

# THICK DISKS WITH NEWTONIAN MULTIPOLAR MOMENTS

## DISCOS GRUESOS CON MOMENTOS MULTIPOLARES NEWTONIANOS

Framsol López-Suspes<sup>1,2</sup>, Guillermo A. González<sup>2</sup>

### RESUMEN

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Se presenta una nueva familia de discos gruesos newtonianos estables a partir del método de desplazamiento, corte, llenado y reflexión construidos en (**González & Letelier**, 2004; **Vogt & Letelier**, 2005). Soluciones de la ecuación de Laplace en coordenadas cilíndricas son consideradas, éstas representarán el campo gravitacional de objetos con momentos multipolares externos solamente. Se definen y calculan las cantidades físicas en el plano del disco, tales como, la frecuencia epicíclica,  $\kappa$ , la frecuencia vertical,  $\nu$ , y la velocidad circular,  $v_c$  de las partículas. Además, se determina la densidad superficial de masa,  $\sigma$ , la densidad del disco grueso,  $\rho$ , y el criterio de estabilidad de los discos gruesos a través del momentum angular o el criterio de Rayleigh (**Rayleigh**, 1917; **Landau**, 1987). Finalmente, se representan las propiedades físicas para algunos parámetros considerando sólo hasta el término cuadrupolar en la expansión multipolar del potencial gravitacional.

**Palabras clave:** Disco grueso, Rayleigh, frecuencia epicíclica, multipolos.

### ABSTRACT

We present a new family of stable thick discs form known *displace, cut, fill and reflect* method (**González & Letelier**, 2004; **Vogt & Letelier**, 2005) is presented. Solutions of the Laplace equation in cylindrical coordinates are considered, these represent the gravitational field of objects with only external multipole moments. The physical quantities in the plane of the disk, are defined and calculated such as, the epicyclic frequency,  $k$ , the vertical frequency,  $\nu$ , and the circular velocity,  $v_c$  of particles. The surface density,  $\sigma$ , density of thick disk,  $\rho$ , and stability of thick disks through of specific angular momentum or Rayleigh criterion (**Rayleigh**, 1917), and (**Landau**, 1987) were calculated. Finally, the physical properties are shown for only some parameters considering only until quadrupolar term in the expasion gravitational potential multipolar.

**Key words:** Thick disk, Rayleigh, epicyclic frequency, and multipole.

<sup>1</sup>Facultad de Ingeniería Mecatrónica, Universidad Santo Tomás, Bucaramanga, Colombia, E-mail: framsol@gmail.com

<sup>2</sup>Escuela de Física, Grupo de Investigación en Relatividad y Gravitación, Universidad Industrial de Santander, A. A. 678, Bucaramanga, Santander, Colombia, E-mail: guillego@uis.edu.co

## 1 Introduction

The obtention and interpretation of solutions of Laplace and Poisson equations constitute an arduous and interesting problem. During many years, it has been obtained by various methods, a large number of solutions of these equations. In particular, over the past four decades the collection and physical analysis of axially symmetric exact solutions of Laplace equations that represent discoidal configurations has raised a variety of works.

The Newtonian gravitational potential of a thick disk can be obtained by a simple procedure such as, the *displace, cut, fill and reflect* method, presented by (González & Letelier, 2004), and then expanded by (Vogt & Letelier, 2005), both cases they were based on known of *displace, cut, reflect* method, considered of (Kuzmin, 1956). Said method mathematically is based on making the transformation  $z \rightarrow h(z) + b$ , wherein  $b$  is a constant and  $h(z)$  an even function of  $z$ .

In the other hand, as is suggested by a wide variety of observational evidences, many astrophysical objects can be modeled as axially symmetric bodies with prolate or oblate deformation (Cooray, 2000; Davies & Birkinshaw, 1986; Fassano & Vio, 1991; Merritt, 1996; & Mosenkov et al, 2010). For example, some galaxies with a large disc component can be assumed as axisymmetric oblate bodies with a large quadrupolar moment and, in some cases, with a significant octupolar deformation due to the remaining components like the halo (Helmi, 2004).

