Stability of slender columns on an elastic foundation with generalised end conditions

Estabilidad de columnas esbeltas sobre fundación elástica con condiciones de apoyo generalizadas

J. D. Aristizabal-Ochoa[1]

ABSTRACT

Slender columns' lateral stability under compressive axial loads is presented, with uninhibited, partially inhibited and totally inhibited end side-sway, including the effects of semi-rigid connections and a uniformly distributed lateral elastic foundation (Winkler's type) throughout its entire span. The proposed classification of prismatic columns on an elastic foundation and the corresponding stability equations are general and relatively simple to apply, yielding exact results when compared to other analytical methods. The buckling load was obtained by making the determinant of a 4 x 4 matrix equal to zero for columns having side-sway uninhibited or partially inhibited at both ends, and of a 3 x 3 matrix for columns having side-sway inhibited at one or both ends. The effect of semi-rigid connections on the buckling load of five classical column cases is fully discussed and the results compared to those arising from other analytical methods.

Keywords: bracing, buckling, column, elastic foundation, pile, semi-rigid connection, stability.

RESUMEN

Se presenta de una manera clásica la estabilidad lateral de columnas esbeltas bajo cargas axiales de compresión; con derivas en los extremos desinhibidas, inhibidas parcialmente y totalmente inhibidas, incluyendo los efectos de las conexiones semirrígidas y una fundación elástica lateral y uniformemente distribuida (tipo Winkler) a lo largo de toda su luz. La clasificación propuesta en las columnas prismáticas, sobre fundación elástica y las ecuaciones correspondientes de estabilidad son generales y relativamente simples de aplicar; obteniéndose resultados exactos cuando se comparan con otros métodos analíticos. La carga de pandeo se obtiene igualando a cero el valor del determinante de una matriz de 4 x 4, para columnas con deriva lateral desinhibida o parcialmente inhibida en ambos extremos, y de una matriz de 3 x 3 para columnas con deriva lateral inhibida en uno o ambos extremos, respectivamente. Los efectos de las conexiones semirrígidas sobre la carga de pandeo, de cinco casos de columna clásicos, son discutidos y los resultados son comparados con los de otros métodos analíticos.

Palabras clave: arriostramiento, pandeo, columnas, fundación elástica, pilas, conexiones semirrígidas y estabilidad.

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Introduction

The stability and second-order static analysis of columns laterally supported on an elastic foundation in generalised end conditions are of importance whit structural engineering. As noted by Hetenyi (1967), there are many applications for such analysis, such as the construction of pile foundations, railroad tracks, bridges, thin revolution shells (pressure vessels, boilers and containers) and large-span cylindrical shells and domes. For instance, a symmetrically loaded longitudinal element of a cylindrical tube regarding its axis can be treated as a beam on an elastic foundation. Thus, the proposed approach can be used for studying a wide range of problems related to elastic stability and second-order static analysis, being of particular importance in structural and geotechnical engineering, specifically in soil-structure interaction problems dealing with piles, drilled shafts, caissons and/or piers subjected to a combinations of axial, moment and lateral loads.

Hetenyi (1967) outlined the classical procedure for resolving the elastic stability of prismatic columns having hinged and clamped connections on an elastic foundation presenting the "exact" solution for particular cases, such as columns having free-free, hinged-hinged and clamped-clamped end conditions. Timoshenko and Gere (1961) used a similar approach to resolving other cases of interest, such as a column under a uniformly-distributed axial load on an elastic foundation. West and Mafi (1984) determined the eigen-values for columns on elastic supports using an initial-value numerical method. Razaqpur (1986) presented the stiffness and equivalent joint load matrices for a column finite element resting on a Winkler-type elastic foundation using "exact" shape func-

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tions. Cheng and Pantelides (1988) derived the differential equations, stiffness coefficients and fixed-end forces of a column on an elastic foundation, including bending and shear deformation. Ang and Wang (1990), as well as Wang, Xiang and Kitipornchai (1991) used finite elements for studying the buckling of columns on an elastic foundation. More recent papers on the stability of beam-columns resting on an elastic foundation, including two and three parameter foundation models, are available in the technical literature, such as those by Struthers and Jayaraman (2010), Morfidis (2007, 2010), Avramidis and Morfidis (2006) and Onu (2006).

Aristizabal-Ochoa (1994, 1996 and 1997) has presented the effects of semi-rigid connections on the stability and second-order analysis of columns and framed structures. Areiza-Hurtado *et al*., (2005) evaluated second-order stiffness matrix and loading vector with generalised end conditions regarding a Timoshenko column on a Winkler foundation. More recently, Arboleda-Monsalve *et al*., (2008) and Zapata-Medina *et al*., (2010) have studied shear effects on elastic stability and the dynamics of beams-columns on an elastic foundation. However, a complete set of stability equations for columns laterally supported on a continuous elastic foundation having generalised end conditions subjected to axial end loads are not available in the technical literature.

