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Weibull accelerated life testing analysis with several variables using multiple linear regression

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Abstract
In Weibull accelerated life test analysis (ALT) with two or more variables $(X_2, X_3, \dots, X_k)$, we estimated, in joint form, the parameters of the life stress model $r\{X(t)\}$ and one shape parameter β. These were then used to extrapolate the conclusions to the operational level. However, these conclusions are biased because in the experiment design (DOE) used, each combination of the variables presents its own Weibull family (β_i, η_i). Thus the estimated β is not representative. On the other hand, since β is determined by the variance of the logarithm of the lifetime data σ_t^2, the response variance σ_y^2 and the correlation coefficient R^2, which increases when variables are added to the analysis, β is always overestimated. In this paper, the problem is statistically addressed and based on the Weibull families (β_i, η_i) a vector Y_η is estimated and used to determine the parameters of $r\{X(t)\}$. Finally, based on the variance σ_{ti}^2 of each level, the variance of the operational level σ_{op}^2 is estimated and used to determine the operational shape parameter β_{op}. The efficiency of the proposed method is shown by numerical applications and by comparing its results with those of the maximum likelihood method (ML).

Keywords: ALT analysis; Weibull analysis; multiple linear regression; experiment design.

Análisis de pruebas de vida acelerada Weibull con varias variables utilizando regresión lineal múltiple

Resumen
En el análisis de pruebas de vida acelerada Weibull con dos o más variables aceleradas $(X_2, X_3, \dots, X_k)$, estimamos en forma conjunta los parámetros del modelo de relación vida esfuerzo $r\{X(t)\}$ y un parámetro de forma β. Después estos parámetros son utilizados para extrapolar las conclusiones al nivel operacional. Como sea, estas conclusiones están sesgadas debido a que dentro del diseño de experimentos (DOE) utilizado, cada combinación de las variables presenta su propia familia Weibull (β_i, η_i). De esa forma la β estimada no es representativa. Por otro lado, dado que β está determinada por la varianza del logaritmo de los tiempos de vida σ_t^2, por la varianza de la respuesta σ_y^2 y por el coeficiente de correlación R^2, el cual crece cuando se agregan variables al análisis, β es siempre sobre estimada. En éste artículo, el problema es estadísticamente identificado y basado sobre las familias Weibull (β_i, η_i) un vector Y_η es estimado y utilizado para determinar los parámetros de $r\{X(t)\}$. Finalmente, basado en la varianza σ_{ti}^2 de cada nivel, la varianza del nivel operacional σ_{op}^2 es estimada y utilizada para determinar el parámetro de forma β_{op} del nivel operacional. La eficiencia del método propuesto es mostrada a través de aplicaciones numéricas y por la comparación de sus resultados con los del método de máxima verosimilitud (ML).

Palabras clave: ALT análisis; análisis Weibull; regresión lineal múltiple; diseño de experimentos.

1. Introduction

In Accelerated Life Testing analysis (ALT) with constant over time and interval-valued variables $(X_2, X_3, \dots, X_k)$, the standard approach of the analysis consists in using higher levels of the stress variables and a life-stress model $r\{X(t)\}$, the lifetime data are obtained as quickly as possible [6]. In this approach, the function $r\{X(t)\}$, which relates the lifetime

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data to the stress variables, is parametrized as

$$r\{X(t)\} = e^{\alpha^t Z} = e^{-b0-\alpha\varphi(x)} \quad (1)$$

Where $\alpha = (\alpha_0, \dots, \alpha_k)$ is a vector of unknown parameters, and $Z = (Z_{0,\dots,}Z_k) = \{\varphi_0(X), \dots, \varphi_k(X)\}$ is a vector of specified functions φ_i with $\varphi_0(t) \equiv 1$. Among the most common models of $r\{X(t)\}$ we have the generalized Eyring model, the temperature-humidity model (T-H), the temperature–non–thermal model, the proportional hazard model and the generalized log–linear model [9] and [13]. On the other hand, in ALT Weibull analysis no matter which model we use, all of them are used to estimate the scale parameter η under different levels of the significant variables. For example if the (T-H) model is used, then in the Weibull probability density function (pdf) ([19] and [16] Chapter 1), given by

