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Discrimination between the lognormal and Weibull distributions by using multiple linear regression

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Abstract
In reliability analysis, both the Weibull and the lognormal distributions are analyzed by using the observed data logarithms. While the Weibull data logarithm presents skewness, the lognormal data logarithm is symmetrical. This paper presents a method to discriminate between both distributions based on: *1)* the coefficients of variation (CV), *2)* the standard deviation of the data logarithms, *3)* the percentile position of the mean of the data logarithm and *4)* the cumulated logarithm dispersion before and after the mean. The efficiency of the proposed method is based on the fact that the ratio of the lognormal (b_{1ln}) and Weibull (b_{1w}) regression coefficients (slopes) b_{1ln}/b_{1w} efficiently represents the skew behavior. Thus, since the ratio of the lognormal (R_{ln}) and Weibull (R_w) correlation coefficients R_{ln}/R_w (for a fixed sample size) depends only on the b_{1ln}/b_{1w} ratio, then the multiple correlation coefficient R^2 is used as the index to discriminate between both distributions. An application and the impact that a wrong selection has on R(t) are given also.

Keywords: Weibull distribution; lognormal distribution; discrimination process; multiple linear regression; Gumbel distribution.

Discriminación entre la distribución lognormal y la distribución Weibull utilizando regresión lineal múltiple

Resumen
En el análisis de confiabilidad, las distribuciones Weibull y lognormal son ambas analizadas utilizando el logaritmo de los datos observados. Debido a que mientras el logaritmo de datos Weibull presenta sesgo, el logaritmo de datos lognormales es simétrico, entonces en este artículo basados en *1)* los coeficientes de variación (CV), *2)* en la desviación estándar del logaritmo de los datos, *3)* en la posición del percentil de la media del logaritmo de los datos y *4)* en dispersión acumulada del logaritmo antes y después de la media, un método para discriminar entre ambas distribuciones es presentado. La eficiencia del método propuesto está basado en el hecho de que el radio entre los coeficientes de regresión (pendientes) b_{1ln}/b_{1w} de la distribución lognormal (b_{1ln}) y de la distribución Weibull (b_{1w}), eficientemente representa el comportamiento del sesgo. De esta manera, dado que el radio de los coeficientes de correlación de la distribución lognormal (R_{ln}) y de la distribución Weibull (R_w), (para un tamaño de muestra fijo), solo depende del radio b_{1ln}/b_{1w}, entonces el coeficiente de correlación múltiple R^2 es utilizado como un índice para discriminar entre ambas distribuciones. Una aplicación y el impacto que una mala selección tiene sobre R(t) son también dadas.

Palabras clave: distribución Weibull; distribución lognormal; proceso de discriminación, regresión lineal múltiple; distribución Gumbel.

1. Introduction

Because of their flexibility to model several behaviors, the Weibull and the lognormal distributions are two of the most used types of distribution in reliability. However, because the Weibull distribution is based on a non-homogeneous Poisson process, it models additive effect behavior [1]. Similarly, because the lognormal distribution is based on a geometric Brownian motion, then it models multiplicative effect behavior [2]. Therefore, they should not be used interchangeably. Hence, a discrimination process between both distributions is needed. In particular, the negative effect on reliability due to a wrong selection between these distributions is shown by using the stress- strength analysis, where the reliability represents all probabilities that the failure

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Table 1
Compression loads

9.6	12.5	13.5	14.4	14.9	15.7	16.6	17.9	20
9.6	12.6	13.8	14.4	15	15.9	16.8	18	20.1
12.3	12.7	14	14.6	15.2	16	16.8	18.1	22.3
12.4	12.7	14.2	14.6	15.3	16.1	16.9	19.1	
12.4	13.2	14.4	14.7	15.3	16.5	17.9	19.7	

Source: Adapted from [4]

Table 2
Strength of the product

21	25	23	31	32
22	26	30	31	33
23	27	30	32	34

Source: Adapted from [4]

Table 3
Stress-strength reliability

Combination	***R***
Lognormal-Lognormal	0.9957
Lognormal-Weibull	0.9860
Weibull-Lognormal	0.9984
Weibull-Weibull	0.9882

Source: The authors

governing strength (S) exceeds the failure governing stress (s) [3]. The stress-strength formulation is given by

$$R = P(S > s) = \int_{-\infty}^{\infty} f(s)\left[\int_{s}^{\infty} f(S)\, dS\right] ds \quad (1)$$

In the stress-strength analysis it is assumed that time is not the cause of failure; instead, failure mechanisms are what cause the part to fail [4]. In addition, as can be seen in eq. (1), the estimated reliability depends entirely on the selected stress and strength distributions. Thus, because a wrong selection will overestimate or underestimate reliability, a wrong selection will largely impact the analysis conclusions. To illustrate the impact of a wrong selection on reliability, following data published in Wessels has been used ([4], sec. 7). Table 1 shows the stress data; and Table 2 the strength data.

Finally, the stress-strength analysis for the four possible combinations between the Weibull and lognormal distributions, is presented in Table 3. The estimation of the stress-strength reliability was performed by using the eq. (40-43) given in section 7.

From Table 3, we conclude that because each combination shows a different reliability index, then the accurate discrimination between the Weibull and the lognormal distributions is an issue that must be solved. To this end, researchers have used several selection procedures. Among the oldest ones are the Chi-square, the Anderson-Darling and the Cramer-Von Mises goodness-of-fit tests [5]. On the other hand, the most widely used methods are those based on the maximum likelihood (ML) function as they are those given in [6-10] and recently in [11-12]. In particular, the methods based on probability plot (PP) tests are in [13-15]. Those based on Kolmogorov-Smirnov (KS) test are in [16] and [17], and those based on Bayes analysis are in [18]. The discrimination process between the Weibull and the lognormal distributions depends *1)* on the relationship between the Coefficient of Variation (CV) of the observed data and their standard deviation (σ_x), *2)* on the mean position of the logarithm of the data (μ_x) and *3)* on the dispersion behavior before and after μ_x. Unfortunately, since none of the above approaches takes into account the skew behavior of the logarithm of the data, then none of them is effective in discriminating between both distributions.

