manner, linked to the mathematical prioritization procedure. Saaty’s *Consistency Ratio* (CR) and Aguarón & Moreno-Jiménez’s *Geometric Consistency Index* (GCI) are the two inconsistency measures usually employed with the EGV and the RGM prioritization procedures, respectively [20].

3. Dependent attributes in TOPSIS. TOPSIS-M

It is well known that all multi-attribute techniques may be improved, depending on the theory on which they are based [21,22]. TOPSIS is no exception. One of its main limitations is the correlation between attributes; in other words, this technique assumes that all the attributes are independent and this is not always the case.

There are several ways for measuring the dependence among the attributes [23]. Some of the most widely used are: the scatterplots matrix, the correlation matrix, variance inflation factors, condition numbers [24,25] and the Gleason-Staelin indicator.

The *scatterplots matrix* is a visual tool that plots each attribute against the other in matrix form; this allows the observation of patterns or linear trends that helps to determine the dependence between attributes. The

Correlation matrix, $\Sigma = (\sigma_{ij})$, is a square ($n \times n$) symmetrical matrix in which each entry σ_{ij} is the correlation between attributes i and j ($\sigma_{ij} \in [-1, 1]$). The *Variance Inflation Factor (VIF)* for the attribute j is given by $VIF_j = 1/(1-R_j^2)$, where R_j^2 is the coefficient of determination of attribute j over others in a multiple linear regression. Values of *VIF* greater than 10 indicate strong dependence in the data [26]. The *Condition Numbers* are calculated [27] as $n_j = \lambda_{max}/\lambda_{min}$, where λ_{max} and λ_{min} are the maximum and minimum eigen-values of the $X^T X$ matrix. If any of the condition numbers calculated for the set of attributes is greater than 1000, it indicates that there is no independence between the attributes. The *Gleason-Staelin* redundancy measure (Φ) is given by [28]:

$$\phi = \sqrt{\frac{\sum_{i=1}^n \sum_{j=i}^n r_{ij}^2 - n}{n(n-1)}} \quad (13)$$

where the r_{ij} ($i, j=1, \dots, n$) are the correlations between the attributes. If this indicator is higher than 0.5, it is an indication of redundancy and dependence in the data.

To deal with the problem of dependence among the attributes, Vega et al. [1] proposed an extension of TOPSIS-T, named TOPSIS-M, which uses the Mahalanobis distance instead of the Euclidean distance in Step 4 of the previous algorithm.

TOPSIS-M solves two of the limitations of TOPSIS-T: (i) the redundancy provoked by the dependence when measuring the proximity with the Euclidean distance and (ii) the problem of selecting the appropriate mode for normalizing the data. Using the Mahalanobis distance is not necessary to normalize the initial data. The value of the Mahalanobis distance is the same, apart from the used normalization mode that is used. The value is also the same without normalization.

The Mahalanobis distance [5,6] determines the similarity between two multi-dimensional random variables as well as considering the existent correlation between them ($m > n$ is required to obtain Σ^{-1}). The Mahalanobis distance between two random variables with the same x and y probability distribution and with Σ variance-covariance matrix is formally defined as:

$$d_m(x, y) = ((x - y)^T \Sigma^{-1} (x - y))^{1/2} \quad (14)$$

Where

$$\Sigma = \frac{1}{n-1} (X_c)^T (X_c) \quad (15)$$

and X is the data matrix with m objects in rows and n columns, X_c the centered matrix, $X_c = (X - \bar{x})$, and \bar{x} the arithmetic mean.

This value coincides with the Euclidean distance if the covariance matrix is the identity matrix, i.e. if all bivariate correlations between variables are zero. When there is some dependence among the attributes, even if it is very small (Gleason-Staelin's $\phi < 0.025$), the rankings obtained with the Euclidean distance and those obtained with the Mahalanobis distance can be significantly different [1]. This is also true for the other distances

of the Minkowsky family and it is especially notable with the Manhattan ($p=1$) and the Tchebycheff distances ($p=\infty$) [1].

In order to deal with this contradiction, in the next section, we advance a new synthesis procedure (Step 5 of the TOPSIS algorithm), based on the Analytic Hierarchy Process (AHP), that allows the consideration of the relative importance of the distances from the ideal and anti-ideal alternatives and provides results that are closer for the distances employed than those obtained with the traditional TOPSIS approach, regardless of the normalization model used.

4. A new synthesis procedure for TOPSIS

4.1. New Proposal (TOPSIS-AHP)

The synthesis procedure followed in TOPSIS combines the distances from the ideal and the anti-ideal solutions using the ratio (12) ($R_i = d_i^+ / (d_i^+ + d_i^-)$). When the distance employed to measure proximity is the Euclidean (Step4), data are previously normalised using the Euclidean mode (11). If the distance employed is the Mahalanobis distance (14), then it is not necessary to normalize the original data (the results obtained when normalizing with any norm and not normalizing are the same).