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] \quad (2)$$

by replacing η with the $(T-H)$ model the Weibull/(T-H) pdf is given by

$$f(t|T,H) = \frac{\beta}{A}e^{-\left(\frac{\phi}{T}+\frac{b}{H}\right)}\left(\frac{t}{A}e^{-\left(\frac{\phi}{T}+\frac{b}{H}\right)}\right)^{\beta-1} e^{-\left(\frac{t}{A}e^{-\left(\frac{\phi}{T}+\frac{b}{H}\right)}\right)^{\beta}} \quad (3)$$

Unfortunately, since in (3) only one shape parameter (β) is estimated and used to represent all the level combinations of the variables, and because the maximum likelihood (ML) or multiple linear regression (MLR) methods perform the estimation as linear combination of β with the coefficients of the vector α, β is always overestimated. As a consequence, the related reliability $R(t)$ is overestimated too. In order to show this problem, in section 2 the generalities of ALT analysis are given. Section 3 presents the problem statement. In section 4 the problem is statistically addressed. Section five details the proposed method and, finally, in section 6 the conclusions are presented.

2. Generalities of Weibull ALT analysis

In ALT analysis, the objective is to obtain life data as quickly as possible. Data are obtained by observing a set of n units functioning under various levels of the explanatory variables $(X_1, X_2, \dots, X_k)$. These levels are chosen to be higher than the normal one. With these life data, we draw conclusions and then, the conclusions are extrapolated to the normal level. Models used to perform the extrapolation are known as life-stress models $r\{X_{Li}(t)\}$. Among the most common $r\{X_{Li}(t)\}$ models we have the parametric accelerated failure time models (AFT) (e.g. Arrhenius model) [8], and the proportional hazard model (PH) (e.g. Weibull proportional hazard model) [5] and [1]. On the other hand, in the analysis, the lifetime data *T* is a nonnegative and absolute continuous random variable. Thus, the survival function is $R(t) = P\{T \geq t\},\ t \geq 0$. And based on this, the corresponding probability density function is $pdf(t) = -dR(t)/dt$ and the hazard rate function is $h(t) = -dR(t)/R(t)$. Additionally, it is important to note that in ALT, the effect that the covariates $(X_1, X_2, \dots, X_k)$ have over *T*, is modeled by $r\{X_{Li}(t)\}$, which as in (3) is included in $h(t)$. On the other hand, for constant over time and interval valued variables, the cumulative risk function is $H(t) = \int_0^t h_{(x)}(u)du = -\ln\{R_{(x)}(t)\}$ with $h(t)$ parametrized as $h(t) = h(0)r\{X_{Li}(t)\} = h(0)e^{\alpha^t Z}$ where α and Z are as they were defined in (1), and $h(0)$ represents the base risk when all the covariates are zero ($\mathrm{X}_1 = \mathrm{X}_2 =, \dots, = \mathrm{X}_\mathrm{k} = 0$). For example, based on this formulation and on the physical principle of Sedyakin [1], pp. 20), the survival function for two different levels of the variables (X_{L1}, X_{L2}) is related by $\{T_1 \geq t_1\} = R_{xL1}(t_1) = R_{xL2}(t_2) = P\{T_2 \geq t_2\}$. Implying that $h_{X_{Li}}(t_1) = h(0)r\left\{X_{Li}(t_1) = h_{X_{L2}}(t_2) = h(0)r\{X_{L2}(t_2)\}\right\}$ which in terms of $R(t)$ mean that $R_X(t) = G\{r(X_{Li})t\}$, and since the survival function G does not depend on X, then the random variable $R = r(X_{Li})T_X$ does not depend on X either. In particular observe that, since the expected value of T_X is $E(T_X) = m/r(X)$ and its variance is $Var(T_X) = \sigma^2/r^2(X)$, then its variation coefficient $VC = m/\sigma$ does not depend on X either. Thus, for any two stress levels (or variables combination), the distribution is the same, implying that only the scale changes (see [1] sec. 2.3). Observe the fact that the scale only changing in the Weibull analysis implies that

$$\hat{\eta}_{Li} = r\{(X_{Li})\} \quad (4)$$

On the other hand, *R(t)* under any two levels X_{L1}, X_{L2} is related by $R_{xL2}(t) = R_{xL1}(t)\{\rho(X_{L1}, X_{L2})t\}$ where $\rho(X_{L1}, X_{L2}) = r\{(x_{L2})\}/r\{(x_{L1})\}$ and that it is called the acceleration factor. Finally, it is important to note that by setting $\varepsilon = \ln\{r(X_{Li})\} - \ln\{T_X\}$ or $\ln\{T_X\} = \ln\{r(X_{Li})\} + \varepsilon$, and because $\ln T_X$ and $\ln\{r(X_{Li})\}$ do not depend on X, then as in (5), the variance of $\ln\{T_X\}$ does not depend on X either.