Based on the fact that the Weibull data logarithm (Gumbel behavior) *always presents negatively skewed behavior*, the logarithm of lognormal data *always presents symmetrical dispersion behavior*, the b_{1ln}/b_{1w} ratio of the estimated lognormal and Weibull coefficients effectively discriminates between the negative and symmetrical dispersion behaviors, a method based on R^2 to effectively discriminate between both distributions is offered by this paper in sec. 4. The reason for the method's efficiency is that the R^2 index for a fixed sample size *(n)* depends only on the b_{1ln}/b_{1w} ratio (see sec. 4.3). That is, because the b_{1ln}/b_{1w} ratio effectively discriminates between negative and symmetrical dispersion behaviors, the R^2 index effectively discriminates between both distributions also.

This paper is structured as follows. Section 2 shows that the behavior of the logarithm of a Weibull variable is always negatively skewed and that the logarithm of a lognormal variable is always symmetrical. In section 3, based on the data behavior log, the characteristics that completely define whether data follow a Weibull or a lognormal distribution are given. Also, in section 3, the case where the dispersion (*Sxx*) contribution is not fulfilled is presented also. Section 4 shows the multiple linear regression (*MLR*) analysis for the Weibull and lognormal distributions. Section 5 presents *1)* how via *MLR*, the b_{1ln}/b_{1w} ratio efficiently captures the *Sxx* dispersion behavior, and *2)* that because the R^2 index for a fixed *n* value only depends on the b_{1ln}/b_{1w} ratio, it captures the *Sxx* dispersion behavior also. The application of a stress-strength analysis is given in section 6, while Section 7 shows the effect that a wrong selection has over the reliability index. Finally, the conclusions are presented in section 8.

2. Behavior of log-Weibull and log-lognormal variables

Since the discrimination method is based on the logarithm of the Weibull or lognormal observed data and on its dispersion behavior, then in this section, we show that the Weibull data logarithm follows a Gumbel distribution and that it is always negatively skewed. Similarly, we show that the logarithm of the lognormal data follows a Normal distribution and that it is always symmetrical.

2.1. Weibull and Gumbel relationship

The Weibull distribution is given by

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\} \quad (2)$$

In eq. (2), $t > 0$ and β and η are the Weibull shape and scale parameters respectively. On the other hand, the Gumbel distribution is given by

$$f(x) = \frac{1}{\sigma_G} \exp\left\{\left(\frac{x-\mu_G}{\sigma_G}\right) - \exp\left\{\frac{x-\mu_G}{\sigma_G}\right\}\right\} \quad (3)$$

In eq. (3) $-\infty < x < \infty$ with $x=ln(t)$ and μ_G is the location parameter and σ_G is the scale parameter [19]. Thus, based on eq. (2) and eq. (3), the relation between both distributions is as follows.

Theorem: If a random variable t follows a Weibull distribution [$t \sim W(\beta, \eta)$], then its logarithm $x=ln(t)$ follows a Gumbel distribution [$x \sim G(\mu_G, \sigma_G)$] [20].

Proof: Let $F(ln(t)) = P(ln(t) \leq ln(T))$ be the cumulative function of $x = ln(t)$, with T representing the failure time value. Thus, in terms of x, $F(ln(t)) = Pr[ln(t) \leq x]$; $F(x) = Pr[t \leq exp(x)]$. Then by substituting $t = exp(x)$, $F(x)$ is

$$F(x) = 1 - \exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\} = 1 - \exp\left\{-\left(\frac{\exp(x)}{\eta}\right)^{\beta}\right\} \quad (4)$$

Finally, based on the relations between the Weibull and Gumbel parameters given by [20].

$$\mu_G = ln(\eta) \quad (5)$$

$$\sigma_G = \frac{1}{\beta} \quad (6)$$

and by taking $W=((x-\mu_G)/\sigma_G)$, eq. (4) is given by $F(x) = 1-exp\{-exp\{(x-ln(\eta))\cdot\beta\}\}$ which in terms of W is

$$F(x) = 1 - \exp\{-\exp\{w\}\} \quad (7)$$

from eq. (7), the reliability function is

$$R(t) = 1 - F(t) = \exp\{-\exp\{w\}\} \quad (8)$$

and the density function is given by

$$f(x) = -\frac{dR(t)}{dt} = \exp\{w - \exp\{w\}\} \quad (9)$$

clearly, eq. (9) in terms of W is

$$f(x) = \frac{1}{\sigma_G} \exp\left\{\left(\frac{x-\mu_G}{\sigma_G}\right) - \exp\left\{\frac{x-\mu_G}{\sigma_G}\right\}\right\} \quad (10)$$

Since eq. (10) is as in eq. (3), we conclude that the logarithm of Weibull data follows a Gumbel distribution. On the other hand, by using the moment method [21] (sec. 1.3.6.6.16), the parameters of eq. (10) are given by:

$$\mu_{EV} = E(x) = \mu_Y + \gamma\sigma_{EV} \quad (11)$$

$$\sigma_{EV} = \frac{\sqrt{6}}{\pi}\sigma_Y \quad (12)$$

where μ_Y and σ_Y are the mean and the standard deviation of the log data.

2.1.1. Dispersion of the Gumbel distribution

In order to show the dispersion of the log Weibull variable, several Weibull probability density functions (pdf) with fixed scale parameter $\eta=50$ and variable shape parameter β are plotted in Fig. 1. Fig. 2, corresponds to the conversion of the Weibull pdf of Fig. 1 on Gumbel pdf.