This publication's main objective was thus to present complete classification and corresponding stability equations for prismatic columns subjected to end axial loads, including the effects of semi-rigid bending connections, bracing stiffness at both ends and a uniformly distributed elastic support (Winkler type). The buckling load was obtained by making the determinant of a 4×4 matrix equal to zero for columns having side-sway uninhibited or partially inhibited at both ends, and of a 3×3 matrix for columns having side-sway inhibited at one or both ends. The present paper is restricted to an elastic static analysis of a single slender column, with three different lateral bracing types, leading to manageable analytical solutions. The effects of shear deformation on the member's buckling capacity are neglected. The importance of shear effects on the static and dynamic response of short columns or columns with low shear stiffness [Areiza-Hurtado et al (2005), Arboleda-Monsalve *et al*., (2008), and Zapata-Medina *et* al. (2010)].

Structural model

Assumptions

A prismatic element is considered that connects points A and B (Fig. 1). AB consists of a column A'B' and lumped flexural connectors AA' and BB' at the top and bottom ends, respectively. It is assumed that:

1) A'B' is made from a homogeneous linear elastic material having a modulus of elasticity E and continuously supported along its span by a uniformly distributed elastic foundation (Winkler type) of magnitude k_s;

2) The centroid axis of the column is initially a straight line in which the elastic centroid coincided with the shear centre of the cross section. Initial geometrical imperfections in the column were not considered;

3) The column is subject to end compressive axial load P applied along its centroid axis and also subjected to overturning moments and shear forces M_a, V_a and M_b, V_b at ends A' and B', respectively;

4) End lateral sway is partially inhibited by springs S_a and S_b located at A' and B', respectively;

5) End rotations are partially inhibited by flexural springs κ_a and κ_b located at A' and B', respectively; and

6) The second-order analysis in the next section is intended to be in the small deflection range (commonly referred to as a "linearised" approach). It should be pointed out that results are valid as long as the lateral deflection of the column remained small.

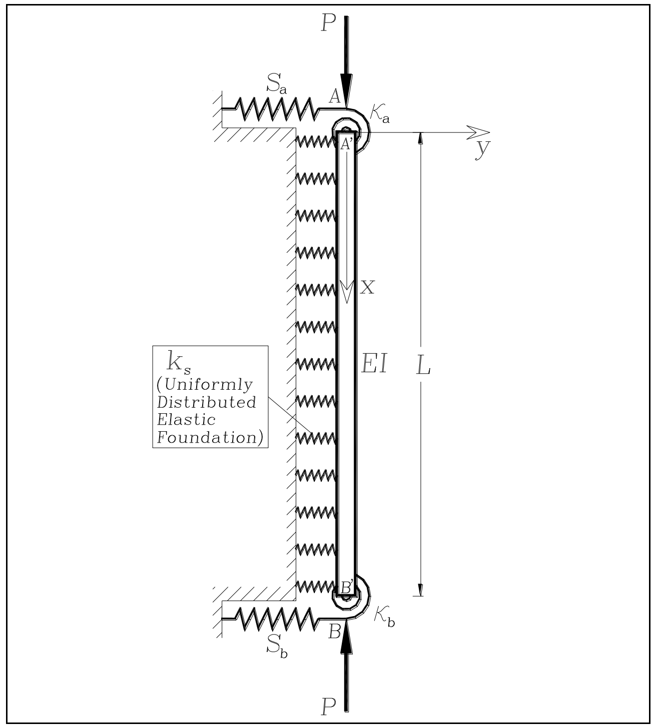

Figure 1. Model of a beam-column resting on an elastic foundation ks with side-sway partially inhibited by end bracing stiffness Sa and Sb and rotationally restrained by end flexural springs κa and κb

Flexural connectors AA' and BB' had stiffness κ_a and κ_b (with units in force-distance/radian), respectively. The ratios $R_a = \kappa_a/(EI/L)$ and $R_b = \kappa_b/(EI/L)$ were denoted as the stiffness indices of the flexural connections, where I = column moment of inertia about the principal axis in question and L = column span. The stiffness indices varied from zero (i.e. $R_a = R_b = 0$) for hinged connections to infinity (i.e., $R_a = R_b = \infty$) for rigid connections or clamped ends.

The main difference between the classical solution presented in this paper and that available in the pertinent technical literature (Timoshenko and Gere, 1961; Hetenyi, 1967; Scott, 1981) is that the solution for the model shown in Fig. 1 included the effects of semi-rigid connections and could be used for both stability and second-order analysis of columns on an elastic foundation subjected to any end load combination.

For convenience, the following two parameters were introduced:

$$\rho_a = 1/(1+3/R_a);\ \text{and}\ \rho_b = 1/(1+3/R_b) \qquad (1a\text{-}b)$$

where ρ_a and ρ_b were called fixity factors. For perfectly hinged connections, both fixity factor ρ and stiffness index R were zero; but, for perfectly rigid connections, fixity factor was 1 and the stiffness index was infinity. Since fixity factor in the elastic range can only vary from 0 to +1 (while stiffness index R may vary form o to +∞), it was more convenient to use it in the analysis of structures having semi-rigid connections, as suggested by Monforton

(1963). Factor 3 in equations (1a-b) came from the slope-deflection analysis of a beam having semi-rigid connections at both ends.