$$Var[\ln\{T_X\}] = \sigma_t^2 = \sum_{i=1}^{n}\{\ln(t_i) - ln(\bar{t})\}^2 \quad (5)$$

Despite of this, because in the estimation of the Weibull life-stress relationship as in (3), we estimate only one β value as a linear combination of the variables $(X_2, X_3, \dots, X_k)$, β is always overestimated as in the following section.

3. Problem statement

Statement 1: Because in ALT failure time data is obtained by increasing the level of the variables $(X_{L1}, X_{L2}, \dots, X_{Lk})$, the variance of the logarithm of the lifetimes (σ_t^2) defined in (5) is diminished (the time to failure is shorted) and as a consequence, β is overestimated.

Statement 2: In multivariate ALT analysis, when significant variables $(X_2, X_3, \dots, X_k)$ are added into $r\{X(t)\}$, the relation between the logarithm of the scale parameter η and $(X_2, X_3, \dots, X_k)$ tends to be one, thus the corresponding sum square error is diminished increasing R^2, and as a consequence, β is overestimated.

Considering these statements, firstly note that β is intrinsically related to the strength characteristics of the product; thus, if the levels of the significant variables $(X_{L1}, X_{L2}, \dots, X_{Lk})$ are selected in such that, for their higher effect combination, they do not generate a foolish failure mode (the effect of the level combinations is lower than the effect that reaches the destructive limits), β *must be constant,* and its value must represent the variance of the strength characteristic σ^2_{SC} which in the estimation processes with constant over time and interval-valued variables is not used (or measured). And second, we can observe that in the estimation process, the shape parameter β is determined by the variance of the logarithm of the lifetime data σ^2_t, the response variance σ^2_y and the correlation coefficient R^2, and that neither of them represent σ^2_{SC}. Thus, due to the dependence of β on σ^2_y, σ^2_t and R^2, adding significant variables (increases R^2) and/or overstressing their levels (diminishes σ^2_t) always overestimates β as in the following section.

4. Statistical analysis

Let us first show that β is completely determined by σ^2_y, σ^2_t and R^2. To see this, let us use the Weibull reliability function given by

$$R(t) = \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] \quad (6)$$

Which in linear form is given by

$$Y = b_0 + \beta X \quad (7a)$$

Where $Y = \ln[-\ln(1 - F(t))]$, $b_0 = -\beta \ln(\eta)$, $\beta = \beta$, $X = \ln(t)$, and $F(t)$ is the cumulative failure function of *t* given by $F(t) = 1 - R(t)$. $F(t)$ based on the median rank approach is estimated as [14].

$$F(t) = (i - 0.3)/(n + 0.4) \quad (7b)$$

For another possible approximation of F(t), see [3], [4] and [21]. On the other hand, observe from (7a) that β is a critical parameter (see [10] and [16] sec. 2.3) and thus, the analysis depends on the accuracy by which it is estimated. Also, observe that (7b) is in function of the sample size n and that for $n > 6$ the F(t) percentile is greater than 90% [2]. Regardless of this, note that η represents the 0.367879 reliability percentile which corresponds to $Y = \ln\{-\ln[\exp(-1)]\} = 0$ implying from (7a) that for center response Y

$$\ln(\eta) = \ln(\bar{t}) \quad (8)$$

Thus, under multiple linear regression the coefficients of (7a) are estimated as

$$b_0 = \bar{Y} - \beta\bar{X} \quad (9a)$$

$$\beta = \begin{bmatrix} n & \sum_{i=1}^{n} Y_i \\ \sum_{i=1}^{n} X_i & \sum_{i=1}^{n} Y_i X_i \end{bmatrix} / \begin{bmatrix} n & \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i & \sum_{i=1}^{n} X_i^2 \end{bmatrix}$$

$$\beta = \frac{n\sum_{i=1}^{n} Y_i X_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2} \quad (9b)$$