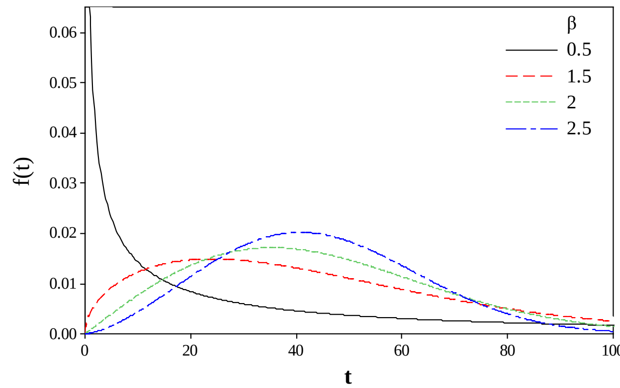

Figure 1 Weibull pdf for $\eta=50$
Source: The authors

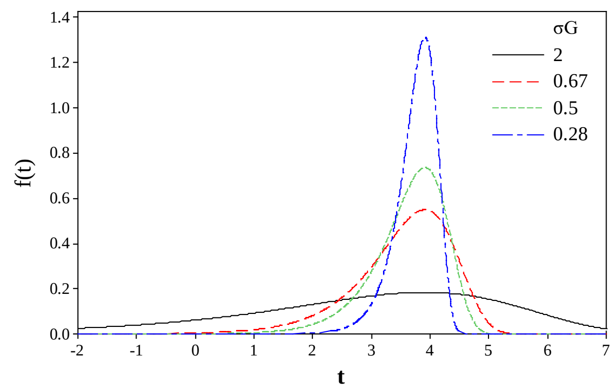

Figure 2 Gumbel pdf for $\mu_G=3.91$
Source: The authors

As can be seen in Fig. 2, the Gumbel distribution is always negatively skewed. Moreover, it is important to highlight that the Gumbel skew is constant at $\gamma_1 = -1.13955$, and as demonstrated by [22], it can be estimated as

$$-2 \cdot 6^{3/2}\zeta(3)/\pi^3 \approx -1.13955.$$

On the other hand, as shown in next section, the logarithm of lognormal data follows a Normal distribution.

2.2. Lognormal and normal relationship

As it is well known, the lognormal data logarithm follows a Normal distribution [19]. If $Y \sim N(\mu, \sigma^2)$, then $X=e^Y$ (non-negative) has a lognormal distribution. Thus, because the logarithm of X yields a Normal variable ($Y=ln(X)$) then the lognormal distribution is given by

$$f(t) = \frac{1}{t\sigma_x\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{ln\, t-\mu_x}{\sigma_x}\right)^2\right\} \quad (13)$$

In eq. (13) μ_x and σ_x are the log mean and log standard deviation. Similarly, the Normal distribution is given by

$$f(t) = \frac{1}{\sigma_N\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{t-\mu_N}{\sigma_N}\right)^2\right\} \quad (14)$$

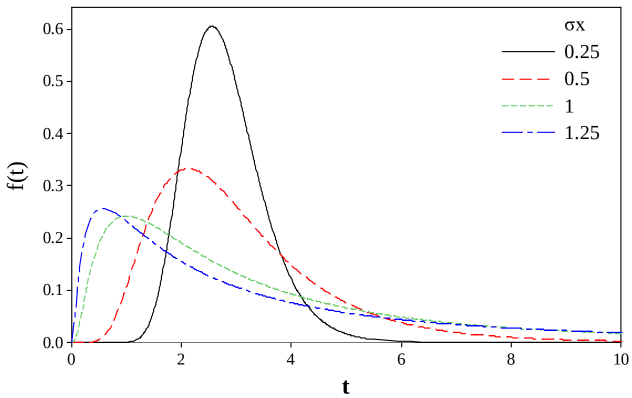

Figure 3 Lognormal pdf for $\mu_x=1$
Source: The authors

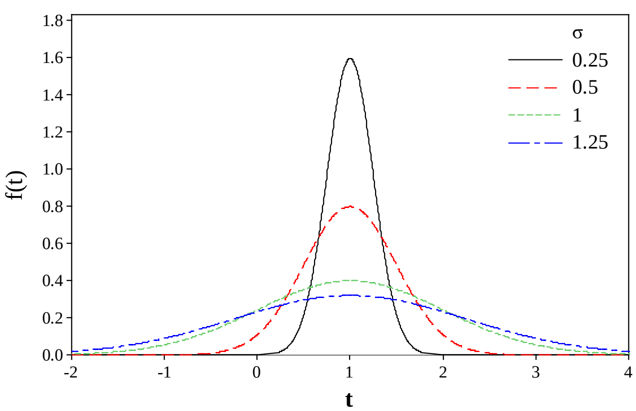

Figure 4 Normal pdf for $\mu=1$
Source: The authors

Note that, although the Normal distribution is the most widely used distribution in statistics, it is rarely used as lifetime distribution. However, in reliability the Normal distribution is used as a model for *ln(t)*, when *t* has a lognormal distribution.

2.2.1. Dispersion of the normal distribution

Fig. 3 represents several lognormal pdf for $\mu_x=1$ and variable σ_x. Plotted Normal pdfs of Fig. 4 correspond to the logarithm of the lognormal pdfs plotted in Fig.3. By comparing Fig.3 and Fig.4, we observe although the lognormal distribution is always positively skewed, its logarithm is always symmetrical.

Therefore, based on the log data behavior, the characteristics that completely define whether data follow a Weibull or lognormal distribution are given in next section.

3. Discrimination properties

This section presents that enough conditions are met in order to show that lognormal data follow a lognormal distribution and that Weibull data follow a Weibull distribution. Additionally, the critical characteristic to discriminate between both distributions when data follow neither a lognormal nor a Weibull distribution is given also.

3.1. Lognormal properties

In order to select the lognormal distribution as the best model to represent the data, the following characteristics have to be met. *First*, the coefficient of variation has to be equal to the log-standard deviation σ_x ($\sigma_x=CV$). Thus, because based on the mean and on the standard deviation of the observed data defined as

$$\mu = \exp\left\{\mu_x + \frac{\sigma_x^2}{2}\right\} \tag{15}$$

$$\sigma^2 = \exp\{2\mu_x + \sigma_x^2\}(\exp\{\sigma_x^2\} - 1) \tag{16}$$

the *CV* index is given by

$$CV = \frac{\sigma}{\mu} = \sqrt{\exp\{\sigma_x^2\} - 1} \tag{17}$$

Then from eq. (17) clearly $\sigma_x \approx CV$. *Second*, the log mean μ_x should be located at the 50th percentile. The reason is that the lognormal data logarithm follows a Normal distribution (see sec. 2.2). *Third*, since the total sum square (*Sxx*) is cumulated by the contribution before (Sxx_-) and after (Sxx_+) the mean μ_x is as follow

$$Sxx = Sxx_- + Sxx_+ = \sum(x-\mu_x)^2 \tag{18}$$

Then, due to the symmetrical behavior of the lognormal data logarithm, then in the lognormal case, the contribution before and after the mean must be equal; it is to say for the lognormal case $Sxx_- = Sxx_+$.