As already mentioned, the results obtained for R_i , and therefore for the associated rankings, using both distances (Euclidean and Mahalanobis) can be clearly different if there is any dependence, even if it is very small. The new, AHP-based, procedure deals with this drawback, as well as the problem of selecting the method for normalizing the data; it further allows the assignation of a different relative importance for both distances. The rest of this section presents the new, AHP-based, procedure.

The Relative Importance Index (W_i) for alternative A_i , $i=1, \dots, m$ is given by:

$$W_i = w^+ W_i^+ + w^- W_i^- \quad (16)$$

where $W_i^+ = w(d_i^+)$ is the priority of d_i^+ derived from the pairwise comparison matrix of the distances (d_i^+) from alternative A_i to the ideal alternative A^+ and w^+ is the relative importance of the priorities of the distances to the ideal alternative. Analogously, $W_i^- = w(d_i^-)$ is the priority of d_i^- derived from the pairwise comparison matrix of the distances (d_i^-) from alternative A_i to the anti-ideal alternative A^- and w^- is the relative importance of the priorities of the distances to the anti-ideal alternative.

In order to derive the priorities of the proximities to the ideal (W_i^+) and to the anti-ideal (W_i^-) solutions for a particular distance (Euclidean, Mahalanobis etc.), the following pairwise comparison matrices should be constructed.

a) In the first case (W_i^+), assuming that $d_{(i)}$ are the distances to the ideal solution ($i \in \{1, \dots, m\}$) in ascending order, the (r, s) -entry of the pairwise comparison matrix, from which the priorities (W_i^+) are derived, includes the judgment from Saaty's fundamental scale [18] that captures the intensity with which $d_{(r)}^+$ is preferred to $d_{(s)}^+$.

b) In the second case (W_i^-), assuming that $d_{(i)}^-$ are the

Table 3.

Data for the Vega et al. (2014)'s example [1]

Alt./Dist.	$d_E(A^+)$	$d_E(A^-)$	$R_E(A_i)$	Ranking $_E$	$d_M(A^+)$	$d_M(A^-)$	$R_M(A_i)$	Ranking $_M$
A1	0.0617	0.0441	0.5832	6	331.53	371.96	0.4712	3
A2	0.0493	0.0607	0.448	2	332.49	371.00	0.4729	4
A3	0.0424	0.0497	0.460	4	330.58	372.88	0.4699	1
A4	0.0489	0.0574	0.460	3	333.63	369.83	0.4742	6
A5	0.0655	0.0492	0.570	5	332.50	370.95	0.4726	5
A6	0.0462	0.0609	0.431	1	331.23	372.24	0.4708	2
Max	0.0655	0.0609	0.583		333.63	372.88	0.4742	
Min	0.0424	0.0441	0.4318		330.58	369.83	0.4699	
Range (R)	0.0230	0.0168	0.1514		3.0544	3.05	0.0043	
d^+/d^0	1.5437	1.3809	1.3506		1.009	1.008	1.009	

Source: Vega et al. (2014) [1]

distances to the anti-ideal solution ($i \in \{1, \dots, m\}$) in descending order, the (r,s)-entry of the pairwise comparison matrix from which the priorities (W_i^-) are derived includes the judgment from the Saaty's fundamental scale [18] that captures the intensity with which $d_{(r)}^-$ is preferred to $d_{(s)}^-$.

By means of any of the existing prioritization procedures (the Row Geometric Mean in this case), we derive for each alternative the priorities of its distances to the ideal and anti-ideal solutions (W_i^+ and W_i^- , respectively). Using these values and the relative importance or weight associated to the distances to the ideal and the anti-ideal solutions (w^+ and w^-), the priority for each alternative is obtained (16) and used to rank them.

4.2. Case Study

This procedure has been applied to the example (Profiles of Graduate Fellowship Applicants) used in Vega et al. [1]. The data corresponding to the distances to the ideal and the anti-ideal as well as the relative proximity (12) and the resulting rankings for the two distances (Euclidean and Mahalanobis) can be seen in Table 3.

It can be observed that the rankings obtained using TOPSIS-T (distance and Euclidean normalization) and TOPSIS-M (Mahalanobis distance and non-normalized data) are clearly different. This is also true for very small dependencies [1]. In our example, the value of the Gleason-Staelin measure of redundancy (Phi) is $\phi = 0.6736$, which is higher than the 0.5 threshold necessary for the existence of dependence.