From (9b) observe that its denominator is the variance of the logarithm of the lifetime data defined in (5), which in terms of the covariates is given by.

$$\sigma^2_t = \sum_{i=1}^{n} (X_i - \bar{X})^2\ ) \quad (9c)$$

On the other hand, the goodness of fit of the polynomial given in (7a) is performed by the anova analysis where its sources of variation are

$$\sigma^2_y = SS\ total = \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \quad (10)$$

$$SS\ regression = \beta^t \sum_{i=1}^{n} (Y_i X_i) \quad (11)$$

$$SS\ error = SS\ total - SS\ regression \quad (12)$$

The goodness of fit index is given by

$$R^2 = \frac{\left(\sum_{i=1}^{n} (Y_i X_i)\right)^2}{\sigma^2_t \sigma^2_y} \quad (13)$$

Finally from (9c), (10), (11) and (13), β is given by

$$\beta = \sum_{i=1}^{n} \frac{(Y_i X_i)}{\sigma^2_t} = \sigma_y R/\sigma_t \quad (14)$$

On the other hand, to see that increasing (or decreasing) η affects β as in ***Statement1***, note from (8) that increasing (or decreasing) η is equivalent to increasing (or decreasing) $\sum_{i=1}^{n} \ln(t_i) = \sum_{i=1}^{n} X_i$ in (9b). That is to say, shortening the time in which the lifetime occurs, decreases their varianceσ^2_t and thus, according to (14), β is overestimated.

In the case of Statement 2, from (11) to (14), it is clear that although the levels of the variables are not stressed, by adding significant variables, since σ_y and σ_t, are fixed as in (5), then in (14) $\sum_{i=1}^{n} Y_i X_i$ is increased, and as a consequence, β is always overestimated.

5. Proposed Method

To see numerically that β is overestimated as in section 4, first note that each combination of the variables, as in Fig. 1, presents its own Weibull family, and that data are gathered by using a replicated experiment design DOE as presented in Fig. 2 (see [15] and [20] Chapter 13).

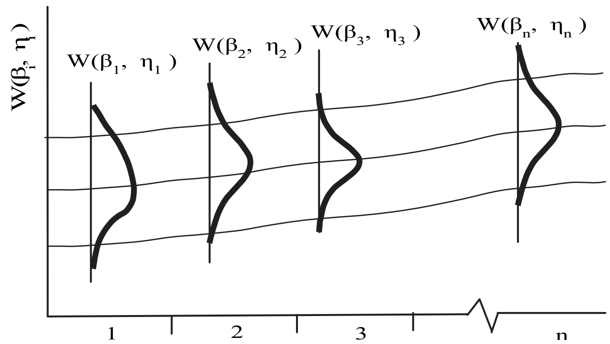

Figure 1: Orthogonla array levels
Source: The authors

Run number	x_1	x_2	x_3	x_4	Outer array Z_1 / Z_2	1 / 1	1 / 2	2 / 1	2 / 2
1	1	1	1	1					
2	1	2	2	2					
3	1	3	3	3					
4	2	1	2	3	(Life time data)				
5	2	2	3	1					
6	2	3	1	2					
7	3	1	3	2					
8	3	2	1	3					
9	3	3	2	1					

Figure 2: Orthogonla Array Levels(9)3^4
Source: The authors

Second, to illustrate this, let us use the DOE data from Table 1, which corresponds to twelve electronic devices. Data were published by [17] p.11.

From these data, the Weibull/(T–H) parameters defined in (3), by using ML are $\hat{\beta} = 5.874$, $\hat{A} = 0.0000597$, $\hat{b} = 0.281$ and $\hat{\phi} = 5630.330$ (the ALTA Pro software was used). In addition, observe that although in Table 1 there are three level combinations among the variables, which as a consequence lead to three Weibull families in this DOE, regardless of this, in the standard approach [eq. (3)], only one shape parameter $(\hat{\beta} = 5.874)$ was estimated.

Thus, it is not representative of the whole set of data. To see this, in Table 2, the scale and shape parameters $(\eta_i,\ \beta_i)$, estimated by ML, and their associated reliability $R(t)$ for t=150 are given.