Thus, because when $\sigma_x \approx CV$, μ_x is located in the 50th percentile and $Sxx_- = Sxx_+$, we should directly fit the lognormal model. Similarly, the characteristics to be met for the Weibull distribution are as follow:

3.2. Weibull properties

In the Weibull case, because the Weibull data logarithm follows a Gumbel distribution, and because the Gumbel distribution is always negatively skewed (See sec 2.1.1), then the following characteristics have to be met. *First*, the coefficient of variation should be different from the standard deviation of the data logarithm ($\sigma_x \neq CV$). *Second*, the log mean μ_x should be located around the 36.21th percentile. Third, the contribution to *Sxx* before μ_x is always greater than the contribution after μ_x; in other words, due to the negative skewness of the Gumbel distribution, in the Weibull case $Sxx_- > Sxx_+$. Thus, because $\sigma_x \neq CV$, μ_x is located around the 36.21th percentile and $Sxx_- > Sxx_+$, then we should directly fit the Weibull distribution. Nonetheless, the next section will describe what happens when the above statements do not hold at all.

3.3. Weibull or lognormal distribution?

The discrimination process, when data neither completely follow a Weibull distribution nor completely follow a lognormal distribution, is based on the following facts. 1) For a Weibull shape parameter $\beta \geq 2.5$, the Weibull *pdf* is similar to the lognormal *pdf* [23]. 2) For $\beta \geq 2.5$, the log-standard deviation σ_x tends to be the *CV* ($\sigma_x \approx CV$), and μ_x tends to be located near the 50th percentile. 3) For Weibull data, regardless of the β value, the contribution before and after the mean tends to be different ($Sxx_{-}>Sxx_{+}$). Now for the Normal distribution we always expect that $Sxx_{-}=Sxx_{+}$ and for the Gumbel distribution we always expect that $Sxx_{-}>Sxx_{+}$; thus, because from eq. (18), Sxx_{-} captures the skewness of the Gumbel distribution, then based on the *MLR* analysis, in the proposed method the product of the *y* vector with the Sxx_{-} and Sxx_{+} contribution is used as the critical variable to discriminate between the Weibull and the lognormal distributions. In order to show that, the linear regression analysis on which the proposed method is based must first be introduced.

4. Weibull and lognormal linear regression analysis

This section shows that by using *MLR*, the ratio of the slopes of the lognormal and Weibull distributions (b_{1ln}/b_{1w}) is indeed efficient to discriminate between the negative and symmetrical skew behavior. Before showing that, the *MLR* analysis for the Weibull and lognormal distributions will first be introduced.

4.1. Weibull linear model

The Weibull and lognormal distributions can be analyzed as a regression model of the form

$$y = b_0 + b_1 x \tag{19}$$

The linear form of the Weibull distribution is based on the cumulative density function, given by

$$F(t) = 1 - R(t) = 1 - \exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\} \tag{20}$$

Thus, by applying double logarithm, its linear form is

$$\begin{aligned} y_i &= \ln\left(-\ln\left(1 - F(t_i)\right)\right) \\ &= -\beta\ln(\eta) + \beta\ln(t_i) = b_0 + b_1 x_i \end{aligned} \tag{21}$$

where $F(t_i)$ is estimated by the median rank approach [24] given by

$$F(t_i) = \frac{i-0.3}{n+0.4} \tag{22}$$

From eq. (21), the shape parameter β is directly given by the slope b_1, and the scale parameter η is given by

$$\eta = \exp\left\{\frac{-b_0}{\beta}\right\} = \exp\left\{\mu_x - \frac{\mu_y}{\beta}\right\} \tag{23}$$

Additionally, it is necessary to note that in eq. (21) $y=ln(-ln(1-F(t)))$ represents the behavior of the Gumbel distribution (negative skew), and that once the Weibull parameters β and η are known, the expected data can be estimated as

$$\ln(t_i) = \frac{y_i}{\beta} + \ln(\eta) = \frac{\ln\left(-\ln\left(1-F(t_i)\right)\right)}{\beta} + \ln(\eta) \tag{24}$$

Clearly, from eq. (24), the $ln(t_i)$ value depends only on *y*. And since from the double logarithm the *y* values before $F(t)=1-e^{-1}=0.6321$ *are always* negatively skewed, then in order for that data follows a Weibull distribution, its logarithm has to be negatively skewed as well. This fact implies that in the Weibull case, $Sxx_{-}>Sxx_{+}$ is always true. On the other hand, the analysis for the lognormal distribution is as follows.

4.2. Lognormal linear model

Since for the lognormal distribution the cumulative density function is given by

$$F(t) = 1 - R(t) = \Phi\left(\frac{\ln t - \mu_x}{\sigma_x}\right) \tag{25}$$

Then the lognormal linear relationship is given by

$$\begin{aligned} y_i &= \Phi^{-1}\left(F(t_i)\right) = Z_i \\ &= -\frac{1}{\sigma_x}\mu_x + \frac{1}{\sigma_x}ln(t_i) = b_0 + b_1 x_i \end{aligned} \tag{26}$$

where μ_x is given by $\mu_x=-b_0/b_1$, and σ_x is given by $\sigma_x=1/b_1$ and $F(t_i)$ is estimated as in eq. (22). On the other hand, μ_x and σ_x can respectively be estimated directly from the data as

$$\mu_x = \frac{1}{n}\sum_{i=1}^{n}\ln t_i \tag{27}$$

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n}(\ln t_i - \mu_x)^2}{n-1}} \tag{28}$$

From eq. (26) $y=\Phi^{-1}(F(t))$ represents the behavior of the Normal distribution (*symmetrical behavior*). Thus, once the lognormal parameters μ_x and σ_x are known, the expected data can be estimated from eq.(26) as follows:

$$\ln(t_i) = \sigma_x y_i + \mu_x = \sigma_x \Phi^{-1}\left(F(t_i)\right) + \mu_x \tag{29}$$

On the other hand, since $ln(t_i)$ in eq. (29) follows a normal distribution, then its behavior is always symmetrical, and as a consequence of the lognormal case, the contribution to the *Sxx* variable is equivalent before and after μ_x. In other words, in the lognormal case, $Sxx_{-}=Sxx_{+}$. Now that it has been seen that for the Weibull distribution $Sxx_{-}>Sxx_{+}$, and that for the lognormal distribution $Sxx_{-}=Sxx_{+}$, let us describe the linear regression analysis to show that the ratio of the Weibull and lognormal regression coefficients efficiently represents the Sxx_{-} and Sxx_{+} behavior.

