Revista Colombiana de Estadística
Junio 2010, volumen 33, no. 1, pp. 127 a 147

Synthesizing the Ability in Multidimensional Item Response Theory Models

Habilidad sintética en modelos multidimensionales de teoría de respuesta al ítem

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Abstract

A central problem associated with Multidimensional Item Response Theory (MIRT) Models is the impossibility of ordering the examinees. In this paper, we obtain two unidimensional synthetic indices that are optimal linear combinations of the ability vector. These synthetic indices are similar to the reference composite commonly used in MIRT models, but they are easier to calculate and interpret. The synthetic indices are compared with the unidimensional ability obtained when a multidimensional data is fitted with an unidimensional IRT (UIRT) model.

***Key words*:** Binary response, Item response theory, Index, Multidimensional data, Synthetic estimator, Latent trait.

Resumen

Un problema central asociado con los Modelos Multidimensionales de Teoría de Respuesta al Item (TRIM) es la imposibilidad de ordenar a los examinados. En este artículo, se obtienen dos índices sintéticos unidimensionales que son combinaciones lineales óptimas del vector de habilidades. Estos índices sintéticos son semejantes a la composición de referencia comúnmente usada en los modelos TRIM, pero son más fáciles de calcular. Los índices sintéticos se comparan con el parámetro de habilidad obtenido cuando un conjunto de datos multidimensionales es ajustado con un modelo TRI unidimensional.

***Palabras clave*:** respuesta binaria, teoría de respuesta al ítem, índice, datos multidimensionales, estimador sintético, trazo latente.

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1. Introduction

This research originated in recent results obtained by Levine (2003) and Carroll & Levine (2007) in the context of Multidimensional Item Response Theory. They proved that any multidimensional model has unidimensional submodels that are equivalent to the original model.

The unidimensional item response theory models (UIRT) consist of models according to which the interactions of persons with items can be adequately represented by a unique parameter describing the characteristics of the person (Reckase 2009).

The multidimensional item response theory (MIRT) models are based on the assumption that people require more than one basic ability to response correctly to an item in a test. There are two major types of MIRT models-the compensatory models Reckase (1985, 1997, 2007) and the non-compensatory or partial compensatory models (Sympson 1978). In this research, we only refer to the compensatory MIRT models, that we will call them simply MIRT models.

Stout (1990) introduced the concept of essential unidimensionality. The central idea of Stout is that even though the ability space is multidimensional, the set of items used in a test may be sensitive, mainly to differences along one of the dimensions. The statistical tests to assess unidimensionalidad can reject the unidimensional assumption. Stout et al. (1999) developed DIMTEST, a procedure to test the assumption of essential unidimensionality of the person's ability.

Several authors tried to determine the relationship between the ability vector $\boldsymbol{\theta}$ and the unidimensional ability denoted θ, obtained by fitting a unidimensional model to data that were generated from multidimensional models. Ansley & Forsyth (1985) examined the unidimensional estimates for two dimensional data using a noncompensatory model. They studied situations in which the θ's were correlated with correlations values of 0.0, 0.3, 0.6, 0.9 and 0.95. Way et al. (1988) also compared the effects of using a UIRT model to estimate two dimensional data for both the noncompensatory and the compensatory MIRT model. Reckase (1990, 1986) reported that, in some situations, where a multidimensional data matrix was fitted with a UIRT model, the dimensionality and the difficulty were confused.

Ackerman (1989) reported that, in his simulations, the unidimensional estimate of θ was highly correlated with $(\theta_1 + \theta_2)/2$; this correlation was better when the correlation of the abilities was increased. Reckase & Ackerman (1988) suggested to build unidimensional tests from multidimensional items by grouping the items that measure more similar the linear combinations of abilities.

Folk & Green (1989) stated that $\widehat{\theta}$ is strongly related to some optimal combination of θ_1 and θ_2, even for simulate samples with low correlations. Doody (1985) reported studies about the robustness of unidimensional fitting applied to two dimensional data. Zhao et al. (2002), in a simulated study of computerized adaptive tests, founded similar results. As Ackerman, they compared the ability $\widehat{\theta}$ with $(\theta_1 + \theta_2)/2$. Walker & Beretvas (2003) compared multidimensional and unidimensional proficiency using real data from a large-scale mathematics test and obtained similar results.

Recently, Sheng (2007) showed that when each one of the items measures essentially only one ability , a multi-unidimensional model fits better the data.