In order to deal with this conflict and after ranking the distances to the ideal and the anti-ideal from the minimum to the maximum, we ask the decision maker to evaluate the relative importance of the distances (Euclidean and Mahalanobis) to the ideal and the anti-ideal solutions. The pairwise comparisons matrices provided by the decision maker are given in Tables 4a and 4b for the Euclidean distance and in Tables 5a and 5b for the Mahalanobis distance.

Using the Row Geometric Mean method as the prioritization procedure, the local priorities for the two distances (Euclidean and Mahalanobis) are obtained in five different scenarios, which depend on the weights assigned to the priorities of the distances to the ideal and the anti-ideal (see Tables 6 and 7).

As can be noted in Tables 6 and 7, the *Relative Importance Index* (W_i) and rankings obtained for the two

Table 4a.

Pairwise comparison matrix for the Euclidean distances to the ideal

Judgments		$d_E(A^+)$	0.042	0.046	0.049	0.049	0.062	0.066
order	$d_E(A^+)$	Alter.	A3	A6	A4	A2	A1	A5
1	0.042	A3	1	3	4	4	5	6
2	0.046	A6	0.333	1	2	2	5	5
3	0.049	A4	0.250	0.500	1	2	4	5
4	0.049	A2	0.250	0.500	0.500	1	4	5
5	0.062	A1	0.200	0.200	0.250	0.250	1	2
6	0.066	A5	0.167	0.200	0.200	0.200	0.500	1

Source: Authors

Table 4b.

Pairwise comparison matrix for the Euclidean distances to the anti-ideal

Judgment		$d_E(A^-)$	0.06	0.06	0.05	0.05	0.04	0.04
order	$d_E(A^-)$	Alter.	A6	A2	A4	A3	A5	A1
1	0.061	A6	1	1	2	4	4	5
2	0.061	A2	0	1	2	4	4	5
3	0.057	A4	0	0	1	4	4	5
4	0.050	A3	0	0	0	1	1	3
5	0.049	A5	0	0	0	0	1	3
6	0.044	A1	0	0	0	0.33	0.33	1

Source: Authors

Table 5a.

Pairwise comparison matrix for the Mahalanobis distances to the ideal

Judgments		$d_M(A^+)$	330.5	331.2	331.5	332.4	332.5	333.6
order	$d_M(A^+)$	Alter.	A3	A6	A1	A2	A5	A4
1	330.58	A3	1	1	2	3	3	5
2	331.23	A6	0.500	1	1	2	2	4
3	331.53	A1	0.333	1.000	1	2	2	3
4	332.49	A2	0.250	0.333	0.333	1	1	2
5	332.51	A5	0.250	0.250	0.333	1.000	1	2
6	333.63	A4	0.167	0.200	0.200	0.250	0.333	1

Source: Authors

distances (Euclidean and Mahalanobis) with the new synthesis procedure (TOPSIS-AHP-E and TOPSIS-AHP-M) differ in the five considered scenarios, which depend on the weights given to the distances from the ideal and the anti-ideal. But both the cardinal (r) and ordinal (Spearman's ρ) correlations between them (TOPSIS-AHP-E and TOPSIS-AHP-M) are greater than those obtained for the TOPSIS-T and TOPSIS-M (see Table 8), except for the linear correlation in the $(\alpha_1, \alpha_2) = (0.25; 0.75)$ situation.

It can also be verified that the cardinal and ordinal correlations between the values obtained with TOPSIS-T and with TOPSIS-AHP-E are greater than 90% in situation $(\alpha_1, \alpha_2) = (0.25; 0.75)$. On the other hand, the greatest values for the correlations between TOPSIS-M and TOPSIS-AHP-M are reached for the situation $(\alpha_1, \alpha_2) = (0; 1)$, with values closer to a 100%.

With respect to the judgment of Saaty's fundamental scale assigned by the decision maker to each comparison of distances, it should be mentioned that these judgments capture the holistic vision of the reality and are given in accordance with the decision maker's experience and culture.

Table 5b.

Pairwise comparison matrix for the Mahalanobis distances to the anti-ideal

Judgments		$d_M(A)$	372.884	372.243	371.966	371.003	370.955	369.834
	$d_M(A)$	Alter.	A3	A6	A1	A2	A5	A4
1	372.884	A3	1	1	2	3	3	5
2	372.243	A6	1.000	1	1	2	2	4
3	371.966	A1	0.500	1.000	1	2	2	3
4	371.003	A2	0.333	0.500	0.500	1	1	2
5	370.955	A5	0.333	0.500	0.500	1.000	1	2
6	369.834	A4	0.200	0.250	0.333	0.500	0.500	1

Source: Authors

Table 6.

