

# A Remark on the Heat Equation and Minimal Morse Functions on Tori and Spheres

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## Abstract

Let  $(M, g)$  be a compact, connected riemannian manifold that is homogeneous, i.e. each pair of points  $p, q \in M$  have isometric neighborhoods. This paper is a first step towards an understanding of the extent to which it is true that for each “generic” initial condition  $f_0$ , the solution to  $\partial f / \partial t = \Delta_g f$ ,  $f(\cdot, 0) = f_0$  is such that for sufficiently large  $t$ ,  $f(\cdot, t)$  is a minimal Morse function, i.e., a Morse function whose total number of critical points is the minimal possible on  $M$ . In this paper we show that this is true for flat tori and round spheres in all dimensions.

**Key words:** morse function, heat equation.

## Highlights

- A novel manifestation of the role played by the heat equation in mathematics is pointed out.
- On some homogeneous riemannian manifolds, evolution by the heat equation “discovers all by itself” minimal Morse functions.

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## Una nota acerca de la ecuación del calor y funciones de Morse minimales en toros y esferas

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### Resumen

Sea  $(M, g)$  una variedad riemanniana que es compacta, conexa y homogénea, es decir, tal que cada par de puntos  $p, q \in M$  tienen vecindades isométricas. Este artículo constituye un primer paso en el estudio de qué tan general es el hecho de que para cada condición inicial "genérica"  $f_0$  en  $(M, g)$ , la solución de  $\partial f / \partial t = \Delta_g f$ ,  $f(\cdot, 0) = f_0$  es tal que para  $t$  suficientemente grande,  $f(\cdot, t)$  es una función de Morse minimal, es decir, una función de Morse cuyo número total de puntos críticos es el mínimo posible en  $M$ . En este artículo se muestra que esto es cierto en el caso de toros planos y esferas redondas, de todas las dimensiones.

**Palabras clave:** función de morse, ecuación del calor.

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## 1 Introduction

The Heat Equation is arguably the most important partial differential equation in mathematics and physics. This equation also seems to be ubiquitous in geometry as well as in many other branches of mathematics. In a suggestive article [1], Serge Lang and Jay Jorgenson called the Heat Kernel "... a universal gadget which is a dominant factor practically everywhere in mathematics, also in physics, and has very simple and powerful properties". In this short note we present evidence corroborating the relevance of the Heat Equation, in a seemingly novel direction. Specifically, we present two concrete cases in which the Heat Equation "discovers all by itself" minimal Morse functions. This could be, however, a manifestation of a phenomenon that might hold on more general homogeneous riemannian manifolds.

Let  $M$  be a closed, connected, oriented smooth manifold, and let  $g$  be a riemannian metric on  $M$ . On each tangent space  $T_p(M)$ , the metric  $g$  determines a bilinear function  $\langle \cdot, \cdot \rangle_g$ . For any given smooth function  $f$  on  $M$ , let us recall that the gradient is defined as the vector field  $\text{grad}(f)$  in  $T(M)$  that satisfies  $\langle \text{grad}(f), \zeta \rangle_g = \zeta(f)$ , for all  $\zeta \in T(M)$ . Let us denote by  $\nabla$  the Levi-Civita connection determined by the metric  $g$ . For each smooth vector field  $X$  on  $M$ , its divergence is defined

as  $\operatorname{div}(X) = \operatorname{trace}(\zeta \rightarrow \nabla_\zeta(X))$ . The Laplacian (or Laplace-Beltrami operator) on  $(M, g)$  of a smooth function  $f : M \rightarrow \mathbb{R}$  is defined as  $\Delta_g f = \operatorname{div}(\operatorname{grad}(f))$ .

The Heat Equation on  $(M, g)$  is the partial differential equation  $\partial f / \partial t = \Delta_g f$ . A solution to the initial condition problem

$$\begin{cases} \partial f / \partial t = \Delta_g f \\ f(\cdot, 0) = f_0 \in L^2(M) \end{cases} \quad (1)$$

is a continuous function  $f : M \times (0, \infty) \rightarrow \mathbb{R}$  such that

1. For each fixed  $t > 0$ ,  $f(\cdot, t)$  is  $C^2$  function, and for each  $x \in M$ ,  $f(x, \cdot)$  is  $C^1$ .
2.  $\partial f / \partial t = \Delta_g f$ , and

$$\lim_{t \rightarrow 0^+} \int_M f(x, t) \psi(x) dV(x) = \int_M f_0(x) \psi(x) dV(x),$$

for all  $\psi \in C^\infty(M)$ .