In agreement with the above considerations, this paper we shall focus on the introduced by González and Letelier method (González & Letelier, 2004). The paper is organized as follows. First, in section 2, we present a summary of the main aspects of the generalized thick disk. Then, in the section 3, we find the physical quantities in the plane the disk such as circular velocity  $v_c$  of particles, the epicyclic frequency,  $\kappa$ , the vertical frequency,  $\nu$ , surface density  $\sigma$ , and specific angular momentum,  $\ell$ . In next section, section 4, we present the differential equations of motion. Later in section 5, we review a new family of thick disks that meet the physical quantities and thus we construct a stable model. Finally, in section 6, we summarize the main results

## 2 Thick Disk Model

The gravitational potential  $\Phi(R, z)$ , in the case of Newtonian gravity in the vacuum, is a solution of the Laplace Equation, this is

$$\nabla^2 \Phi = \Phi_{,RR} + \frac{\Phi_{,R}}{R} + \Phi_{,zz} = 0, \quad (1)$$

where  $(R, \varphi, z)$  are the usual cylindrical coordinates. After we make the transformation  $z \rightarrow h(z) + b$ , the above equation becomes (González & Letelier, 2004)

$$\nabla^2 \Phi = h'' \Phi_{,h} + [(h')^2 - 1] \Phi_{,hh} = 4\pi G \rho, \quad (2)$$

where premiums indicate differentiation with respect to  $z$ ,  $G$  is the Gravitational constant and  $\rho = \rho(R, z)$  is the mass density. To ensure that the disc is located in  $|z| \geq a$ , the function  $h(z)$  outside the disk must be a linear function of  $z$  such that  $h'(z) = 1$ .

For the case of thick disks the function  $h(z)$  must be selected in such a way that  $\Phi$  and its first derivatives be continuous across the plane  $z = 0$ . Let us take a function  $h(z)$  defined as (González & Letelier, 2004)

$$h(z) = \begin{cases} z - a/2 & , \quad z \geq a, \\ z^2/2a & , \quad -a \leq z \leq a, \\ -z - a/2 & , \quad z \leq -a. \end{cases} \quad (3)$$

Hence, by taking the function  $h(z)$  above defined we can generate disks of thickness  $2a$  located in the region  $-a \leq z \leq a$ .

## 3 Physical Quantities

The physical quantities of interest are evaluated in the plane the disk, these are the circular velocity,  $v_c$ , the epicyclic frequency,  $\kappa$ , and the vertical frequency,  $\nu$ , of small oscillations about the equilibrium circular orbit. They are found with the equations (Binney & Tremaine, 2008)

$$v_c^2 = R \Phi_{,R}, \quad (4)$$

$$\kappa^2 = \Phi_{,RR} + \frac{3}{R} \Phi_{,R}, \quad (5)$$

$$\nu^2 = \Phi_{,zz}, \quad (6)$$

$$\ell^2 = R^2 v_c^2, \quad (7)$$

where all quantities are evaluated in the plane  $z = 0$ . The stability conditions are set by  $\kappa^2 \geq 0$ ,  $\nu^2 \geq 0$ ,

and  $\ell^2 \geq 0$ , wherein  $\ell$  is the specific angular momentum known as Rayleigh criterion (Rayleigh, 1917; Landau, 1987).

In the case of thin disk the expression for the surface density,  $\sigma$ , can be obtained taking the function  $h(z) = |z|$ , and using the well-known relation of Gauss's theorem, one can write

$$\sigma = \frac{1}{2\pi G} \Phi_{,z}, \quad (8)$$

evaluated at  $z \rightarrow 0^+$ .

When  $|z| \geq a$  the function  $h(z)$  is a linear function of  $z$  such that  $h'(z) = 1$ . Hence, its second derivative is zero. Then the mass density vanish outside the disk. Since the first derivative is continuous at  $|z| = a$  and the second derivative piecewise constant we have that the mass density,  $\rho$ , will be well defined inside the disk, (González & Letelier, 2004)

$$\rho(R, z) = \frac{1}{4\pi G} \{ h'' \Phi_{,h} + [(h')^2 - 1] \Phi_{,hh} \}. \quad (9)$$

In this expression  $|z| \leq a$  and  $\rho = 0$  for  $|z| > a$ .