Deriving lateral equilibrium equations

The second-order governing of a differential equation for a prismatic column laterally supported on an elastic foundation k_s subjected to compressive axial loads P at both ends as well as to overturning moments and shears M_a, V_a and M_b, V_b at A' and B', respectively, is as follows:

$$EI\frac{d^4y}{dx^4}+P\frac{d^2y}{dx^2}+k_s y=0 \tag{2}$$

Eq. (2) is the equation governing a column laterally supported on an elastic foundation subject to compressive axial loads P at both ends in deformed conditions. Its solution is known as *second-order analysis* (see Chen and Lui, 1987, p. 2).

For a column having **side-sway partially inhibited** at both ends (Fig. 2a), the solution to Eq. (2) has to be subject to the following four boundary conditions:

1) At $x= L$ $$EI\frac{d^2y}{dx^2}+\kappa_b\frac{dy}{dx}=M_b \tag{3a}$$

2) At $x= 0$ $$-EI\frac{d^2y}{dx^2}+\kappa_a\frac{dy}{dx}=M_a \tag{3b}$$

3) At $x= L$ $$-EI\frac{d^3y}{dx^3}-P\frac{dy}{dx}+S_b y=V_b \tag{3c}$$

4) At $x= 0$ $$EI\frac{d^3y}{dx^3}+P\frac{dy}{dx}+S_a y=V_a \tag{3d}$$

The solution to Eq. (2) depends on the relative value of compressive P regarding $\sqrt{4k_s EI}$, as explained by Hetenyi (1967). There were two types of solution in the compressive range (i.e., for $P > 0$), as follows:

for $0 \le P < \sqrt{4k_s EI}$

$$y = (C_1e^{\alpha x}+ C_2e^{-\alpha x}) \cos\beta x + (C_3e^{\alpha x} + C_4e^{-\alpha x}) \sin\beta x \tag{4}$$

where: $\alpha=\sqrt{\sqrt{\frac{k_s}{4EI}}-\frac{P}{4EI}}$; and $\beta=\sqrt{\sqrt{\frac{k_s}{4EI}}+\frac{P}{4EI}}$

for $P > \sqrt{4k_s EI}$

$$y= C_1 \cos\gamma x + C_2 \cos\varphi x + C_3 \sin\gamma x + C_4 \sin\varphi x \tag{5}$$

where: $\gamma=\sqrt{\frac{P}{2EI}-\sqrt{\left(\frac{P}{2EI}\right)^2-\frac{k_s}{EI}}}$; and $\varphi=\sqrt{\frac{P}{2EI}+\sqrt{\left(\frac{P}{2EI}\right)^2-\frac{k_s}{EI}}}$

The four constants C_1, C_2, C_3 and C_4 in Eqs. (4) and (5) is determined from the four natural boundary conditions given by Eqs. (3a)-(3d). These is reduced to Eqs. (6a)-(6d) after some algebraic manipulation:

$$A_{11}\, C_1 + A_{12}\, C_2 + A_{13}\, C_3 + A_{14}\, C_4 = (1-\rho_b)M_b/(EI/L) \tag{6a}$$

$$A_{21}\, C_1 + A_{22}\, C_2 + A_{23}\, C_3 + A_{24}\, C_4 = (1-\rho_a)M_a/(EI/L) \tag{6b}$$

$$A_{31}\, C_1 + A_{32}\, C_2 + A_{33}\, C_3 + A_{34}\, C_4 = V_b/EI \tag{6c}$$

$$A_{41}\, C_1 + A_{42}\, C_2 + A_{43}\, C_3 + A_{44}\, C_4 = V_a/EI \tag{6d}$$

where A_{ij} coefficients were as follows:

I) for $0\le P < \sqrt{4k_s EI}$

$$A_{11}= e^{\alpha L}\{-[3\rho_b + 2\alpha L(1-\rho_b)]\beta \sin\beta L + [3\alpha\rho_b + L(1-\rho_b)(\alpha^2-\beta^2)]\cos\beta L\} \tag{7a}$$

$$A_{12}= -e^{-\alpha L}\{[3\rho_b-2\alpha L(1-\rho_b)]\beta \sin\beta L + [3\alpha\rho_b-L(1-\rho_b)(\alpha^2-\beta^2)] \cos\beta L\} \tag{7b}$$

$$A_{13}= e^{\alpha L}\{[3\alpha\rho_b + L(1-\rho_b)(\alpha^2-\beta^2)] \sin\beta L + \beta[3\rho_b + 2\alpha L(1-\rho_b)] \cos\beta L\} \tag{7c}$$

$$A_{14}= -e^{-\alpha L}\{[3\alpha\rho_b- L(1-\rho_b)(\alpha^2-\beta^2)] \sin\beta L - [3\rho_b-2\alpha L(1-\rho_b)]\beta \cos\beta L\} \tag{7d}$$