In this paper, we review the previous works about the important issue of synthesizing the latent ability vector in MIRT models. We derive two optimal linear combinations of the components of the ability vector, which are synthetic indices of the abilities. Through a simulation study, we compared the proposed indices with the others proposed previously, and we infer that all the synthetic indices are similar. Our indices are easier to compute and interpret by the experts. The synthetic indices obtained are also estimations of the linear combination of the latent ability vector that is best measured by a test. We state how the covariance of the latent ability vector affects the synthetic index. Finally, we infer through a second simulation study that when the multidimensional data is fitted with a unidimensional model, the unidimensional latent ability is precisely the synthetic index of the ability vector. In the paper, the terms latent ability and latent trait are used as synonyms.

2. The geometrical facts

When a UIRT model is used to fit data set, it is usual to assume a normal standard distribution for the abilities of the individuals. Clearly, if the data is multidimensional, the correlation matrix of the ability vector is the identity matrix. But, if really the correlation matrix is not the identity, the obvious question is what happens with the item and the ability parameters when this information about the correlation matrix of the abilities is omitted?

The works reviewed in Section 1 suggested us that when a data set is generated from a MIRT model and the correlation matrix of the ability vector is not the identity, a unidimensional model can fit well the data. This lead us to conjecture that if the unidimensional model is used with the assumption that the abilities have a normal standard distribution, the correlation matrix of the abilities transforms the direction of the items in such a way that in the extreme case all of them must be aligned. The direction of an item is discussed in Section 3. Also, the results reported in Section 1 seems to suggest that in the extreme case the unique direction of the items is just $\frac{1}{\sqrt{d}}\mathbf{1}_d$, where d is the dimension of the ability space. This conjecture lead us to propose and prove the results of this Section. The required facts from d-dimensional geometry can be consulted in the Appendix.

Theorem 1. *Let Σ be a $d \times d$ symmetric and positive definite matrix, such that all its diagonal elements are 1 and the off-diagonal elements are nonnegative. Let $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ be unitary vectors of $\mathbb{R}^d$, such that all their elements are nonnegative. Let $|\boldsymbol{\Sigma}|$ be the determinant of $\boldsymbol{\Sigma}$, then*

$$\left[\frac{\boldsymbol{\beta_1^t \Sigma \beta_2}}{\sqrt{(\boldsymbol{\beta_1^t \Sigma \beta_1})(\boldsymbol{\beta_2^t \Sigma \beta_2})}}\right]^2 \geq 1 - |\boldsymbol{\Sigma}|(1 - (\boldsymbol{\beta_1^t \beta_2})^2) \tag{1}$$

***Proof*.** Let $\Sigma^{1/2}$ be the squared root of Σ. Let $\gamma_i = (\Sigma^{1/2}\beta_i)/\sqrt{\beta_i^t \Sigma \beta_i}$, $i = 1, 2$. Then, the vectors γ_1 and γ_2 are unitary. Let $vol(\gamma_1, \gamma_2)$ be the volume of the parallelotope determined by the vectors γ_1 and γ_2. From equations (34), (36) and (37) in the Appendix, it follows that

$$vol^2(\gamma_1, \gamma_2) = 1 - \left[\frac{\boldsymbol{\beta_1^t \Sigma \beta_2}}{\sqrt{(\boldsymbol{\beta_1^t \Sigma \beta_1})(\boldsymbol{\beta_2^t \Sigma \beta_2})}}\right]^2 \tag{2}$$

and

$$vol^2(\gamma_1, \gamma_2) = \frac{|\boldsymbol{\Sigma}| vol^2(\boldsymbol{\beta_1}, \boldsymbol{\beta_2})}{(\boldsymbol{\beta_1^t \Sigma \beta_1})(\boldsymbol{\beta_2^t \Sigma \beta_2})} \tag{3}$$

The properties of matrix Σ permit us to conclude that $\beta_i^t \Sigma \beta_i \geq 1, \ i = 1, 2$. The result follows from this fact and also from the previous two equations and Lemma 3 in the Appendix. □

Corollary 1. *Under the conditions of Theorem 1, we have that*

$$\left[\frac{\boldsymbol{\beta_1^t \Sigma \beta_2}}{\sqrt{(\boldsymbol{\beta_1^t \Sigma \beta_1})(\boldsymbol{\beta_2^t \Sigma \beta_2})}}\right] \geq (\boldsymbol{\beta_1^t \beta_2}) \tag{4}$$

***Proof*.** The result follows from the fact that $|\boldsymbol{\Sigma}| \leq 1$. □

In the next result, we assume that $\boldsymbol{\Sigma_m^{1/2}}$ is the squared root of $\boldsymbol{\Sigma_m}$.