## 4 Equations of Motion

The motion of a test particle in a gravitational field described by (14), obeys the relations (Ramos-Caro, González & López-Suspes, 2008)

$$\dot{R} = V_R, \quad \dot{z} = V_z, \quad (10)$$

$$\dot{V}_R = -\frac{\partial}{\partial R} \Phi_{eff}, \quad \dot{V}_z = -\frac{\partial}{\partial z} \Phi_{eff}, \quad (11)$$

where  $\Phi_{eff}$  is the effective potential, given by

$$\Phi_{eff}(R, z) = \Phi(R, z) + \frac{\ell^2}{2R^2}. \quad (12)$$

Here,  $\ell = R^2 \dot{\phi}$  is the axial specific angular momentum that is conserved as a consequence of the axial symmetry. The second integral of motion is the total specific energy

$$E = \frac{1}{2}(V_R^2 + V_z^2) + \Phi_{eff}(R, z). \quad (13)$$

According to eqs. (10)-(13), the motion is restricted to a three dimensional phase space  $(R, z, V_R)$ . This fact enable us to introduce the Poincaré surfaces of section method, in order to investigate the trajectories of test particles. Note that the orbit of particle is confined to the plane defined by the effective potential, this plane is known as meridional plane.

## 5 One New Family of Thick Disk

We consider the axially symmetric gravitational field generated by a stellar body multipolar moments. In spherical coordinates  $(\mathcal{R}, \theta)$ , its potential has the form

$$\Phi_m(\mathcal{R}, \theta) = -\sum_{n=0}^m \frac{C_n P_n(\cos \theta)}{\mathcal{R}^{(n+1)}}, \quad (14)$$

where  $m \in \mathbb{Z}^+$  and  $C_n$  are constants related with the multipolar moments ( $2^n$  poles), we are interested in the future in test particle motion in the exterior of source, thus we regard only external multipolar moments. The relation with cylindrical coordinates are

$$\mathcal{R}^2 = R^2 + z^2, \quad \cos \theta = z/R. \quad (15)$$

with  $\mathcal{R} \geq 0$ ,  $R \in [0, \infty)$ ,  $z \in (-\infty, \infty)$  and  $0 \leq \theta \leq \pi$ .

As an example, if we chose the monopolar, dipolar, and quadrupolar terms, ie  $m = 2$ , thus we obtained the quantities

$$\nu^2 = \frac{2bC_0(b^2 + R^2)^2 + 3bC_2(2b^2 - 3R^2) + 2C_1(2b^4 + b^2R^2 - R^4)}{2a(b^2 + R^2)^{7/2}}, \quad (16)$$

$$\kappa^2 = \frac{3(C_1(8b^5 + 6b^3R^2 - 2bR^4) + C_2(16b^4 - 18b^2R^2 + R^4))}{2(b^2 + R^2)^{9/2}} + \frac{2C_0(4b^2 + R^2)(b^2 + R^2)^2}{2(b^2 + R^2)^{9/2}}, \quad (17)$$

$$\ell^2 = \frac{R^4(2C_0(b^2 + R^2)^2 + 6bC_1(b^2 + R^2) + 3C_2(4b^2 - R^2))}{2(b^2 + R^2)^{7/2}}, \quad (18)$$

$$v_c^2 = \frac{R^2(2C_0(b^2 + R^2)^2 + 6bC_1(b^2 + R^2) + 3C_2(4b^2 - R^2))}{2(b^2 + R^2)^{7/2}}, \quad (19)$$

$$\sigma = \frac{2bC_0(b^2 + R^2)^2 + 3bC_2(2b^2 - 3R^2) + 2C_1(2b^4 + b^2R^2 - R^4)}{4\pi G(b^2 + R^2)^{7/2}}. \quad (20)$$