Table 1.
Weibull (T-H) Data

Sub-set	Time	Temperature	Humidity
1	190	378	0.8
	208	378	0.8
	230	378	0.8
	298	378	0.8
2	310	378	0.4
	316	378	0.4
	329	378	0.4
	411	378	0.4
3	108	398	0.4
	123	398	0.4
	166	398	0.4
	200	398	0.4

Source: Vassiliou and Metas, 2003.

Table 2.
Weibull parameters and reliability; ML approach.

Sub-set	Time	Temp	Hum	Eta (η)	Beta (β)	R (t)
1	150	378	0.8	249.666	5.874	0.951
2	150	378	0.4	354.736	5.874	0.994
3	150	398	0.4	167.818	5.874	0.596

Source: The authors

Table 3.
Weibull parameters and reliability; MLR approach.

Sub-set	Time	Temp	Hum	Eta (η)	Beta (β)	R (t)
1	150	378	0.8	228.134	4.84061	0.877
2	150	378	0.4	339.251	6.42755	0.995
3	150	398	0.4	144.916	3.49029	0.324

Source: The authors

In order to compare the standard results of Table 2, with those found in the DOE, Table 3 presents the Weibull family and R(t) for each DOE combination, using (8) and (9b) with centered response (Y).

By comparing these results, we observe that the estimated η_i in Table 2, in contrast to the estimated η_i from Table 3, does not represent the expected 0.367879 percentile as defined in (8). And that $\hat{\beta} = 5.874$ does not represent the shape parameter of the levels found in the DOE. Thus the proposed method to avoid this issue, using MLR, is as in the following section.

5.1. Regression approach for statement 1.

In ALT with one interval valued and constant over time variables, as is the case of Weibull/Arrhenius, Weibull/Inverse power law and Weibull/Eyring, it is possible to estimate their parameters by applying (8), (9a) and (9b) by following the next steps.

Step 1. For each replicated level of the stress variable (We must have almost 4 replicates, although 10 are recommended), determine the corresponding β and η_i parameters by using (7a), (8), (9a) y (9b). (In this one variable approach, β is generally constant). If β is not constant, proceed as in section 5.2.

Step 2. Take the effect of the corresponding linear transformation (see next section) of the time/stress model $r\{X(t)\}$ defined in (1) as (X) (e.g. in Arrhenius $X = 1/T$) and the corresponding logarithm of the scale parameter η_i of the i-th level of the variable estimated in step 1 as Y $(Y_i = \eta_i)$.

Step 3. Using (9a) and (9b), estimate by regression between the variables X and Y defined in step 2, the parameters of the life/stress model $\{X(t)\}$.

Note: In the Eyring case, do not forget to subtract the logarithm of the reciprocal of the temperature $(1/T)$ from the logarithm of η_i before you perform the regression.

Step 4. Using the regression parameters of $r\{X(t)\}$ estimated in step 3, estimate the logarithm of η_{op} for the operational (or desired) level (see next section). Finally, form the Weibull family of the operational (or desired) level $W(\beta, \eta_{op})$ with the shape parameters estimated in step1 and the scale parameter estimated in this step. And with these Weibull parameters, determine the desired reliability indexes.

5.1.1 Let us exemplify the above methodology, through the

Weibull/Arrhenius and Weibull/Eyring relationship, which are parametrized as in (1). In the case of Arrhenius, the infinitesimal characteristic (see [18]) is given by $\delta(T) = \alpha/T^2$, thus the primitive (integral) $\varphi 1(T)$ of $\delta(T)$, is given by $\varphi 1(T) = -1/T$. Since $\varphi 1(T)$ shows the form in which the variable affects the time, in the Arrhenius model the effect is $X = 1/T$ (see step 2 of section 5.1). Thus from (1) and (4), the Arrhenius model is given by: (for details see [1], Chapter 5).

$$\eta_i = Ce^{\left(\frac{B}{T_i}\right)} \quad (15a)$$

In (15a), $C = e^{-b_o}$ and $B = \beta_1$ are the parameters to be estimated, and T is the absolute temperature (Kelvin). The linear form of (15a) is given by

$$\ln(\eta_i) = \ln(C) + \frac{B}{T_i} = Y = b_0 + \beta_1 X_i \quad (15b)$$

Using (15a) the Weibull/Arrhenius pdf is given by

$$f(t|T) = \frac{\beta}{Ce^{\frac{B}{T}}}\left(\frac{t}{Ce^{\frac{B}{T}}}\right)^{\beta-1} \exp\left[-\left(\frac{t}{Ce^{\frac{B}{T}}}\right)^{\beta}\right] \quad (16)$$

As a numerical application, consider the data in Table 4. Data were published by [17]. The Weibull parameters of step 1 are given in Table 5a. The effect for step 2, $X = 1/T$ and $Y_i = \ln(\eta_i)$ are given in Table 5b.