$$A_{21}= 3\alpha\rho_a-L(1-\rho_a)(\alpha^2-\beta^2) \tag{7e}$$

$$A_{22}= -3\alpha\rho_a-L(1-\rho_a)(\alpha^2-\beta^2) \tag{7f}$$

$$A_{23}= \beta[3\rho_a-2\alpha L(1-\rho_a)] \tag{7g}$$

$$A_{24}= \beta[3\rho_a+2\alpha L(1-\rho_a)] \tag{7h}$$

$$A_{31}= e^{\alpha L}\{-[P/EI-(\beta^2-3\alpha^2)]\beta \sin\beta L + [(S_b-\alpha P)/EI-\alpha(\alpha^2-3\beta^2)] \cos\beta L\} \tag{7i}$$

$$A_{32}= e^{-\alpha L}\{[P/EI-(\beta^2-3\alpha^2)]\beta \sin\beta L + [(S_b + \alpha P)/EI+\alpha(\alpha^2-3\beta^2)] \cos\beta L\} \tag{7j}$$

$$A_{33}= e^{\alpha L}\{[(S_b-\alpha P)/EI-\alpha(\alpha^2-3\beta^2)] \sin\beta L - [P/EI-\alpha(\beta^2-3\alpha^2)]\beta \cos\beta L\} \tag{7k}$$

$$A_{34}= e^{-\alpha L}\{[(S_b+\alpha P)/EI + \alpha(\alpha^2-3\beta^2)]\sin\beta L-[P/EI-(\beta^2-3\alpha^2)]\beta \cos\beta L\} \tag{7l}$$

$$A_{41}= (S_a+\alpha P)/EI + \alpha(\alpha^2-3\beta^2) \tag{7m}$$

$$A_{42}= (As-\alpha P)/EI - \alpha(\alpha^2-3\beta^2) \tag{7n}$$

$$A_{43}= A_{44}= [P/EI-(\beta^2-3\alpha^2)]\beta \tag{7o}$$

where: $\alpha=\sqrt{\sqrt{\frac{k_s}{4EI}}-\frac{P}{4EI}}$; and $\beta=\sqrt{\sqrt{\frac{k_s}{4EI}}+\frac{P}{4EI}}$

II) for $P > \sqrt{4k_s EI}$

$$A_{11}= -[3\rho_b\gamma \sin\gamma L + L(1-\rho_b)\gamma^2 \cos\gamma L] \tag{8a}$$

$$A_{12}= -[3\rho_b\varphi \sin\varphi L + L(1-\rho_b)\varphi^2 \cos\varphi L] \tag{8b}$$

$$A_{13}= 3\rho_b\gamma \cos\gamma L - L(1-\rho_b)\gamma^2 \sin\gamma L \tag{8c}$$

$$A_{14}= 3\rho_b\varphi \cos\varphi L - L(1-\rho_b)\varphi^2 \sin\varphi L \tag{8d}$$

$$A_{21}= L(1-\rho_a)\, \gamma^2 \tag{8e}$$

$$A_{22}= L(1-\rho_a)\, \varphi^2 \tag{8f}$$

$$A_{23}= 3\rho_a\gamma \tag{8g}$$

$$A_{24}= 3\rho_a\varphi \tag{8h}$$

$$A_{31}= (S_b/EI)\cos\gamma L - (\gamma^2-P/EI)\gamma \sin\gamma L \tag{8i}$$

$$A_{32}= (S_b/EI)\cos\varphi L - (\varphi^2-P/EI)\varphi \sin\varphi L \tag{8j}$$

$$A_{33}= (S_b/EI)\sin\gamma L + (\gamma^2-P/EI)\gamma \cos\gamma L \tag{8k}$$

$$A_{34}= (S_b/EI)\sin\varphi L + (\varphi^2-P/EI)\, \varphi \cos\varphi L \tag{8l}$$

$$A_{41} = A_{42} = S_a/EI \tag{8m}$$

$$A_{43} = -(\gamma^2 - P/EI)\gamma \tag{8n}$$

$$A_{44} = -(\varphi^2 - P/EI)\varphi \tag{8o}$$

where: $\gamma = \sqrt{\dfrac{P}{2EI} - \sqrt{\left(\dfrac{P}{2EI}\right)^2 - \dfrac{k_s}{EI}}}$; and $\varphi = \sqrt{\dfrac{P}{2EI} + \sqrt{\left(\dfrac{P}{2EI}\right)^2 - \dfrac{k_s}{EI}}}$