4.3. Multiple linear regression analysis

In order to discriminate between the Weibull and lognormal distributions, first, the Weibull parameters of eq. (21) and the lognormal parameters of eq. (26) have to be estimated by using linear regression analysis as follows

$$b_0 = \mu_y - b_1\mu_x \tag{30}$$

$$b_1 = \frac{\sum_{i=1}^{n} y_i(x_i - \mu_x)}{\sum_{i=i}^{n}(x_i - \mu_x)^2} = \frac{Sxy}{Sxx} \tag{31}$$

The related multiple determination coefficient (R^2), is

$$R^2 = \frac{b_1 \sum_{i=1}^{n} y_i(x_i - \mu_x)}{\sum_{i=1}^{n} (y_i - \mu_y)^2} = \frac{b_1 Sxy}{Syy} \tag{32}$$

Thus, since from eq. (30) and eq. (31), we observe that the estimated coefficients are based on the *key variable Sxx*, then we conclude that the regression coefficients b_0 and b_1 represent the *Sxx* behavior also. Based on these parameters, the proposed method is outlined in the next section.

5. Proposed method

The proposed method is based on the fact that the critical characteristic to discriminate between the Weibull and the lognormal distributions is the *Sxx* contribution to the log standard deviation σ_x. Thus, in order to present the steps of the proposed method to discriminate between the Weibull and lognormal distributions, it is necessary first to show that via *MLR*, the regression coefficients (slopes) b_{1ln}/b_{1w} ratio completely incorporates the negative skew and the symmetrical behavior of the observed data, and that the multiple linear regression coefficient R^2 completely depends on the b_{1ln}/b_{1w} ratio.

5.1. *The ratio b_{1ln}/b_{1w} efficiently capture the Sxx behavior*

The analysis for the Weibull and lognormal distributions is given below.

5.1.1. Weibull analysis

In order to show that the regression coefficients (slopes) b_{1ln}/b_{1w} ratio completely incorporates the skew behavior of the Weibull distribution represented by *Sxx*, it is necessary to first show that based on the b_{1ln}/b_{1w} ratio given by

$$\frac{b_{1ln}}{b_{1w}} = \frac{\sum_{i=1}^{n} y_{iln}(x_i - \mu_x)}{\sum_{i=1}^{n} y_{iw}(x_i - \mu_x)} = \frac{Sxy_{ln}}{Sxy_w} \tag{33}$$

For the Weibull distribution, $Sxy_w > Sxy_{ln}$. To observe this, it should be remembered that because the Weibull response variable y_w given by $y_w = ln[-ln(1-F(t))]$ is higher weighted in the initial values (lower percentiles), and because for Weibull data, Sxx_- tends to be greater than Sxx_+, then the impact of Sxx_{--} over Sxy_w given by $Sxy_w = y_w(x-\mu)$ is higher in the initial values. As should be noted, this fact implies that when data follows a Weibull distribution, the difference between Sxy_w and Sxy_{ln} tends to be higher. Likewise, from eq. (33), this fact implies that for Weibull data the b_{1ln}/b_{1w} ratio or Sxy_{ln}/Sxy_w decreases.

5.1.2. Lognormal analysis

In the lognormal case, because the lognormal response variable y_{ln}, given by $y_{ln} = \Phi^{-1}(F(t))$, is symmetrical around the 50th percentile, then for lognormal data Sxx_{-} it tends to be Sxx_+ (see sec 3.1). As a consequence, the impact of Sxx_{--} on $Sxy_{ln} = y_{ln}(x-\mu)$ is lower than that of the Weibull distribution. This fact implies that for lognormal data, the difference between Sxy_w and Sxy_{ln} tends to be lower than when data is Weibull. As a result of this lower impact, when data is lognormal in eq. (33), the b_{1ln}/b_{1w} ratio or its equivalent Sxy_{ln}/Sxy_w increases.

Thus, because based on the *Sxx* behavior, for Weibull data the b_{1ln}/b_{1w} ratio decreases, and for lognormal, data it increases, then we conclude that because $Sxy = y(x-\mu)$ clearly captures the behavior of *Sxx*, then the b_{1ln}/b_{1w} ratio efficiently captures the behavior of *Sxx* also.

Now it will be shown that because the R^2 index depends only on the b_{1ln}/b_{1w} ratio, then it also captures the behavior of *Sxx*. Consequently, the R^2 index can also be used to discriminate between the Weibull and the lognormal distributions.

5.2. *The R^2 index is completely defined by the b_{1ln}/b_{1w} ratio*

In order to show that the R^2 index is completely defined for the b_{1ln}/b_{1w} ratio, it will be first be noted that based on eq. (32), the relationship between b_{1w} parameter and the Weibull R_w index and the relationship between the b_{1ln} parameter and the lognormal R_{ln} index can be formulated by the following relation

$$b_1 = \frac{R^2 Syy}{Sxy} \tag{34}$$

Secondly, in doing this it should be observed that by taking away $Sxy = b_1 Sxx$ from eq. (31), and by replacing it in eq. (34), b_1 is directly related with σ_x, σ_y and R^2, as follows

$$b_1 = \frac{R^2 Syy}{b_1 Sxx} = \frac{R\,\sigma_y}{\sigma_x} \tag{35}$$

where

$$\sigma_y = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \mu_y)^2}{n-1}} \tag{36}$$

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{n-1}} \tag{37}$$

Thus, from eq. (35), the Weibull b_{1w} and R_w values are related with the lognormal b_{1ln} and R_{ln} values as follows

$$b_{1w} = \frac{R_w\,\sigma_{yw}}{\sigma_x} \tag{38}$$

$$b_{1ln} = \frac{R_{ln}\,\sigma_{yln}}{\sigma_x} \tag{39}$$

Next, it will be shown that because the R_{ln}/R_w ratio depends only on the b_{1ln}/b_{1w} ratio, then the R^2 index can be used to efficiently discriminate between the Weibull and the lognormal distributions. Having done this, it should also be noted from eq. (38) and eq. (39) that σ_x is the standard deviation of the data logarithm, and that it is the same for both distributions. This fact ($\sigma_x = \sigma_x$) implies from eq. (38) and eq. (39) that

$$\frac{R_w\,\sigma_{yw}}{b_{1w}} = \frac{R_{ln}\,\sigma_{yln}}{b_{1ln}} \tag{40}$$