It is a non trivial fact that for each  $t > 0$ ,  $f_t = f(\cdot, t)$  is smooth [2], [3]. It is well known that the solution to problem (1) can be obtained in the following way. First, it can be seen that the eigenvalues of the operator  $\Delta_g$ , understood as those  $\lambda \in \mathbb{R}$  for which  $\Delta_g f + \lambda f = 0$  for some smooth function  $f$  nonidentically zero, are nonnegative and form a discrete set ([2], [3]),  $\lambda_0 = 0 < \lambda_1 < \dots < \lambda_j < \dots$ . Moreover, for each  $j \geq 0$ , the corresponding eigenspace  $E_j$  has finite dimension  $m_j$  and  $E_j \subset C^\infty(M)$ . By the assumptions about  $M$ ,  $E_0$  is the one dimensional vector space of constants functions. For each  $j \geq 0$  let  $B_j = \{\phi_{j,i} : i = 1, \dots, m_j\}$  be an orthonormal basis for  $E_j$ . Their union  $B = \cup B_j$  is an orthonormal basis for  $L^2(M)$ . Then, the solution to problem (1) can be written as

$$f = \sum_{j \geq 0} e^{-\lambda_j t} \sum_{i=1}^{m_j} \langle f_0, \phi_{j,i} \rangle_{L^2(M)} \phi_{j,i}, \quad (2)$$

where  $\langle -, - \rangle_{L^2(M)}$  denotes the inner product of  $L^2(M)$ .

We want to address the following question: to what extent is it true that for each "generic" initial condition  $f_0$ , the solution to (1) is such that for sufficiently large  $t$ ,  $f_t$  is a minimal Morse function, i.e., a Morse function whose total number of critical points is less or than equal to that of any other Morse function on  $M$ ? Below, we show that the answer to this question is affirmative for flat tori and round spheres. A key ingredient in the proof is the following result which is a corollary of *Mather's Stability Theorem* [4].

**Theorem 1.1** (Stability of Morse functions). *Let  $M$  be a smooth manifold and let  $h : M \rightarrow \mathbb{R}$  be any Morse function. Then, there exists an open neighborhood  $W_h$  in the  $C^\infty$  topology (of  $C^\infty(M)$ ) such that any function  $\varphi \in W_h$  is Morse with the same number of critical points as  $h$ . When  $M$  is compact, the  $C^\infty$  topology (of  $C^\infty(M)$ ) is the union of all  $C^r$  topologies (of  $C^\infty(M)$ ) [5], so in this case the conclusion can be restated as follows: There exists  $r > 0$  and  $\epsilon > 0$  such that, if  $\|\varphi - h\|_r < \epsilon$  for  $\varphi \in C^\infty(M)$  and  $\|\cdot\|_r$  being the  $C^r$ -norm, then  $\varphi$  is also a Morse function with the same number of critical points as  $h$ .*

## 2 Tori

Let  $T^n$  denote the  $n$ -dimensional flat torus obtained as the quotient of  $\mathbb{R}^n$  by the obvious action of  $(2\pi\mathbb{Z})^n$  by isometries. In this case the spectrum of the Laplacian is the set  $\{\lambda_j : j \geq 0\}$  where  $\lambda_j$  is the  $j$ -th integer that can be written as a sum of squares  $k_1^2 + \dots + k_n^2$  for some nonnegative integers  $k_1, \dots, k_n$  [6]. Notice that  $\lambda_0 = 0$  (as it should) and  $\lambda_1 = 1$ . An orthonormal basis  $B_j$  for  $E_j$  is given by the functions in  $T^n$  induced by the functions in  $\mathbb{R}^n$  of the form