For columns having side-sway **totally inhibited at A' and partially inhibited at B'** (Fig. 2b), the fourth natural boundary condition given by Eq. (3d) or (6d) has to be substituted by forced boundary condition $y = 0$ at A or simple $C_1 = -C_2$, which, when introduced into the other three natural boundary conditions given by Eqs. (6a)-(6c), became:

$$(A_{11} - A_{12})C_1 + A_{13}\,C_3 + A_{14}\,C_4 = (1 - \rho_b)M_b/(EI/L) \tag{9a}$$

$$(A_{21} - A_{22})C_1 + A_{23}\,C_3 + A_{24}\,C_4 = (1 - \rho_a)M_a/(EI/L) \tag{9b}$$

$$(A_{31} - A_{32})C_1 + A_{33}\,C_3 + A_{34}\,C_4 = V_b/EI \tag{9c}$$

For columns having lateral side-sway **totally inhibited at both ends A' and B'** (Fig. 2c), the third natural boundary condition given by Eqs. (3c) or (9c) has to be substituted by forced boundary conditions $y = 0$ at B. This condition became:

for $0 \le P < \sqrt{4k_s EI}$

$$(e^{2\alpha L} - 1)\cos\beta L\, C_1 + e^{2\alpha L}\sin\beta L\, C_3 + \sin\beta L\, C_4 = 0 \tag{10}$$

for $P > \sqrt{4k_s EI}$

$$(\cos\gamma L - \cos\varphi L)\, C_1 + \sin\gamma L\, C_3 + \sin\varphi L\, C_4 = 0 \tag{11}$$

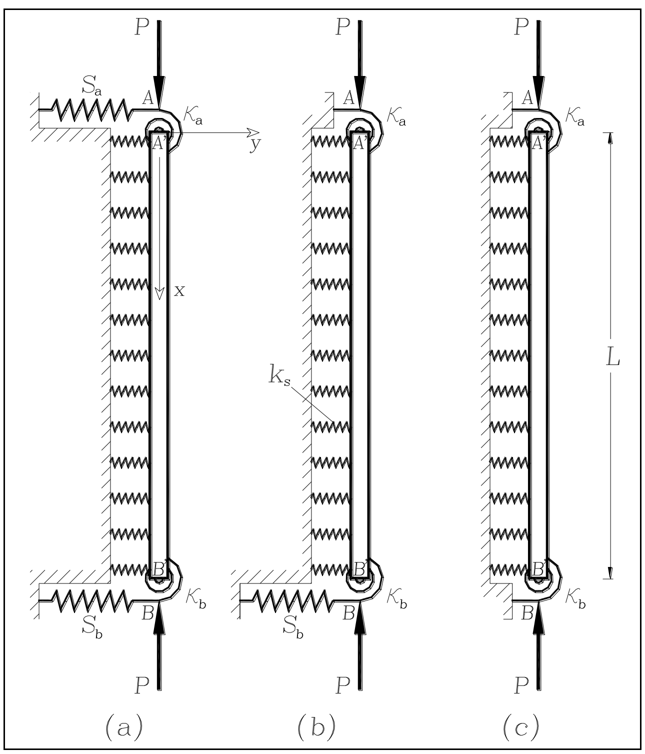

Figure 2. Beam-column classification according to the side-sway allowed at each end: a) side-sway uninhibited or partially inhibited at both ends; b) side-sway inhibited at A and partially inhibited at B; and c) lateral side-sway inhibited at both ends

Proposed stability equations

According to the type of end side-sway just described, three major buckling modes is distinguished in columns having semi-rigid connections and being laterally supported on a continuous Winkler´s foundation. They were columns having lateral sway which was uninhibited or partially inhibited at both ends (Fig. 2a), totally inhibited at one end and partially inhibited at the other end (Fig. 2b) and/or totally inhibited at both ends (Fig. 2c). The corresponding stability equations are listed and explained below for easy reference:

Type 1: *Columns with side-sway uninhibited or partially inhibited at both ends*

For columns in which side-sway at A' and B' is partially inhibited by springs S_a and S_b (Fig. 2a), respectively, the stability equation consisted of the eigen-value solution to the 4×4 determinant given by Eq. (12):

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} = 0 \tag{12}$$

Type 2: *Columns having side-sway inhibited at one end and partially inhibited at the other end*

For columns in which lateral sway at A' is totally inhibited and partially inhibited at B' by lateral spring S_b (Fig. 2b), the stability equation consists of the eigen-value solution to the 3×3 determinant given by Eq. (13):

$$\begin{vmatrix} A_{11} - A_{12} & A_{13} & A_{14} \\ A_{21} - A_{22} & A_{23} & A_{24} \\ A_{31} - A_{32} & A_{33} & A_{34} \end{vmatrix} = 0 \tag{13}$$

Type 3: *Columns with lateral side-sway inhibited at both ends*

For columns having lateral sway totally inhibited at both ends (Fig. 2c), the stability equation consists of the eigen-value solution to the 3×3 determinant given by Eqs. (14) or (15):

for $0 \le P < \sqrt{4k_s EI}$:
$$\begin{vmatrix} A_{11} - A_{12} & A_{13} & A_{14} \\ A_{21} - A_{22} & A_{23} & A_{24} \\ (e^{2\alpha L} - 1)\cos\beta L & e^{2\alpha L}\sin\beta L & \sin\beta L \end{vmatrix} = 0 \tag{14}$$

for $P > \sqrt{4k_s EI}$:
$$\begin{vmatrix} A_{11} - A_{12} & A_{13} & A_{14} \\ A_{21} - A_{22} & A_{23} & A_{24} \\ \cos\gamma L - \cos\psi L & \sin\gamma L & \sin\psi L \end{vmatrix} = 0 \tag{15}$$

The A_{ij} coefficients in Eqs. (12), (13), (14) and (15) is given by expression (7) or (8), depending on the magnitude of compressive load P when compared to $\sqrt{4k_s EI}$ (the value of elastic buckling load of an infinitely long column, as explained by Hetenyi (1967), p. 136).