Therefore, based in eq. (40), the relationship between the R_{ln} and the R_w indices is given by

$$R_{ln} = R_w \left(\frac{b_{1ln}}{b_{1w}}\right)\left(\frac{\sigma_{y\,w}}{\sigma_{y\,ln}}\right) \tag{41}$$

And because the $\sigma_{y\,w}/\sigma_{y\,ln}$ ratio is constant in the analysis, we conclude that the R_{ln}/R_w ratio depends only on the b_{1ln}/b_{1w} ratio. Consequently, the R^2 index is efficient to discriminate between the Weibull and the lognormal distributions. Additionally, it is important to highlight that the σ_{yw}/σ_{yln} ratio in eq. (41) is constant also, and that this is so because σ_y defined in eq. (36) depends only on the sample size *n*. Thus, once *n* is known (or selected, see [25] eq. (13)), σ_y is constant.

5.3. Steps of the proposed method

Because based on the observed data, the R^2 index efficiently represents the Sxx behavior, then based on the observed data, the steps of the proposed method to discriminate between the Weibull and the lognormal distributions are as follows.

By using the Weibull y vector defined in eq. (21) (or the lognormal y vector defined in eq. (26)) and the observed data logarithm (ln(t)=x), the Weibull (or lognormal) correlation is estimated as $Sxy=\sum y_i(x_i-\mu_x)$.

1) From the logarithm of the observed data, estimate the variance of *x* as $Sxx=\sum(x_i-\mu_x)^2$.
2) By using the Weibull (or lognormal) *Sxy* value from step 1 and the *Sxx* value from step 2 into eq. (31), estimate the Weibull (or lognormal) slope b_1 coefficient.
3) By using the Weibull *y* vector defined in eq. (21) (or the lognormal *y* vector defined in eq. (26)), estimate the Weibull (or lognormal) variance of *y* as $Syy=\sum(y_i-\mu_y)^2$.
4) By using the Weibull (or lognormal) slope b_1 coefficient from step 3, Weibull (or lognormal) *Sxy* value from step 1 and the Weibull (or lognormal) *Syy* value from step 4 into eq. (32), estimate the Weibull (or lognormal) coefficient R^2.
5) Compare the Weibull and the lognormal R^2 indices, select the distribution with higher R^2 value. If $R_w^2>R_{ln}^2$ select Weibull distribution; otherwise select lognormal distribution.

6. An application

The efficiency of the R^2 index to discriminate between the Weibull and the lognormal distribution is shown in a stress-strength analysis by using data in section 1. Table 1 Data corresponds to the stress load in a machine that uses a plunger to press a shaft into a bushing. Table 2 Data corresponds to the strength of the plunger when it is subjected to compression loads [26]. Thus, the selection of the stress distribution by using the proposed method is as follows.

6.1. Stress data analysis

From the stress observed data shown in Table 4, we note that because 1) the $\sigma_x \approx CV$ ($\sigma_x=CV=0.0055$), 2), μ_x is located near the 50th percentile, and 3) $Sxx_{-} = 53\% \approx Sxx_{+} = 47\%$. Then, from section 3.1, it is reasonable to expect that the lognormal distribution represents the data.

The above statement is verified by applying the proposed method to the Table 1 data. The required values *Sxy*, *Sxx* and *Syy* to apply the method are estimated by applying the *MLR* analysis to the stress data. The values for the Weibull and the lognormal distributions are given in Table 4.

Thus, by using data of Table 4, the lognormal analysis is as follows. From **step 1**, $Sxy_{ln}=7.1714$ (column 9). From **step 2**, $Sxx=1.3412$ (column 10). From **step 3**, and eq. (31), $b_{1ln}=5.3471$. From **step 4**, $Syy_{ln}=39.4812$ (column 13). Therefore, from **step 5**, and eq. (32), $R_{ln}^2=0.9712$.

Similarly, by applying the proposed method to the Weibull distribution, we have from **step 1**, $Sxy_w=8.8996$ (column 8). From **step 2**, *Sxx*, as in the lognormal case, is also $Sxx=1.3412$ (column 10). From **step 3**, and eq. (31), $b_{1w}=6.6357$. From **step 4**, $Syy_w=61.9775$ (column 12). Therefore, from **step 5**, and eq. (32), $R_w^2=0.9528$.

Finally, as expected, by comparing the Weibull and lognormal R^2 indices, in **step 6**, we have that $R_{ln}^2=0.9712>R_w^2=0.9528$. Thus, we conclude that the failure governing the stress distribution is the lognormal distribution. On the other hand, the selection of the strength distribution by using the proposed method is as follows.

6.2. Strength data analysis

The strength data is given in Table 5. From this data, we note that while 1) the $\sigma_x \approx CV$ ($\sigma_x=CV=0.0077$) and 2) the μ_x is located near the 50th percentile, 3) the Sxx_{-} contribution is greater than the Sxx_{+} contribution $Sxx_{-}=61\%> Sxx_{+}=39\%$. Thus, because from section 3.3 the characteristics of the lognormal distribution are not completely met, then we conclude that data can be better represented by the Weibull distribution. However, the estimation of the R^2 index is necessary. When doing this, the values of *Sxy*, *Sxx* and *Syy* are estimated by using an *MLR* analysis of the strength data. The *MLR* analyses for the Weibull and the lognormal distributions are summarized in Table 5.

By using Table 5 data and by applying the proposed method for the lognormal distribution, we have that, from **step 1**, $Sxy_{ln}=1.9402$ (column 9). From **step 2**, $Sxx=0.3298$ (column 10). From **step 3**, and eq. (31), $b_{1ln}=5.8820$. And from **step 4**, $Syy_{ln}=12.2451$ (column 13). Therefore, from **step 5**, and eq. (32) $R^2=0.9319$.