$$g_{1,\mathbf{k}} = \cos(k_1x_1 + \dots + k_nx_n) \tag{3}$$

$$g_{2,\mathbf{k}} = \sin(k_1x_1 + \dots + k_nx_n), \tag{4}$$

where  $\mathbf{k}$  denotes a  $n$ -tuple  $(k_1, \dots, k_n)$ , such that  $k_1^2 + \dots + k_n^2 = \lambda_j$ . The set  $B = \cup_{j \geq 0} B_j$  is an orthonormal basis of  $L^2(T^n)$ . In particular,  $B_1$  is  $\{\cos(x_1), \sin(x_1), \dots, \cos(x_n), \sin(x_n)\}$ .

**Lemma 1.** *The function induced on  $T^n$  by  $h = \sum_{k=1}^n (a_k \cos(x_k) + b_k \sin(x_k))$  is a Morse function, if and only if,  $a_k^2 + b_k^2 \neq 0$  for every  $k$ . Moreover, if it is Morse it is a minimal Morse function.*

*Proof.* First, we observe that a smooth function on  $T^n$  is Morse if and only if its pullback is Morse. Now, each partial derivative  $\partial h / \partial x_k = -a_k \sin(x_k) + b_k \cos(x_k)$  can be rewritten in the form  $A_k \sin(x_k + \theta_k) = 0$ , where  $A_k = \sqrt{a_k^2 + b_k^2}$ , and the  $\theta_k$  are suitable constants. We notice that  $h$  is Morse if and only if whenever  $(x) = (x_1, \dots, x_n) \in \mathbb{R}^n$  is a solution to the system  $A_k \sin(x_k + \theta_k) = 0, k = 1, \dots, n$ ,  $(x)$  is *not* a zero of the the determinant of the Hessian matrix, which can be readily computed as

$$\det \text{Hess}(h) = (-1)^n (a_1 \cos(x_1) + b_1 \sin(x_1)) \cdots (a_n \cos(x_n) + b_n \sin(x_n)).$$

It is easy to see that this is the case if and only if  $a_k^2 + b_k^2 \neq 0$ , for each  $k = 1, \dots, n$ . On the other hand, it is a general fact that the sum of the betti numbers of a manifold  $M$  is a lower bound for the number of critical points of any Morse function on  $M$ . Using Künneth's formula we see that the sum of the betti numbers of  $T^n$  equals  $2^n$ . Finally, the system  $A_k \sin(x_k + \theta_k) = 0, k = 1, \dots, n$ , under the assumption that  $a_k^2 + b_k^2 \neq 0$ , for all  $k$ , has  $2^n$  solutions in the box  $[0, 2\pi)^n$ . This allows us to conclude that  $h$  is a minimal Morse function.  $\square$

**Theorem 2.1.** *There exists a set  $S \subset C^\infty(T^n)$ , that is dense and open in the  $C^\infty$  topology, such that for any initial condition  $f_0 \in S$ , if  $f$  is corresponding solution to (1), then there exists  $T > 0$ , depending on  $f_0$ , so that for each  $t \geq T$ , the function  $f_t = f(\cdot, t)$  is minimal Morse.*

*Proof.* Let  $f_0 = h_0 + h_1 + \cdots$  where each  $h_j$  is the projection of  $f_0$  on  $E_j$ . Let us fix a nonnegative integer  $r$ , and let  $\|\cdot\|_r$  be the  $C^r$  norm on  $C^\infty(M)$  (see [5]). As noticed in the introduction,  $f_t = \sum_{j=0}^\infty e^{-\lambda_j t} h_j$ . In order to prove that  $f_t$  is minimal Morse for all  $t$  sufficiently large, it

suffices to show that the same is true for  $e^{\lambda_1 t}(f_t - h_0)$ . We have

$$\|e^{\lambda_1 t}(f_t - h_0) - h_1\|_r = e^{-(\lambda_2 - \lambda_1)t} \left\| \sum_{j=2}^{\infty} e^{-(\lambda_j - \lambda_2)t} h_j \right\|_r. \quad (5)$$