Theorem 2. *Let $\boldsymbol{\Sigma_m}$ be a sequence of $d \times d$ matrices that have the same properties than $\boldsymbol{\Sigma}$ in Theorem 1, and such that their determinants are decreasing and that $|\boldsymbol{\Sigma_m}| \to 0$ as $m \to \infty$. Let $\boldsymbol{\beta_m} = \boldsymbol{\Sigma_m^{1/2}\beta}$, where $\boldsymbol{\beta}$ is any not-zero vector, where all of its components are nonnegative. Thus, $\boldsymbol{\beta_m}/||\boldsymbol{\beta_m}|| \to \frac{1}{\sqrt{d}}\mathbf{1}_d$, where $\mathbf{1}_d$ is the vector with 1's at all its components.*

***Proof*.** It is easy to see that $|\boldsymbol{\Sigma}| = 0$, if and only if $\boldsymbol{\Sigma} = J_d$, where J_d is the matrix with 1's in all of its components. Thus, $\boldsymbol{\Sigma_m^{1/2}} \to \frac{1}{\sqrt{d}} J_d$. □

Suppose that $\boldsymbol{\Sigma}$ is a correlation matrix. It can be shown that if the off-diagonal elements of the matrix $\boldsymbol{\Sigma}$ become large, then the determinant of the matrix $\boldsymbol{\Sigma}$ decreases due to the relationship

$$|\ \boldsymbol{\Sigma}\ | = (1 - R^2_{p.1\cdots p-1})(1 - R^2_{p-1.1\cdots p-2}) \cdots (1 - R^2_{2.1})$$

where $R^2_{k\cdots.d}$ is the squared multiple correlation coefficient between the variable k and the following variables. See, for example (Peña 2002), (Peña & Rodríguez 2003).

From theorems 1 and 2 we conclude that, if the off-diagonal elements of the matrix $\mathbf{\Sigma}$ are increased, all the transformed vectors $\mathbf{\Sigma}_{m}^{1/2}\boldsymbol{\beta}$ have a smaller angle between them than the original vectors, and the respective transformed normalized vectors have a greater orthogonal projection between them. Also, all the transformed vectors are conducted toward the unitary vector $\frac{1}{\sqrt{d}}\mathbf{1}_d$. In the limit case, all the transform vectors align with that unitary vector.

3. The nature of the items in the MIRT model

In this Section, we show that any item in a compensatory MIRT model is essentially unidimensional and prove that the item response hypersurface of an item in a MIRT model is monotonic along any direction. This property allows exchanging the item response function (IRF) and the item response hypersurface (IRHS) as in the unidimensional case, but also permits us to determine what an item really measures in a MIRT model.

In the logistic two parameter model (Baker & Seok-Ho 2004), (Bock 1972), (Bock & Jones 1968), (Hambleton et al. 1991), the probability of a correct response for the unidimensional case is given by

$$p_j(\theta_i) = P(X_{ij} = 1 \mid \theta_i, a_j, b_j) = \frac{1}{1 + e^{-a_j(\theta_i - b_j)}} \tag{5}$$

where X_{ij} is the response of person i to item j; $X_{ij} = 1$ if the examinee i responses correctly to item j, and $X_{ij} = 0$ otherwise; θ_i is the unidimensional ability parameter for person i. The scale parameter a_j is called the discrimination parameter of item j, and b_j is the difficulty or position parameter of item j.

The function $f_j(\theta) = p_j(\theta)$ is called the item response function (IRF) and its graph is the item response curve (IRC). Note that

$$f_j(b_j) = \frac{1}{2} \tag{6}$$

and,

$$f'(b_j) = \frac{1}{4}a_j \tag{7}$$

so, except by the term $1/4$, a_j represents the slope of the IRC at the point b_j.