Similarly, by applying the proposed method for the Weibull distribution, we have that, from **step 1**, $Sxy_w=2.4320$ (column 8). From **step 2**, *Sxx* as in the lognormal case, is $Sxx=0.3298$ (column 10). From **step 3**, and eq. (31), $b_{1w}=7.3730$. And from **step 4**, $Syy_w=18.5330$ (column 12). Therefore, from **step 5**, and eq. (32), $R_w^2=0.9675$.

Finally, by comparing the R^2 indices as in **step 6**, we have that $R_w^2=0.9675>R_{ln}^2=0.9319$. Thus, the failure governing the strength distribution is the Weibull distribution.

As a summary, because the stress data follows a lognormal distribution and the strength data follows a Weibull distribution, then for the stress-strength analysis the lognormal-Weibull combination has to be used. Therefore, from Table 3, the corresponding lognormal-Weibull reliability is $R(t)=0.9860$. Finally, the effect that a wrong selection of the distribution has over the estimated reliability is given in Table 3. Although the reliability values given in Table 3 were estimated by using the Weibull++ software, the next section provides the formulas to estimate such values.

Table 4
Load data analysis

1	2	3	4	5	6	7	8	9	10	11	12	13
th perc	***i***	***T***	***x=ln(t)***	***F(t)***	y_w	y_{ln}	Sxy_w	Sxy_{ln}	***Sxx***	***∑Sxx***	Syy_w	Syy_{ln}
	1	9.6	2.2618	0.0161	-4.1190	-2.1412	1.8617	0.9678	0.2043		12.6740	4.5847
	2	9.6	2.2618	0.0392	-3.2199	-1.7604	1.4553	0.7956	0.2043		7.0807	3.0990
	3	12.3	2.5096	0.0622	-2.7453	-1.5365	0.5604	0.3136	0.0417		4.7799	2.3607
	4	12.4	2.5177	0.0853	-2.4179	-1.3706	0.4740	0.2687	0.0384		3.4557	1.8785
	5	12.4	2.5177	0.1083	-2.1661	-1.2356	0.4246	0.2422	0.0384		2.5830	1.5268
	6	12.5	2.5257	0.1313	-1.9604	-1.1201	0.3686	0.2106	0.0353		1.9641	1.2546
	7	12.6	2.5337	0.1544	-1.7857	-1.0178	0.3215	0.1832	0.0324		1.5049	1.0360
	8	12.7	2.5416	0.1774	-1.6332	-0.9252	0.2811	0.1593	0.0296		1.1539	0.8561
	9	12.7	2.5416	0.2005	-1.4974	-0.8400	0.2577	0.1446	0.0296		0.8806	0.7056
	10	13.2	2.5802	0.2235	-1.3745	-0.7604	0.1835	0.1015	0.0178		0.6651	0.5782
	11	13.5	2.6027	0.2465	-1.2620	-0.6854	0.1401	0.0761	0.0123		0.4943	0.4698
	12	13.8	2.6247	0.2696	-1.1579	-0.6141	0.1031	0.0547	0.0079		0.3587	0.3771
	13	14	2.6391	0.2926	-1.0607	-0.5457	0.0792	0.0408	0.0056		0.2518	0.2978
	14	14.2	2.6532	0.3157	-0.9694	-0.4798	0.0586	0.0290	0.0037		0.1685	0.2303
	15	14.4	2.6672	0.3387	-0.8829	-0.4160	0.0411	0.0193	0.0022		0.1050	0.1730
	16	14.4	2.6672	0.3618	-0.8007	-0.3538	0.0372	0.0165	0.0022		0.0584	0.1252
	17	14.4	2.6672	0.3848	-0.7220	-0.2929	0.0336	0.0136	0.0022		0.0266	0.0858
	18	14.6	2.6810	0.4078	-0.6463	-0.2331	0.0211	0.0076	0.0011		0.0076	0.0543
	19	14.6	2.6810	0.4309	-0.5733	-0.1741	0.0188	0.0057	0.0011		0.0002	0.0303
	20	14.7	2.6878	0.4539	-0.5026	-0.1158	0.0130	0.0030	0.0007		0.0032	0.0134
	21	14.9	2.7014	0.4770	-0.4337	-0.0578	0.0054	0.0007	0.0002	∑=0.7109	0.0157	0.0033
	22	15	2.7081	0.5000	-0.3665	0.0000	0.0021	0.0000	0.0000	53%	0.0370	0.0000
50	23	15.2	2.7213	0.5230	-0.3007	0.0578	-0.0023	0.0004	0.0001		0.0667	0.0033
	24	15.3	2.7279	0.5461	-0.2359	0.1158	-0.0033	0.0016	0.0002		0.1043	0.0134
	25	15.3	2.7279	0.5691	-0.1721	0.1741	-0.0024	0.0025	0.0002		0.1497	0.0303
	26	15.7	2.7537	0.5922	-0.1088	0.2331	-0.0043	0.0093	0.0016		0.2026	0.0543
63.21	27	15.9	2.7663	0.6152	-0.0460	0.2929	-0.0024	0.0154	0.0028		0.2631	0.0858
	28	16	2.7726	0.6382	0.0167	0.3538	0.0010	0.0208	0.0035		0.3313	0.1252
	29	16.1	2.7788	0.6613	0.0794	0.4160	0.0052	0.0271	0.0042		0.4075	0.1730
	30	16.5	2.8034	0.6843	0.1424	0.4798	0.0128	0.0430	0.0080		0.4919	0.2303
	31	16.6	2.8094	0.7074	0.2061	0.5457	0.0197	0.0522	0.0092		0.5853	0.2978
	32	16.8	2.8214	0.7304	0.2707	0.6141	0.0291	0.0661	0.0116		0.6883	0.3771
	33	16.8	2.8214	0.7535	0.3366	0.6854	0.0362	0.0738	0.0116		0.8021	0.4698
	34	16.9	2.8273	0.7765	0.4044	0.7604	0.0459	0.0864	0.0129		0.9280	0.5782
	35	17.9	2.8848	0.7995	0.4745	0.8400	0.0812	0.1437	0.0293		1.0679	0.7056
	36	17.9	2.8848	0.8226	0.5477	0.9252	0.0937	0.1583	0.0293		1.2246	0.8561
	37	18	2.8904	0.8456	0.6251	1.0178	0.1104	0.1798	0.0312		1.4019	1.0360
	38	18.1	2.8959	0.8687	0.7080	1.1201	0.1290	0.2041	0.0332		1.6053	1.2546
	39	19.1	2.9497	0.8917	0.7988	1.2356	0.1885	0.2916	0.0557		1.8435	1.5268
	40	19.7	2.9806	0.9147	0.9010	1.3706	0.2405	0.3658	0.0712		2.1315	1.8785
	41	20	2.9957	0.9378	1.0214	1.5365	0.2880	0.4333	0.0795		2.4977	2.3607
	42	20.1	3.0007	0.9608	1.1755	1.7604	0.3374	0.5052	0.0824	∑=0.6302	3.0084	3.0990
	43	22.3	3.1046	0.9839	1.4176	2.1412	0.5541	0.8369	0.1528	47%	3.9067	4.5847
	μ	15.319	2.7137		-0.5590	0.0000						
	Σ	2.6934	0.1787									
	CV	0.1758				∑	8.8996	7.1714	1.3412		61.9775	39.4812