It may be noticed that the buckling modal shapes for any of the three types of buckling could be determined directly using either Eq. (4) or Eq. (5) once the value of the buckling load had been determined from the corresponding characteristic equations listed above (i.e. Eqs. (12)-(15)] and corresponding A_{ij} coefficients given by either Eqs. (7) or (8) calculated). This solution was identical to that for any standard eigen-value problem.

Classical column cases

Variations in the critical load of a column supported by a uniformly distributed elastic foundation of stiffness k_s under different boundary conditions (five cases shown in Fig. 3a-e), including the effects of semi-rigid connections, were studied. The critical loads for different values of $\sqrt{k_s L^4/(EI)}$, as suggested by Hetenyi (1967), and for five different fixity factors ranging from hinged to clamped conditions, are listed in Tables 1-5 for each case shown in Fig. 3a-e, respectively. The values listed in these tables were calculated to five significant figures (for comparison with those obtained by Hetenty's formulae) using the proposed model and Eqs. (12)-(15).

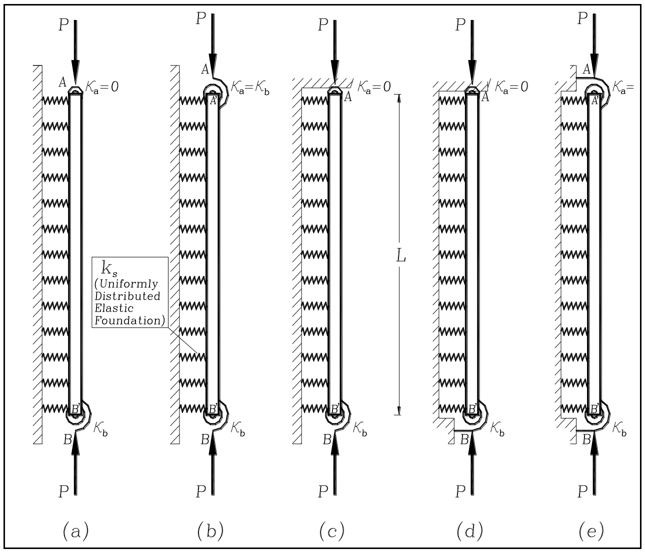

Figure 3. Five classical beam-column cases: a) side-sway uninhibited at both ends, hinged at A and restrained against rotation at B; b) side-sway uninhibited and equally restrained against rotation at both ends; c) side-sway inhibited and rota-tionally restrained at A and hinged at B; d) side-sway inhibited at both ends, rotationally restrained at A and hinged at B; and e) side-sway inhibited and equally restrained against rotation at both ends

The values listed in Tables 1 and 2 for $\rho = 0$ (i.e. hinged) were verified with those by Hetenyi (1967) [Eq. 125 and Fig. 117, p. 143], yielding identical results, whereas those in Table 3 compared very well to the simplified formula $P_{cr}=\sqrt{k_s EI}$ for infinitely long columns and any value of rotational restraint at B reported by Hetenyi (1967). Similarly, values listed in Tables 4 and 5 for $\rho = 0$ (i.e. hinged) and $\rho = 1$ (i.e. perfectly clamped) were verified with those by Hetenyi (1967) [Eqs. 127 and 128, and Figs. 118 and 119, respectively, pages 145-147], identical results being yielded too. The values in Table 4 also compared very well to the simplified formula $P_{cr}=\sqrt{4k_s EI}$ for infinitely long columns and any value of end rotational restraint reported by Hetenyi (1967).

Table 1 indicates that for columns having side-sway uninhibited at both ends and rotational restraint at one end, it was only effective in columns on elastic foundations having $k_s < 25EI/L^4$. For larger values of k_s, increased buckling load regarding rotational restraint was rather small (less than 20% for values of $k_s < 900EI/L^4$ and reaching increases lower than 3% for $k_s > 2500EI/L^4$). Table 2 indicates that critical load increased over 500% for $k_s < 25EI/L^4$ by perfectly clamping both ends as compared to perfectly hinged ends. For larger values of k_s, the buckling load increased by just over 100%.