In the classical compensatory MIRT model, there is more that one ability measured by a test. Let $\boldsymbol{\theta}_i$ be a vector of $\mathbb{R}^d$ that represents the ability vector of the examinee i. The parameters of item j in this case are: $\boldsymbol{a}_j$, a vector of $\mathbb{R}^d$ related with the discrimination of the item and γ_j, a scalar related with the difficulty of the item. The probability that an examinee with ability vector $\boldsymbol{\theta}_i$ responses correctly to item j is given by

$$P(X_{ij} = 1 \mid \boldsymbol{\theta}_i, \boldsymbol{a}_j, \gamma_j) = \frac{1}{1 + e^{-(\boldsymbol{a}_j^t\boldsymbol{\theta}_i + \gamma_j)}} \tag{8}$$

The component θ_{ik} of $\boldsymbol{\theta_i}$ represents the ability of the person i in the k-th dimension. The interpretations of $\boldsymbol{a}_j$'s and γ_j's parameters are a little different from those in the unidimensional case. Reckase (1985, 1997, 2007) states that the MIRT model does not provide a direct interpretation about the parameters $\boldsymbol{a}_j$ and γ_j. In this case, the item response function $f_j(\boldsymbol{\theta}) = p_j(\boldsymbol{\theta})$ is a multivariate function and its graph is a hypersurface. Let α_j be the norm of the vector $\boldsymbol{a}_j$, that is,

$$\alpha_j = \sqrt{\sum_{k=1}^{d} a_{jk}^2}$$

where the a_{jk}'s are the components of vector $\boldsymbol{a}_j$. Then, the vector $\boldsymbol{a}_j$ can be rewritten as

$$\boldsymbol{a}_j = \alpha_j \boldsymbol{\beta}_j \tag{9}$$

where $\boldsymbol{\beta_j} = (\beta_{j1}, \beta_{j2}, \dots, \beta_{jd})^t$, $\beta_{jk} = a_{jk}/\alpha_j$. Clearly, $\boldsymbol{\beta}_j$ is a unitary vector of $\mathbb{R}^d$. Thus, the model given by Equation (8) can be rewritten as

$$P(X_{ij} = 1 \mid \boldsymbol{\theta}_i, \alpha_j, \boldsymbol{\beta}_j, b_j) = \frac{1}{1 + e^{-\alpha_j(\boldsymbol{\beta_j^t}\boldsymbol{\theta_i} - b_j)}} \tag{10}$$

where $b_j = -\gamma_j/\alpha_j$. Reckase (1985) defined the value α_j as the multidimensional discrimination (MDISC) parameter and the value b_j as the multidimensional difficulty (MDIFF) parameter. He showed that α_j is the slope at the point of the steepest slope in the direction specified by the vector $\boldsymbol{\beta}_j$, called the *direction of item j*.

Additionally, he proved that b_j is the distance from the origin to the point of the steepest slope. We will show in this Section why the MDISC and MDIFF names are justified.

At this point, we introduce the concept of item response hypersurface. In the UIRT models, one may use the item response function (IRF) and its geometrical representation-the item response curve (IRC)-almost interchangeable. In the multidimensional case, however, the matter is not so straightforward.

First, we fix some notations. For any $v \in \mathbb{R}^d$, the ray of v is defined to be the line $\mathbb{R} \cdot v$ in $\mathbb{R}^d$ determined by $\mathbb{R} \cdot v = \{tv \in \mathbb{R}^d \mid t \in \mathbb{R}\}$. Similarly, for $v, w \in \mathbb{R}^d$ the directed line going through w is defined by

$$w + \mathbb{R} \cdot v = \{w + tv \in \mathbb{R}^d \mid t \in \mathbb{R}\}$$

Definition 1. A dichotomous item response hypersurface is a d-dimensional smooth submanifold M of $\mathbb{R}^d \times [0,1]$, so that for any two vectors $v, w \in R^d$ the intersection of $(w + \mathbb{R} \cdot v) \times [0,1]$ and M is the graph of a monotonic function $f_{v,w} : w + R \cdot v \to [0,1]$.

We shall use the notation $f_v = f_{v,0}$. Definition 1 and the notation were taken from Antal's paper (Antal 2007).

Lemma 1. *The graph of the item response function given by,*

$$f(\boldsymbol{\theta}) = \frac{1}{1 + e^{-\alpha_j(\boldsymbol{\beta}_j^t\boldsymbol{\theta}+\gamma_j)}} \tag{11}$$

is a dichotomous item response hypersurface.

Proof. Let $\boldsymbol{v}, \boldsymbol{w}$ be two arbitrary vectors of $\mathbb{R}^d$ and consider the line given by $\boldsymbol{\eta}(t) = \boldsymbol{w} + t\boldsymbol{v}, t \in \mathbb{R}$. Clearly, $\boldsymbol{\beta}_j^t\boldsymbol{\eta}(t) = \boldsymbol{\beta}_j^t w + (\boldsymbol{\beta}_j^t\boldsymbol{v})t$ is a monotonic function of t and then $f(\boldsymbol{\eta}(t))$ is a monotonic function along the direction $\boldsymbol{v}$ through $\boldsymbol{w}$. □

As a consequence of Lemma 1, the item response function (11) defines a dichotomous item response hypersurface and the MIRT model is completely determined by these hypersurfaces.