The Weibull/Arrhenius parameters of step 3 using Minitab® and data of Table 5b, are $B = 1862.4$ and $C = \exp(3.9483) = 51.86271$ with $R^2 = 99.5\%$. Finally, by using these parameters, the Weibull family mentioned in step 4, for a level of 323K is $W(4.092, 16{,}556.67)$.

Table 4.
Weibull/Arrhenius data.

Stress	393K	408K	423K
Time	3850	3300	2750
	4340	3720	3100
	4760	4080	3400
	5320	4560	3800
	5740	4920	4100
	6160	5280	4400
	6580	5640	4700
	7140	6120	5100
	7980	6840	5700
	8960	7680	6400

Source: Vassiliou and Metas, 2003.

Table 5a.
Data step 1

Stress	Eta	Beta
393	5890.059	4.092
408	5048.622	4.092
423	4207.185	4.092

Source: The authors

Table 5b. Data step 2

b0	X	Y
1	0.002545	8.68102134
1	0.002451	8.52687066
1	0.002364	8.3445491

Source: The authors

Table 6.
Data for Weibull/Eyring model.

ln (1/T)	ln (Eta)	X	Y
-5.97381	8.68102	0.002545	14.654831
-6.01127	8.52687	0.002451	14.538138
-6.04737	8.34455	0.002364	14.391921

Source: The authors

5.1.2 In the case of the Weibull/Eyring relationship the infinitesimal characteristic is given by $\delta(T) = \frac{1}{T} + \alpha/T^2$, with primitive $\varphi 1(T)$ of $\delta(T)$, given by $\varphi 1(T) = \ln(T) - 1/T$, thus $r(T) = e^{-b_0 - \beta_1 \ln(T) - \beta_2/T}$. This formulation with $\beta_1 = -1$ is used in the Eyring model when the temperature is used. The Eyring model is given by:

$$\eta = \frac{1}{T} e^{-\left(A - \frac{B}{T}\right)} \quad (17)$$

In (17), $A = -b_0$ and $B = \beta_2$ are parameters to be estimated and T is the absolute temperature. The linear relationship of (17) is

$$\ln(\eta) = -A + \ln(1/T) + B/T \quad (18a)$$

And the linear relationship used to estimate the parameters is given by

$$\ln(\eta) - \ln\left(\frac{1}{T}\right) = -A + B/T = Y = b_0 + \beta_2 X \quad (18b)$$

The Weibull/Eyring pdf using (17) is

$$f(t|T) = \beta T e^{\left(A - \frac{B}{T}\right)} \left(tTe^{\left(A - \frac{B}{T}\right)}\right)^{\beta-1} \exp - \left(tTe^{\left(A - \frac{B}{T}\right)}\right)^{\beta} \quad (19)$$

Using data of Table 4, the Weibull/Eyring parameters of step 3 using Minitab® with data of Table 6, are $B = 1454.2$ and $A = -10.9606$ with $R^2 = 99.3\%$. By using these parameters the Weibull family mentioned in step 4, for a level of 323K is $(4.092, 16{,}076.52)$.

Finally, for the one variable case, when the shape parameter is not constant for all the stress levels proceed as in the multivariate case of the following section.

5.2. Regression approach for statement 2.

For the multivariate ALT analysis, as in Fig. 1, each covariate combination presents its own Weibull family. Thus, because in the standard ALT analysis, the estimated β value does not represent the whole set of data, in MLR, we propose to estimate the Weibull/life/stress parameters through the following steps.

Step 1. For each replicated combination level of the stress variables (We must have almost 4 replicates; 10 is recommended; see comment below eq. (7b)), determine the corresponding Weibull family (β_i, η_i). This could be performed by ML, but MLR is recommended. (ML is a biased estimator and n is small).

Step 2. Take the effect of the corresponding linear transformation of the variables as the independent variables $(X_1, \dots, X_k)$ and the corresponding logarithm of the scale parameter η_i of the *i*-th Weibull family of step1 as the dependent variable Y $(Y_i = \eta_i)$.