Now we show that the second factor in the right hand side of (5) is bounded on  $T^n \times [1, \infty)$ . Let us write  $h_j = \sum_{\mathbf{k}} (c_{j,\mathbf{k}} g_{1,\mathbf{k}} + d_{j,\mathbf{k}} g_{2,\mathbf{k}})$ , where the  $g_{1,\mathbf{k}}$  and  $g_{2,\mathbf{k}}$  are as in (3). It can be seen inductively that for  $i = 1, 2$ , then  $\partial g_{i,\mathbf{k}} / \partial x_{i_s} \cdots \partial x_{i_1}$  is equal to  $\pm k_{i_1} \cdots k_{i_s} g_{l,\mathbf{k}}$ , for some  $l = 1, 2$ . Using these we can estimate the sum in (5) as follows:

$$\begin{aligned} |\partial h_j / \partial x_{i_s} \cdots \partial x_{i_1}| &\leq \left| \sum_{\mathbf{k}} c_{j,\mathbf{k}} \partial g_{1,\mathbf{k}} / \partial x_{i_s} \cdots \partial x_{i_1} \right| \\ &\quad + \left| \sum_{\mathbf{k}} d_{j,\mathbf{k}} \partial g_{2,\mathbf{k}} / \partial x_{i_s} \cdots \partial x_{i_1} \right| \\ &\leq \left( \sum_{\mathbf{k}} c_{j,\mathbf{k}}^2 \right)^{1/2} \left( \sum_{\mathbf{k}} (\partial g_{1,\mathbf{k}} / \partial x_{i_s} \cdots \partial x_{i_1})^2 \right)^{1/2} \\ &\quad + \left( \sum_{\mathbf{k}} d_{j,\mathbf{k}}^2 \right)^{1/2} \left( \sum_{\mathbf{k}} (\partial g_{2,\mathbf{k}} / \partial x_{i_s} \cdots \partial x_{i_1})^2 \right)^{1/2} \end{aligned}$$

by the Cauchy-Schwartz inequality.

Since  $\partial g_{i,\mathbf{k}} / \partial x_{i_s} \cdots \partial x_{i_1} = \pm k_{i_1} \cdots k_{i_s} g_{l,\mathbf{k}}$ , for some  $l = 1, 2$ . Now, the number of all possible  $n$  tuples  $\mathbf{k} = (k_1, \dots, k_n)$  such that  $k_1^2 + \dots + k_n^2 = \lambda_j$  is clearly bounded above by  $\lambda_j^n$ . Thus,

$$|\partial g_{i,\mathbf{k}} / \partial x_{i_s} \cdots \partial x_{i_1}| \leq \lambda_j^{(r+n)/2} \leq \lambda_j^{r+n}.$$

Hence,

$$|\partial h_j / \partial x_{i_s} \cdots \partial x_{i_1}| \leq \lambda_j^{r+n} \left( \left( \sum_{\mathbf{k}} c_{j,\mathbf{k}}^2 \right)^{1/2} + \left( \sum_{\mathbf{k}} d_{j,\mathbf{k}}^2 \right)^{1/2} \right),$$

and consequently,  $|\partial h_j / \partial x_{i_s} \cdots \partial x_{i_1}| \leq 2\lambda_j^{r+n} \|f_0\|_{L^2(T^n)}$ . It follows immediately that  $\|h_j\|_r \leq 2\lambda_j^{r+n} \|f_0\|_{L^2(T^n)}$ . From this we get the estimate

$$\left\| \sum_{j=2}^{\infty} e^{-(\lambda_j - \lambda_2)t} h_j \right\|_r \leq 2 \|f_0\|_{L^2(T^n)} \sum_{j=2}^{\infty} \lambda_j^{r+n} e^{-(\lambda_j - \lambda_2)t}. \quad (6)$$

It is to verify the convergence of the series on the right hand side of 6 for each fixed value of  $t \geq 1$ . Since the sum of this series decreases as  $t$  increases, we deduce that  $\|\sum_{j=2}^{\infty} e^{-(\lambda_j - \lambda_2)t} h_j\|_r$  is bounded on  $T^n \times [1, \infty)$ . Finally, this implies that

$$\lim_{t \rightarrow \infty} \|e^{\lambda_1 t}(f_t - h_0) - h_1\|_r = 0 \tag{7}$$

for each  $r \geq 0$ .