Lemma 2. *The item response function $f_j(\boldsymbol{\theta})$ of a MIRT model is constant in the orthogonal complement of vector $\boldsymbol{\beta}_j$.*

Proof. For any vector $\boldsymbol{\eta}$ in the orthogonal subspace of $\boldsymbol{\beta}_j$, $\boldsymbol{\beta}_j^t\boldsymbol{\eta} = 0$, so, $f_j(\boldsymbol{\eta}) = 1/(1 + e^{-\alpha_j\gamma_j})$. □

The next Corollary can be directly proven.

Corollary 2. *Given $\boldsymbol{w} \in \mathbb{R}^d$, the item response function $f_j(\boldsymbol{\theta})$ is constant in the hyperplane parallel to the orthogonal complement of vector $\boldsymbol{\beta_j}$ that contains $\boldsymbol{w}$.*

This Corollary is well-known. It states that the contours of equiprobability are hyperplanes, and that all of them are parallel. However, the important fact is that they are orthogonal to the vector $\boldsymbol{\beta_j}$. Theorem 3 is the main result of this Section. It establishes that the item response function $f_j(\boldsymbol{\theta})$ is a trivial extension of a unidimensional item response function (UIRF). According to Equation (9) we will use the expression $\boldsymbol{a_j} = \alpha_j\boldsymbol{\beta_j}$ in the Proof. It is not necessary, but is useful to understand the result.

Theorem 3. *The multidimensional IRF $f_j(\boldsymbol{\theta})$ of a MIRT model is a trivial extension of a classical UIRF.*

Proof. Let $\boldsymbol{\theta}$ be a vector in $\mathbb{R}^d$, and let $\{\boldsymbol{\beta}_j, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-1}\}$ be a normed orthogonal basis of $\mathbb{R}^d$ that contains the vector $\boldsymbol{\beta}_j$. Then, there exist real numbers $t, t_1, \dots, t_{d-1}$ such that

$$\boldsymbol{\theta} = t\boldsymbol{\beta}_j + t_1\boldsymbol{v}_1 + \cdots + t_{d-1}\boldsymbol{v}_{d-1}$$

then,

$$\boldsymbol{\beta}_j^t\boldsymbol{\theta} = (\boldsymbol{\beta}_j^t\boldsymbol{\beta}_j)t = t \tag{12}$$

Hence,

$$f_j(\boldsymbol{\theta}) = \frac{1}{1 + e^{-\alpha_j\boldsymbol{\beta}_j^t\boldsymbol{\theta}-\gamma_j}} = \frac{1}{1 + e^{-\alpha_j t-\gamma_j}} = \frac{1}{1 + e^{-\alpha_j(t-b_j)}} = p_{\boldsymbol{\beta}_j}(t) \tag{13}$$

□

The notation $p_{\boldsymbol{\beta}_j}$ is used to emphasize the direction $\boldsymbol{\beta}_j$, and that $f_j(\boldsymbol{\theta})$ is an extension of a UIRF. Theorem 3 shows an explicit way to construct the hypersurface defined by $f_j(\boldsymbol{\theta})$ from a unidimensional IRC. Let $p_j(t)$ be the UIRF defined by

$$p_j(t) = \frac{1}{1 + e^{-\alpha_j(t-b_j)}} \tag{14}$$

The function $p_j(t)$ can be trivially extended to a multivariate function by $p_j(t_1, \dots, t_d) = p_j(t_1)$. The original hypersurface is obtained by a rigid rotation of the hypersurface defined by $p_j(t_1, t_2, \dots, t_d)$ on the hyperplane defined by the canonical vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_d$, which aligns vector $\boldsymbol{e}_1$ with vector $\boldsymbol{\beta}_j$. This is a general result, since any rotation in $\mathbb{R}^d$ can be done in this way. The theory of rigid rotations in d-dimensional spaces can be found in Aguilera & Pérez-Aguila (2004) and Mortari (2001). A direct and important consequence of Theorem 3 is stated in the next Corollary.

Corollary 3. *Let's suppose that the directions of all items in a MIRT model coincide, that is, $\boldsymbol{\beta}_i = \boldsymbol{\beta}$, for all i. Then, the model is essentially unidimensional. In other words, the MIRT model is a trivial extension of a UIRT model.*

The result of Corollary 3 was first proven by Stout and Reckase in a paper presented at a meeting of the Psychometric Society (Reckase & Stout 1995). Reckase (2009) reproduced the result (Theorem 1, page 197).