Priorities and positions for the Euclidean distance in five different scenarios

Euclidean	(α_1, α_2)		(1;0)		(0.75;0.25)		(0.5;0.5)		(0.25;0.75)		(0;1)	
	W_i^+	W_i^-	Priority	Position	Priority	Position	Priority	Position	Priority	Position	Priority	Position
A1	0.050	0.040	0.050	5	0.048	5	0.045	6	0.042	6	0.040	6
A2	0.126	0.298	0.126	4	0.169	4	0.212	3	0.255	2	0.298	1
A3	0.409	0.077	0.409	1	0.326	1	0.243	2	0.160	4	0.077	4
A4	0.159	0.211	0.159	3	0.172	3	0.185	4	0.198	3	0.211	3
A5	0.036	0.077	0.036	6	0.046	6	0.056	5	0.067	5	0.077	4
A6	0.219	0.298	0.219	2	0.238	2	0.258	1	0.278	1	0.298	1

Source: Authors

Table 7.

Priorities and positions for the Mahalanobis distance in five different scenarios

Mahalanobis	(α_1, α_2)		(1;0)		(0.75;0.25)		(0.5;0.5)		(0.25;0.75)		(0;1)	
	W_i^+	W_i^-	Priority	Position	Priority	Position	Priority	Position	Priority	Position	Priority	Position
A1	0.200	0.194	0.200	3	0.199	3	0.197	3	0.196	3	0.194	3
A2	0.098	0.107	0.098	4	0.100	4	0.103	4	0.105	4	0.107	4
A3	0.337	0.305	0.337	1	0.329	1	0.321	1	0.313	1	0.305	1
A4	0.046	0.058	0.046	6	0.049	6	0.052	6	0.055	6	0.058	6
A5	0.094	0.107	0.094	5	0.097	5	0.100	5	0.104	5	0.107	5
A6	0.225	0.229	0.225	2	0.226	2	0.227	2	0.228	2	0.229	2

Source: Authors

Table 8.

Cardinal (r) and ordinal (p) correlations

Correlations	Cardinal	Ordinal
AHP (E vs M)	(r)	(p)
(1;0)	0.725	0.533
(0.75;0.25)	0.598	0.533
(0.5;0.5)	0.339	0.467
(0.25;0.75)	-0.010	-0.067
(0;1)	-0.278	-0.267
TOPSIS T vs M	0.047	-0.067

Source: Authors

Table 9.

A suggestion for the judgments assigned to the ratio of distances

a_{rs}	With d_r/d_s between			
	Euclidean	Mahalanobis		
1	1,000	1,025	1,000	1,002
2	1,025	1,075	1,002	1,004
3	1,075	1,150	1,004	1,006
4	1,150	1,300	1,006	1,008
5	1,300	1,500	1,008	1,010
6	1,500	1,750		
7	1,750	2,000		
8	2,000	4,000		
9	4,000			

Source: Authors

It is not easy to assign these judgments in a systematic way because of the diversity of the distances for both metrics.

For the values of our example, we can suggest the following practical procedure, which depends on the values of the Range of the distances ($R = d^* - d^0$) and the ratio d^*/d^0 , where $d^* = \text{Max}_i d_i$ and $d^0 = \text{Min}_i d_i$ with $i=1, \dots, m$.

Let d_r and d_s be the two compared distances ($d_r \geq d_s$) and assuming that the criterion for the comparisons is the higher the better, the judgment (a_{rs}) associated to the comparison between d_r and d_s for the Euclidean distance ($d^*/d^0 = 1.544$ to the ideal and 1.381 to the anti-ideal) and the Mahalanobis distance are given in the scales included in Table 9.

Obviously, as it has already been mentioned, this suggestion depends on the considered distances ($d^* - d^0$ and d^*/d^0) and a more detailed study would be necessary in order to establish a systematic rule.

5. Conclusions

One of the limitations of TOPSIS in its initial proposal (Euclidean distance and normalization), known as traditional TOPSIS (TOPSIS-T), is the problem of dependence among the attributes. In order to solve this problem and capture the dependence among the attributes, [1] proposed the use of the

Mahalanobis distance (no need to normalize the data) instead of the Euclidean. This extension of TOPSIS-T, known as TOPSIS-M, provides rankings for the alternatives being compared which can be significantly different, even for small degrees of dependence.

To deal with this problem, this paper proposes a new synthesis procedure for the distances of the alternatives from the ideal and the anti-ideal ones. The new *Relative Importance Index* integrates the relative importance of the distances to these two points and provides results for the Euclidean and the Mahalanobis distances that are closer than those obtained with TOPSIS-T and TOPSIS-M.

The new proposal aims to be a stepping-stone in the process of obtaining a synthesis procedure for the distances to the ideal and the anti-ideal that allows us to reduce the gap between the results obtained with dependent and independent attributes. The results, that appear to be justified by the relative importance captured by the AHP, should be tested with some other examples.

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