Here  $C_0, C_1, C_2$  are monopolar, dipolar, and quadrupolar terms, respectively. According to these equations we find the range

$$b > 0, \quad \text{and} \quad C_2 > \frac{1}{6}(-b^2C_0 - 3b) \quad (21)$$

wherein  $C_1 \in \mathbb{R}$  and  $C_0 > 0$ , so that the specific angular momentum and circular velocity are positives. Whereas the frequencies epicyclic and vertical we find the intervals

$$C_2 \leq \frac{b^2C_0}{3}, \quad \text{and} \quad C_1 > \frac{-b^2C_0 - 6C_2}{3b}, \quad (22)$$

$$C_2 > \frac{1}{3}b^2C_0, \quad \text{and} \quad C_1 > \frac{-b^2C_0 - 3C_2}{2b} \quad (23)$$

respectively. Finally, for surface density we obtained

$$b > 0, \quad \text{and} \quad C_2 > \frac{1}{3}(-b^2C_0 - 2bC_1) \quad (24)$$

For example, we choose the following values for parameters  $C_0 = 5$ ,  $C_1 = 2$ ,  $C_2 = 3$  (prolate deformed objects),  $a = 10$ ,  $b = 3$  and  $G = 1$ , which satisfied all quantities physicals and the model is stable. In the Fig.1, we show (a) Circular velocity. (b) Epicyclic frequency. (c) Vertical frequency. (d) Specific angular momentum. In Fig.2 we presented (a) Surface density. (b) Density of thick disk, for our stable model. The disk is stable in the plane to radial and vertical perturbations of its radius, Fig. 1(b) o Fig.1(d) and Fig.1(c), respectively. The circular velocity, and densities also have a behaviour appropriate Fig.1(a), Fig.2(a) and Fig.2(b).

## 6 Conclusiones

We present the mass density  $\rho$ , of a new family of thick disk considered *displace, cut, fill and reflect* method (**González & Letelier**, 2004; **Vogt & Letelier**, 2005). Also calculated of the physical quantities in plane the disk as circular velocity  $v_c$  of particles, the epicyclic frequency,  $\kappa$ , the vertical frequency,  $\nu$ , surface density  $\sigma$ ,

and specific angular momentum,  $\ell$ , the latter is used to determine the stability of the model (Rayleigh criterion).

We consider only the first three terms of multipolar expansion potential, ie the mass monopole, and terms dipolar and quadrupolar. Thus, we can find an interval for which it meets all the physical quantities of the model, section 5. Then we chose some particular parameters for graphics of physical quantities of a specific model (Fig.1 and Fig.2). Additionally, we presented the equations of evolution of motion (section 4), work that we think developed in a later article to complete the models.