Let us assume that  $h_1$  is a minimal Morse function. By Theorem 1.1 there exist  $r = r(h_1) > 0$  and  $\epsilon = \epsilon(h_1) > 0$  such that every  $\varphi \in C^\infty(M)$  with  $\|\varphi - h_1\|_r < \epsilon_0$  is Morse and has the same number of critical points as  $h_1$ . Since  $\lim_{t \rightarrow \infty} \|e^{\lambda_1 t}(f_t - h_0) - h_1\|_r = 0$ , there is a  $T_0 > 0$  such that if  $t \geq T_0$ ,

$$\|e^{\lambda_1 t}(f_t - h_0) - h_1\|_r < \epsilon.$$

This implies that if  $t \geq T_0$ , the function  $e^{\lambda_1 t}(f_t - h_0) - h_1$  is Morse and has the same number of critical points as  $h_1$ . This allows us to conclude that for  $t \geq T_0$ ,  $e^{\lambda_1 t}(f_t - h_0) - h_1$  is minimal Morse, and so it is  $f_t$ . The proof is finished by verifying that the subset  $S \subset C^\infty(T^n)$  of all functions whose projection on  $E_1$  is minimal Morse is open and dense in the  $C^\infty$  topology. Since the  $C^0$  norm bounds from above the  $L^2$  norm ( $T^n$  is compact), then two functions close in the  $C^0$  sense are also close in the  $L^2$  sense and therefore the coefficients in their Fourier expansions are also close. A combination of this with Lemma 1 shows that a smooth function that is close to a smooth function whose projection on  $E_1$  is minimal Morse, must also have projection on  $E_1$  that is minimal Morse. Finally, let us show that  $S$  is dense in  $C^\infty(T^n)$ . It suffices to prove that for any smooth  $f_0 = h_0 + h_1 + \dots$ ,  $r > 0$ , and  $\epsilon > 0$ , there is  $\tilde{f}_0 \in S$  such that  $\|f_0 - \tilde{f}_0\|_r < \epsilon$ . It is enough to define  $\tilde{f}_0$  with the same Fourier expansion as  $f_0$  except that  $h_1$  is replaced by a minimal Morse function  $\tilde{h}_1$ , such that  $\|h_1 - \tilde{h}_1\|_r < \epsilon$ .  $\square$

### 3 Spheres

In this section we will prove a result similar to Theorem 2.1 for the  $n$ -dimensional sphere  $S^n = \{(x_1, \dots, x_{n+1}) : x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ ,  $n \geq 1$ , with the metric induced from  $\mathbb{R}^{n+1}$ . In this case, it is well known that the eigenvalues of the Laplacian operator are given by  $\lambda_j = j(j + n - 1)$ ,  $j \geq 0$  and the corresponding eigenfunctions, the so called *spherical harmonics*, are given as the restriction to the sphere of homogeneous polynomials  $H(x_1, \dots, x_{n+1})$  in the coordinates of  $\mathbb{R}^{n+1}$ , of total degree  $j$ , which satisfy the condition

$$\partial^2 H / \partial x_1^2 + \dots + \partial^2 H / \partial x_{n+1}^2 = 0. \quad (8)$$

The corresponding eigenspace of  $\lambda_j$ ,  $E_j$ , turns out to be a space of dimension  $\binom{n+j}{n} - \binom{n+j-2}{n}$  [7]. (In the latter formula it is understood that  $\binom{a}{b} = 0$  in case  $a < b$ .) When  $j = 1$ ,  $\lambda_1 = n$ , and a basis for  $E_1$  is given by the  $n + 1$  coordinate functions  $x_1, \dots, x_{n+1}$ . Each nonzero linear form in these variables is a Morse function with two critical points, which is the minimal possible, since this is precisely the sum of the betti numbers of  $S^n$ .