Table 1. Buckling load/($\pi^2 EI/L^2$)Column with side-sway uninhibited at both ends, hinged at A and restrained against rotation at B

	ρ_a= 0 and 0≤ ρ_b ≤ 1				
$\sqrt{k_s L^4/(EI)}$	0	0.20	0.50	0.80	1.00
0	0	0.06028	0.14407	0.21318	0.25000
5	0.20891	0.27179	0.35189	0.41080	0.43978
10	0.80557	0.86948	0.91621	0.93663	0.94443
15	1.43173	1.51215	1.54143	1.54992	1.55277
20	1.75911	1.88170	2.00042	2.05988	2.08293
25	2.16856	2.30282	2.44727	2.52784	2.56055
30	2.65042	2.79168	2.93408	3.00846	3.03785
40	3.77770	3.91537	3.99683	4.02601	4.03630
50	5.00904	5.05753	5.06157	5.06260	5.06294
60	5.96797	6.06041	6.07769	6.08230	6.08385
70	6.90228	7.03763	7.07976	7.09193	7.09606
80	7.90451	8.04711	8.08985	8.10207	8.10622
90	8.95261	9.07986	9.10775	9.11531	9.11785
100	10.02396	10.11580	10.12715	10.13008	10.13106
200	20.25175	20.26407	20.26418	20.26421	20.26422
500	104.82612	104.84274	104.89287	105.09434	105.85696
1,000	206.39871	206.40872	206.43888	206.56051	206.90558
2,500	510.52741	510.52744	510.52752	510.52775	510.52848

Table 2. Buckling load/($\pi^2 EI/L^2$) Column with side-sway uninhibited and restrained against rotation at both ends

	ρ_a= ρ_b				
$\sqrt{k_s L^4/(EI)}$	0	0.20	0.50	0.80	1.00
0	0	0.13475	0.39581	0.73814	1.00000
5	0.20891	0.35335	0.63180	0.99049	1.25665
10	0.80557	0.98480	1.32721	1.74568	2.02659
15	1.43173	1.73172	2.40731	2.99527	3.30985
20	1.75911	2.07375	2.78342	3.97126	5.02660
25	2.16856	2.50282	3.25907	4.51969	5.60406
30	2.65042	3.00979	3.82701	5.18633	6.30985
40	3.77770	4.20395	5.19333	6.85964	8.10639
50	5.00904	5.51779	6.74164	8.93369	10.41624
60	5.96797	6.51097	7.84518	10.51335	13.10639
70	6.90228	7.48365	8.92394	11.84295	14.58926
80	7.90451	8.52998	10.09601	13.34020	16.30025
90	8.95261	9.62578	11.33206	14.98057	18.23938
100	10.02396	10.74576	12.59879	16.72462	20.40665
200	20.25175	21.29899	24.12361	31.59736	41.42557
500	104.82612	104.85951	104.96207	105.40719	107.29154
1,000	206.39871	206.41878	206.47986	206.73737	207.74052
2,500	510.52741	510.52747	510.52766	510.52843	511.01519

Table 3 indicates that for columns having side-sway inhibited at one end, rotational restraint at the same end was only effective in columns on an elastic foundation where $k_s \approx 0$. For $25EI/L^4 < k_s < 400EI/L^4$ the increase in buckling load was less than 12% and for values of $k_s > 900EI/L^4$ the increase in buckling load was less than 1%.

Tables 4 and 5 indicate similar trends to those observed in Tables 1 and 2 for columns having side-sway inhibited at both ends. Rotational restraints at one or both ends were only effective in columns on an elastic foundation where $k_s < 25EI/L^4$. For larger values of k_s, increased buckling load regarding rotational restraint was not significant. It was obvious that it was more effective to restrain both ends simultaneously against rotation than just one end. The values listed in Tables 1, 3 and 4 varied linearly with $\sqrt{k_s L^4/(EI)}$, having very small variations in P_{cr} with the degree of flexural restraint, making rotational conditions of secondary importance in long columns.

Table 3. Buckling load/($\pi^2 EI/L^2$) Column with side-sway inhibited and rotationally restrained at A, and free and hinged at B

	$0 \le \rho a \le 1$ and $\rho b = 0$				
$\sqrt{k_s L^4/(EI)}$	0	0.20	0.50	0.80	1.00
0	0	0.06029	0.14407	0.21318	0.25000
5	0.63111	0.63295	0.63512	0.63681	0.63773
10	1.08744	1.10806	1.14333	1.18449	1.21549
15	1.49199	1.51474	1.55680	1.61175	1.65795
20	1.97613	1.99163	2.02080	2.06007	2.09435
25	2.50599	2.51318	2.52686	2.54571	2.56270
30	3.04149	3.04334	3.04695	3.05208	3.05691
40	4.06982	4.07007	4.07057	4.07135	4.07216
50	5.06917	5.07050	5.07330	5.07794	5.08317
60	6.07473	6.07595	6.07859	6.08315	6.08864
70	7.08892	7.08951	7.09081	7.09316	7.09612
80	8.10517	8.10532	8.10567	8.10630	8.10714
90	9.11997	9.11997	9.11999	9.12002	9.12006
100	10.13318	10.13319	10.13322	10.13328	10.13337
200	20.26426	20.26426	20.26426	20.26427	20.26429
500	102.42982	102.72692	103.39819	105.10425	105.99062
1,000	203.68868	203.97683	204.62458	205.66899	206.20415
2,500	507.57870	507.85635	508.47716	509.50189	510.30875

Summary and conclusions

An analytical method has been presented for evaluating the critical buckling axial load of columns having side-sway uninhibited, partially inhibited and totally inhibited and laterally supported by a uniformly distributed elastic foundation (Winkler's model). The proposed column classification and corresponding stability equations were general and relatively simple to apply, yielding exact results when compared to other linear elastic analytical methods. The proposed formulae can also be applied to the stability of columns having rigid, semi-rigid and simple connections, with or without elastic supports. Semi-rigid connection effects were explicitly included. The end fixity factors were selected to consider the effects of semi-rigid connections in stability analysis, since they were practical and convenient.