Source: The authors

Table 5
Strength data analysis

1	2	3	4	5	6	7	8	9	10	11	12	13
th perc	***I***	***t***	***x=ln(t)***	***F(t)***	y_w	y_{ln}	Sxy_w	Sxy_{ln}	***Sxx***	***∑Sxx***	Syy_w	Syy_{ln}
	1	21	3.0445	0.0455	-3.0679	-1.6906	0.8862	0.4884	0.0834		6.4067	2.8582
	2	22	3.0910	0.1104	-2.1458	-1.2245	0.5200	0.2967	0.0587		2.5892	1.4993
	3	23	3.1355	0.1753	-1.6463	-0.9333	0.3258	0.1847	0.0392		1.2311	0.8711
	4	25	3.2189	0.2403	-1.2918	-0.7055	0.1479	0.0808	0.0131		0.5701	0.4977
	5	26	3.2581	0.3052	-1.0103	-0.5095	0.0761	0.0384	0.0057		0.2242	0.2596
	6	27	3.2958	0.3701	-0.7717	-0.3315	0.0290	0.0124	0.0014	∑=0.2015	0.0552	0.1099
	7	28	3.3322	0.4351	-0.5603	-0.1635	0.0007	0.0002	0.0000	61%	0.0006	0.0267
50	8	30	3.4012	0.5000	-0.3665	0.0000	-0.0249	0.0000	0.0046		0.0290	0.0000
	9	30	3.4012	0.5649	-0.1836	0.1635	-0.0125	0.0111	0.0046		0.1247	0.0267
63.21	10	31	3.4340	0.6299	-0.0061	0.3315	-0.0006	0.0334	0.0101		0.2815	0.1099
	11	31	3.4340	0.6948	0.1713	0.5095	0.0172	0.0513	0.0101		0.5012	0.2596
	12	32	3.4657	0.7597	0.3549	0.7055	0.0470	0.0934	0.0175		0.7950	0.4977
	13	32	3.4657	0.8247	0.5545	0.9333	0.0734	0.1235	0.0175		1.1908	0.8711
	14	33	3.4965	0.8896	0.7902	1.2245	0.1289	0.1997	0.0266	∑=0.1283	1.7606	1.4993
	15	34	3.5264	0.9545	1.1285	1.6906	0.2178	0.3262	0.0372	39%	2.7730	2.8582
	μ	28.3333	3.3334		-0.5367	0.0000						
	Σ	4.1519	**0.1535**									
	CV	**0.1465**				∑	2.4320	1.9402	0.3298		18.5330	12.2451

Source: The authors

7. Stress-Strength reliability

The stress-strength reliability values of Table 3 were estimated as follow. For the lognormal-lognormal stress-strength, the formulation given in eq. (42) was used

$$R = \phi\left(\frac{\mu_{xS}-\mu_{xs}}{\sqrt{\sigma_S^2+\sigma_s^2}}\right) \tag{42}$$

For the lognormal-Weibull stress-strength, the formulation given in eq. (43) was used

$$R = \int_0^\infty \frac{1}{s\sigma_x\sqrt{2\pi}} \exp\left\{-\left[\frac{1}{2}\left(\frac{\ln s-\mu_x}{\sigma_x}\right)^2 + \left(\frac{s}{\eta_S}\right)^{\beta_S}\right]\right\} ds \tag{43}$$

For the Weibull-lognormal stress-strength, the formulation given in eq. (44) was used

$$R = 1-\int_0^\infty \frac{1}{S\sigma_{xS}\sqrt{2\pi}} \exp\left\{-\left[\frac{1}{2}\left(\frac{\ln S-\mu_{xS}}{\sigma_{xS}}\right)^2 + \left(\frac{S}{\eta_s}\right)^{\beta_s}\right]\right\} dS \tag{44}$$

Finally, for the Weibull-Weibull stress-strength, the formulation given in eq. (45) was used

$$R = 1-\int_0^\infty \exp\left\{-\left[w + \left(\frac{\eta_S w^{1/\beta_S}}{\eta_s}\right)^{\beta_s}\right]\right\} dw \tag{45}$$

Where

$$w = \left(\frac{S}{\eta_S}\right)^{\beta_S} \tag{46}$$

8. Conclusions

The reliability analysis for the Weibull and the lognormal distributions is performed by using the data logarithm. For the Weibull distribution, the logarithm data is negatively skewed. For the lognormal distribution, the logarithm data is symmetrical. Because for the Weibull distribution, the contribution to the variance before the mean is always greater than the contribution after the mean [$Sxx_{-}>Sxx_{+}$], then this behavior is used to discriminate between the Weibull and the lognormal distributions. Since the b_{1ln}/b_{1w} ratio efficiently represents the contribution behavior, and since the R^2 index depends only on this ratio, then the R^2 index is indeed efficient to discriminate between the Weibull and the lognormal distributions. Finally, it is important to highlight that when in the observed data, $\sigma_x=CV$, μ_x tends to the 50th percentile and $Sxx_{-}=Sxx_{+}$, then the lognormal distribution can be directly fitted. And when for the observed data, $\sigma_x\neq CV$, μ_x tends to the 36.21th percentile and $Sxx_{-}>Sxx_{+}$, then the Weibull distribution can be directly fitted.

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