Moreover, if  $h$  denotes the restriction of a harmonic homogeneous polynomial  $H$  to  $S^n$ , the following estimate holds for the sup norm ( $C^0$ -norm in  $M$ ):

$$\|h\|_0 \leq C_n \binom{n+j}{n}^{1/2} \|h\|_{L^2(S^n)}, \quad (9)$$

where  $C_n$  is a constant that only depends on  $n$  ([7], Section 7). Moreover, in the  $C^r$ -norm on  $S^n$ , the following more general estimate holds ([8])

$$\|h\|_r \leq C_n \binom{n+j}{n}^{1/2} (1 + \lambda_j)^{r/2} \|h\|_{L^2(S^n)}. \quad (10)$$

With these preliminaries we can state the following theorem.

**Theorem 3.1.** *There exists a set  $S \subset C^\infty(S^n)$ , that is dense and open in the  $C^\infty$  topology, such that for any initial condition  $f_0 \in S$ , if  $f$  is*



corresponding solution to (1), then there exists  $T > 0$ , depending on  $f_0$ , so that for each  $t \geq T$ , the function  $f_t = f(\cdot, t)$  is minimal Morse.

*Proof.* Let  $f_0 = h_0 + h_1 + \dots$  where each  $h_j$  is the projection of  $f_0$  on  $E_j$ , and let  $f_t$  denote the solution to the Heat Equation with initial condition  $f_0$ . In order to prove that  $f_t$  is minimal Morse for all  $t$  sufficiently large, it suffices to show that the same is true for  $e^{\lambda_1 t}(f_t - h_0)$ . Let us fix a positive integer  $r$ . As in the previous proof, the key issue is to show that  $\left\| \sum_{j=2}^{\infty} e^{-(\lambda_j - \lambda_2)t} h_j \right\|_r$  is bounded on  $S^n \times [1, \infty)$ . Let us notice that  $\lambda_j - \lambda_2 = j(j + n - 1) - 2(n + 1)$  becomes greater than  $n + j$  for any  $j$  sufficiently large. On the other hand,  $e^x$  is also greater than  $x^2 \binom{n+x}{n} (1+x)^r$  for all  $x \geq N$ . Thus, for all  $j \geq N' \geq N$ , and  $t \geq 1$

$$e^{-2(\lambda_j - \lambda_2)t} \binom{n+j}{n} (1+j)^r < 1/j^2.$$

This implies that

$$\begin{aligned} \left\| \sum_{j=N'}^m e^{-(\lambda_j - \lambda_2)t} h_j \right\|_r &\leq \sum_{j=N'}^m e^{-(\lambda_j - \lambda_2)t} \|h_j\|_r \\ &\leq C_n \sum_{j=N'}^m e^{-(\lambda_j - \lambda_2)t} \binom{n+j}{n}^{1/2} (1+\lambda_j)^{r/2} \|h_j\|_{L^2(S^n)} \\ &\leq C_n \left( \sum_{j=N'}^m e^{-2(\lambda_j - \lambda_2)t} \binom{n+j}{n} (1+\lambda_j)^r \right)^{1/2} \left( \sum_{j=N'}^m \|h_j\|_{L^2(S^n)}^2 \right)^{1/2} \\ &\leq C_n \|f_0\|_{L^2(S^n)} \sum_{j=1}^{\infty} 1/j^2. \end{aligned}$$

Hence,

$$\left\| \sum_{j=2}^{\infty} e^{-(\lambda_j - \lambda_2)t} h_j \right\|_r \leq K_0 + C_n \|f_0\|_{L^2(S^n)} \sum_{j=1}^{\infty} 1/j^2,$$

where  $K_0 = \left\| \sum_{j=2}^{N'-1} e^{-(\lambda_j - \lambda_2)t} h_j \right\|_r$ , which proves the claim. The rest of the proof is exactly as that of the theorem above.  $\square$

## 4 Conclusions

In this article a novel manifestation of the role played by the heat equation in mathematics is pointed out; namely, that in some homogeneous riemannian manifolds, evolution by the heat equation of a generic initial datum “discovers all by itself” a minimal Morse function for the manifold, i.e. one whose total number of critical points is less than or equal to that of any other Morse function admitted by the manifold. In the future the authors intend to verify whether the same phenomenon takes place on other families of homogeneous riemannian manifolds.

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