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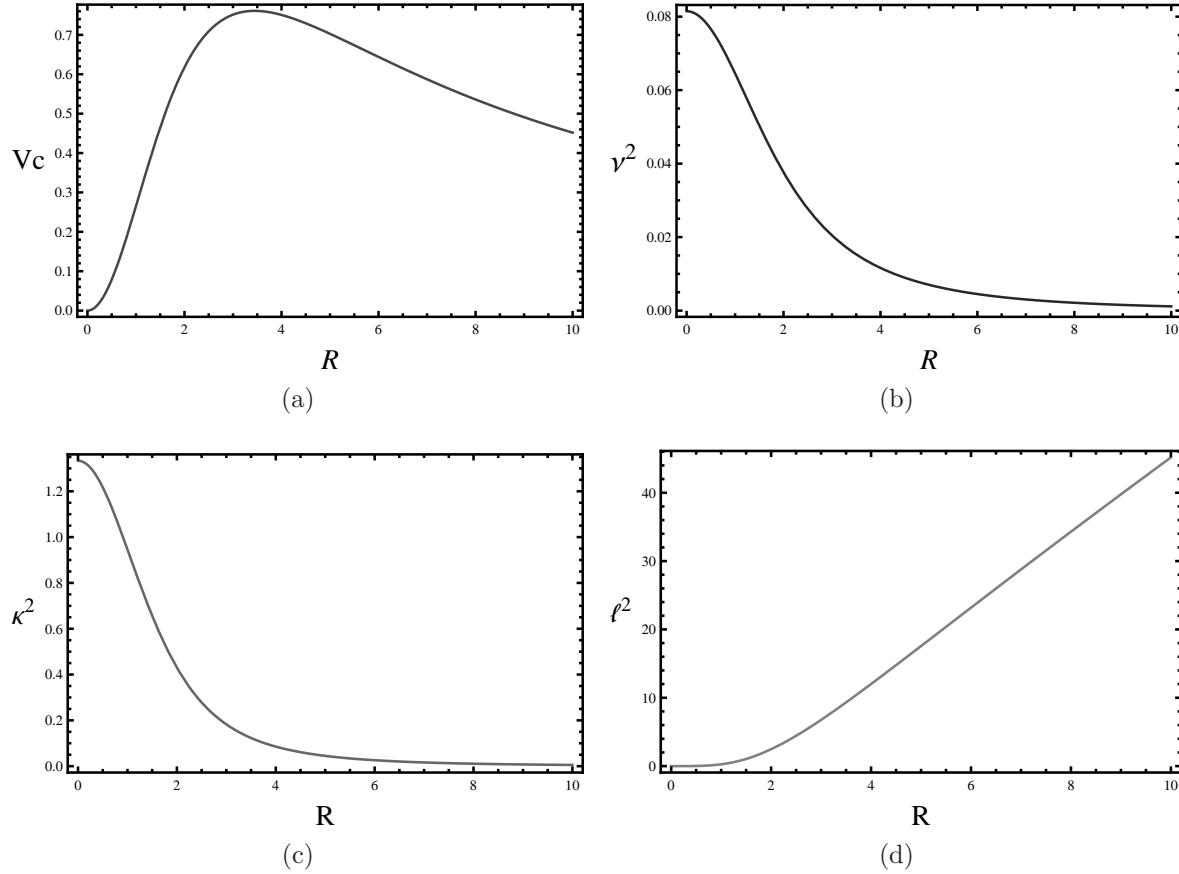


Figure 1: (a) Circular velocity. (b) Epicyclic frequency. (c) Vertical frequency. (d) Specific angular momentum. The parameters utilized are  $C_0 = 5$ ,  $C_1 = 2$ ,  $C_2 = 3$ ,  $a = 10$ ,  $b = 3$ .

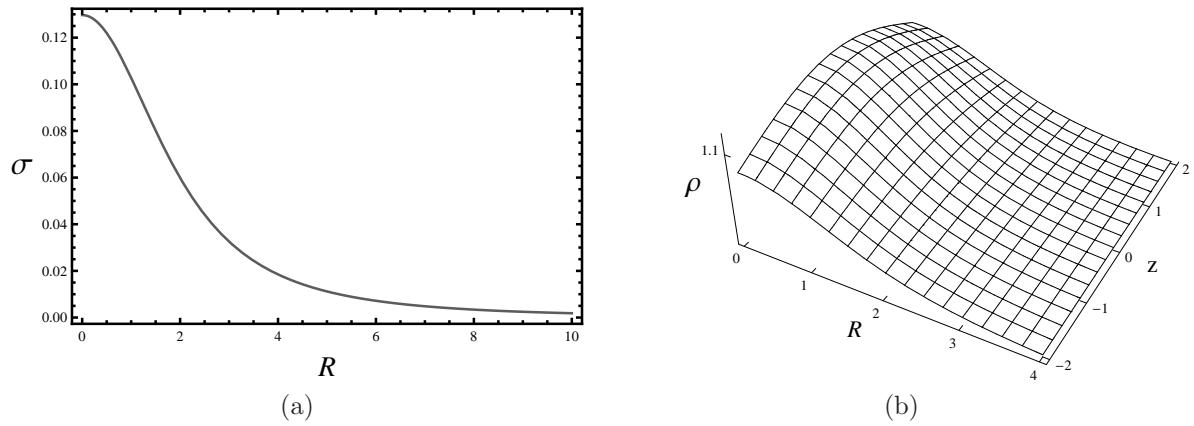


Figure 2: (a) Surface density. (b) Density of thick disk. The parameters used are  $C_0 = 5$ ,  $C_1 = 2$ ,  $C_2 = 3$ ,  $a = 10$ ,  $b = 3$ .