Step 3. Estimate the parameters of the life/stress model $r\{X(t)\}$ by regression between the set of variables $(X_1, \dots, X_k)$ and Y defined in step 2. If there are not enough degrees of freedom to perform the analysis, proceed as follows.

a) Estimate a vector $Y_\eta = (\eta_{11} \dots \eta_{n1}, \eta_{12}, \dots, \eta_{n2}, \eta_{n1}, \dots \eta_{nk})$ by reordering (7a) as

$$Y_\eta = -\frac{Y_i}{\beta_j} + \ln(t_{ij}) \tag{20}$$

Estimate the parameters of $r\{X(t)\}$ by performing a regression between $(X_1, \dots, X_k)$ and Y_η. In (20), Y_i is as in (7b), and β_j and $\ln(t_{ij})$ are the shape parameter and the logarithm of the lifetime data of the *i*-th Weibull families of step 1.

b) Based on (5) and on the fact that $\ln(S^2) \sim N\{\ln(S^2), 2/(n-1)\}$ where S^2 is the sample variance of the lifetime data, form the logarithm vector

$\ln(\sigma_{tY}) = \left\{\left(\ln((n-1)\sigma_{L1}^2)\right)^{1/2}, \left(\ln((n-1)\sigma_{L2}^2)\right)^2, \dots, \left(\ln((n-1)\sigma_{Lk}^2)\right)^{1/2}\right\}$ where σ_{Li}^2 is the variance of the *i*-th level defined in (9c) and n is the number of replicates of the *i*-th level of step1.

c) Take the inverse of the effect of the covariates of step 2, as the independent variables $(X_{\sigma 1}, X_{\sigma 2}, \dots, X_{\sigma k})$ and σ_{tY}^2 as the response variable and perform a regression between $(X_{\sigma 1}, X_{\sigma 2}, \dots, X_{\sigma k})$ and σ_{tY}^2.

Observe that Y_η and σ_{tY}^2 are vectors for the complete DOE data (or families).

Step 4. Using the regression parameters of $r\{X(t)\}$ estimated in step 3-a), estimate the scale parameter η_{op} for the operational level by applying (4). By using the regression parameters of step 3-b), estimate the value of σ_{top}^2 of the operational level, and by applying (14), with σ_Y^2 of step 1 and a desired R^2 index, estimate the corresponding β_{op} value. (β_{op}, η_{op}) are the parameters of the Weibull family of the desired stress level and they could be used to determine any desired reliability index. Observe that the estimation of β_{op} using (14) is robust (almost insensible) to the selected R^2 index.

As a numerical application consider the data in Table 7. Data were published in [17].

On the other hand, data of step 3 using (20) are given in Table 8. By using Minitab, the parameters of W(T–H) model by regression between (X_1, X_2) and Y_η are $A = \exp(-11.894) = 6.831E-06$, $\phi = 6398.3$, and $b = 0.31745$ with $R^2 = 96.4\%$. The parameters of the regression between (X_1, X_2) and $\ln(\sigma_{tY})$ are $\alpha_0 = -16.3645$, $\alpha_1 = 0.038294$ and $\alpha_2 = 1.001203$ with $R^2 = 100\%$. To show the method, suppose that the operational level is $T = 258K$ with $H = 0.2$, then, by using the above parameters as in step 4, $\eta_{op} = 1930.62$, and by taking $\sigma_{top} = 0.08587$, $\sigma_Y = 3.046497$ and $R^2 = 95.0\%$, $\beta_{op} = 5.8054$. Thus, the operational Weibull family is $W(5.8054, 1930.62)$.

Table 7.
Data for the (T-H) model.

Level	Time	Temp	Hum	b0	X	Y	W(β,η)
	190	378	0.8	1	5.2470	-1.275132	4.84061
	208	378	0.8	1	5.3375	-0.238955	228.13413
1	230	378	0.8	1	5.4381	0.427496	$3^*\sigma^2_{L1}$= 0.11343
	298	378	0.8	1	5.6971	1.086592	
	310	378	0.4	1	5.7366	-1.275132	6.42755
	316	378	0.4	1	5.7557	-0.238955	339.25138
2	329	378	0.4	1	5.7961	0.427496	$3^*\sigma^2_{L2}$= 0.05092
	411	378	0.4	1	6.0186	1.086592	
	108	398	0.4	1	4.6821	-1.275132	3.49029
	123	398	0.4	1	4.8122	-0.238955	144.91613
3	166	398	0.4	1	5.1120	0.427496	$3^*\sigma^2_{L3}$= 0.23558
	200	398	0.4	1	5.2983	1.086592	

Source: The authors

Table 8.
Data for step 3 of the (T-H) model.