The buckling load is obtained by making the determinant of a 4×4 matrix equal to zero for columns having side-sway uninhibited or partially inhibited at both ends, and of a 3×3 matrix for columns having end side-sway inhibited at one or both ends. The effects of lateral bracing one or both ends of the column have been presented. The buckling load of five classical cases was presented and compared to those by other analytical methods to demonstrate the use and relative simplicity of the proposed column classification and corresponding stability equations. Tables 1-5 can be used directly in stability analysis of columns having semirigid connections on an elastic foundation. The proposed method and its equations can be programmed, facilitating calculations and efficient computer coding and avoiding cumbersome procedures. It should be emphasised that the proposed model is just an approximation of the real load transfer mechanism on piles, drilled shafts, caissons and piers. It neglects the soil-pile friction skin contribution to load capacity and is unable to calculate vertical settlement and stresses in foundation soils which are generally the controlling factors during design.

Table 4. Buckling load/($\pi^2 EI/L^2$) Column with side-sway inhibited at both ends, rotationally restrained at A and hinged at B

	$0 \le \rho_a \le 1$ and $\rho_b = 0$				
$\sqrt{k_s L^4/(EI)}$	0	0.20	0.50	0.80	1.0
0	1.00000	1.13608	1.40694	1.77129	2.04575
5	1.25665	1.39216	1.65772	2.00342	2.25403
10	2.02660	2.15990	2.40569	2.68845	2.86806
15	3.30985	3.43578	3.62369	3.77387	3.84515
20	5.02660	5.06093	5.06474	5.06572	5.06605
25	5.60406	5.73286	5.94515	6.15132	6.26753
30	6.30985	6.44373	6.69767	7.01894	7.25656
40	8.10639	8.24068	8.49664	8.81665	9.04279
50	10.41624	10.54413	10.74624	10.92273	11.01150
60	13.10639	13.15798	13.16812	13.17082	13.17173
70	14.58926	14.71960	14.94319	15.17431	15.31092
80	16.30025	16.43473	16.69354	17.02918	17.28330
90	18.23938	18.37385	18.63186	18.96087	19.20093
100	20.40665	20.53839	20.77181	21.02286	21.17358
200	41.42557	41.50133	41.53007	41.53868	41.54161
500	101.37746	101.51125	101.76321	102.07252	102.28860
1,000	202.65982	202.79426	203.05225	203.38296	203.62765
2,500	510.16617	511.18858	511.81888	513.49927	515.62152

Table 5. Buckling load/($\pi^2 EI/L^2$) Column with side-sway inhibited and equally restrained against rotation at both ends

	$\rho_a = \rho_b$				
$\sqrt{k_s L^4/(EI)}$	0	0.20	0.50	0.80	1.00
0	1.00000	1.28208	1.91659	2.99066	4.00000
5	1.25665	1.53836	2.16884	3.22204	4.19205
10	2.02660	2.30710	2.92422	3.91072	4.76276
15	3.30985	3.58777	4.17716	5.03690	5.69423
20	5.02660	5.31335	5.91043	6.95059	6.55234
25	5.60406	5.89044	6.57283	7.90956	8.46691
30	6.30985	6.59572	7.27227	8.57226	10.04787
40	8.10639	8.39077	9.04988	10.24535	11.47225
50	10.41624	10.69796	11.32695	12.35858	13.26391
60	13.10639	13.39200	14.06788	14.85438	15.38058
70	14.58926	14.87533	15.55595	16.90354	17.75189
80	16.30025	16.43473	17.26383	20.16972	18.58612
90	18.23938	18.52452	19.19332	20.45562	21.86351
100	20.40665	20.69046	21.34371	22.51090	23.68742
200	41.42557	41.71010	42.37368	43.40027	44.04601
500	101.37746	101.66276	102.33422	103.62254	105.13392
1,000	202.65982	202.94548	203.62132	204.94061	206.55531
2,500	510.16617	511.39897	512.77221	516.39050	522.52653

The analytical results indicated that the effects of a uniformly distributed lateral support of magnitude k_s became more significant in columns having free ends than in columns having both ends restrained against rotation and side-sway. The buckling load of a column, laterally supported by a very soft elastic foundation, could be substantially increased by either restraining its ends against rotation or by bracing both ends against side-sway.

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