Sub-set	b0	X_1=1/T	X_2=1/H	Yη
	1	0.0026455	1.25	5.51045
1	1	0.0026455	1.25	5.38690
	1	0.0026455	1.25	5.34976
	1	0.0026455	1.25	5.47262
	1	0.0026455	2.5	5.93496
2	1	0.0026455	2.5	5.79292
	1	0.0026455	2.5	5.72955
	1	0.0026455	2.5	5.84954
	1	0.0025126	2.5	5.04747
3	1	0.0025126	2.5	4.88065
	1	0.0025126	2.5	4.98951
	1	0.0025126	2.5	4.98700

Source: The authors

On the other hand, the ML parameters using the ALTA routine are $A = 5.9701e-05$, $\phi = 5630.3299$, $b = 0.2806$ with $\beta_{0L} = 5.8744$ and $\eta_{0L} = 1642.4765$. With operational Weibull Family given by $(5.8744, 1642.48)$. A comparison of the Weibull parameters and reliability index of the ML and the proposed Method is given in Table 9.

In Table 9, we can see that the shape parameter is not representative of the observed Weibull families as it is in the proposed method. The same occurs with the estimated reliability. In particular, it is important to note that the proposed method is based on the observed variance and thus it is directly related to the operational factors of the process.

6. Conclusions

In Weibull multivariate ALT analysis, each combination of the significant variables presents its own behavior, thus the standard approach of estimating only one shape parameter to represent all the Weibull families is suboptimal. Since β depends on R^2, which increases when variables are added to the analysis, in the multivariate case β is always overestimated. Clearly, since the change in the scale parameter η is reflected in $\sum_{i=1}^{n} \ln(t_i)^2$, thus the proposed method could easily be generalized to the right censured case by reflecting the censured data on $\sum_{i=1}^{n} \ln(t_i)^2$ and by substituting n for r in (9b) where r is the number of failure. Although the proposed method depends greatly on the accuracy in which σ_t^2 is estimated, because $\ln(S^2) \sim N\{\ln(S^2), 2/(n-1)\}$ stabilize the variance as defined in step 3-b, the proposed method could be considered robust for this issue. It is important to mention that β in (14) is not highly sensitive to the selected R^2 index. Knowing (14), it seems to be possible to generalize the proposed method to the ML approach by formulating a log-likelihood function based on the β values of the Weibull families, but more research must be undertaken. Since the shape parameter β is inversely related to σ_t^2, and because σ_t^2 is the standard deviation of the lognormal distribution, which presents a flexible behavior and similar analysis to the Weibull distribution [11], it seems to be possible to extend the present method to the lognormal analysis. On the other hand, although the proposed method is practical and its application could easily be performed by using a standard software routine, as Minitab does, a more detailed method could be proposed by using a copula to modeling in joint form the Weibull families behavior, but because the Weibull distribution is determined by an non-homogeneous Poison processes [7] and its convolutions do not have a closed form [12], more research must be undertaken.

Table 9.
Comparison between ML and proposed method.

ML Parameters				Proposed method (PM)					Reliability index	
(T-H) Parameters	Level	Eta	Beta	Eta	Beta	$3*\sigma^2_L$	σ^2_Y	R^2	R(ti) (ML)	R(ti) (PM)
	1	249.5454	5.8744	228.13413	4.8406	0.1134336	3.046497	0.872453	0.7615	0.5893
A=0.000059701	2	354.3875	5.8744	339.25138	6.4276	0.0509200	3.046497	0.690524	0.9936	0.9947
φ=5630.3299	3	167.6529	5.8744	144.91613	3.4903	0.2355755	3.046497	0.942006	0.9531	0.7604
b=0.28060	Op	1642.4765	5.8744	1930.61980	5.8054	0.0858736	3.046497	0.950000	0.5561	0.7937

Source: The authors

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