Other useful properties of the MIRT model follow. On the hyperplane $\boldsymbol{\beta}_j^t\boldsymbol{\theta} - b_j = 0$ we have that

$$f_j(\boldsymbol{\theta}) = 1/2 \tag{15}$$

It is straightforward to verify that for all $\boldsymbol{\theta}$ in that hyperplane

$$\frac{\partial f_j}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}) = \frac{1}{4}\alpha_j \boldsymbol{\beta}_j \tag{16}$$

So, as in Equation (7), the parameter α_j, except by the constant $1/4$, is the slope of the IRHS for all $\boldsymbol{\theta}$ in the hyperplane $\boldsymbol{\beta}_j^t\boldsymbol{\theta} - b_j = 0$. The slope in the direction $\boldsymbol{\beta}_j$ is maximum when the IRHS crosses the hyperplane (Reckase 1985).

From equations (10), (15) and (16), we can conclude that IRHS of item j in the MIRT model is a trivial extension of a unidimensional IRC whose parameters of discrimination and difficulty are respectively α_j and $b_j = -\gamma_j/\alpha_j$. Also, it is clear that item j measures the linear combination of the abilities given by $\boldsymbol{\beta}_j^t\boldsymbol{\theta}$.

4. Synthesizing the latent ability

A unidimensional synthetic index of the latent trait vector in a MIRT model is usually called a composite. The formal concept is given in Definition 2.

Definition 2. A *composite* Θ_β of the complete latent trait vector $\boldsymbol{\Theta}$ is a linear combination of $\boldsymbol{\Theta}$, that is $\Theta_\beta = \boldsymbol{\beta^t}\boldsymbol{\Theta} = \sum_{k=1}^{d} \beta_k \Theta_k$, where $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_d)^t$ is a constant vector called the direction of the composite Θ_β. If $Var(\Theta_\beta) = 1$, Θ_β will be called a normalized composite.

Some authors have done theoretically developments to construct a unidimensional synthetic index of the latent trait vector. Yen (1985) considered an approximation of a MIRT model by a UIRT, using a least squares (LS) approach. She used the objective function

$$G\left[\widehat{a},\widehat{b},\widehat{\theta}\right]=\sum_{i}\sum_{j}\left[\widehat{a_j}(\widehat{\theta_i}-\widehat{b_j})-\alpha_j\boldsymbol{\beta}_j^t\boldsymbol{\theta}_i-\gamma_j\right]^2 \tag{17}$$

where $\widehat{a}=(\widehat{a_1},\dots,\widehat{a_p})^t$, $\widehat{b}=\left(\widehat{b_1},\dots,\widehat{b_p}\right)^t$, $\widehat{\theta}=\left(\widehat{\theta_1},\dots,\widehat{\theta_N}\right)^t$ are the corresponding parameters in an approximate UIRT model, and p is the number of items. The respective LS equations do not have a closed solution. Then, she assumed the particular case where $\boldsymbol{\beta}_i=\boldsymbol{\beta}, i=1,\dots,p$, to obtain the solution

$$\widehat{\theta_i}=\frac{\boldsymbol{\beta}^t\boldsymbol{\theta}_i}{\sqrt{\boldsymbol{\beta}^t\boldsymbol{\Sigma}\boldsymbol{\beta}}} \tag{18}$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of the latent trait $\boldsymbol{\theta}$. This result can be obtained as a direct consequence of Theorem 3, since in this particular case all directions of the items coincide, and then we have essentially a UIRT along the direction $\boldsymbol{\beta}$.

Let $\{X_1,\dots,X_m\}$ be a subtest, and let $Y=\sum_{j=1}^m X_j$ be the subtest number correct score, let $\xi(\boldsymbol{\theta})=\sum_{j=1}^m p_j(\theta)$ be the true subtest score. Zhang & Stout (1999) defined the direction of score Y as the vector $\boldsymbol{\xi}$ that maximizes the *expected multidimensional critical ratio* (EMCR) defined as

$$EMCR(\boldsymbol{\xi},\boldsymbol{\theta};Y)=E\left[\frac{\nabla_{\boldsymbol{\xi}}\xi(\boldsymbol{\theta})}{[Var(Y\mid\boldsymbol{\theta})]^{\frac{1}{2}}}\right] \tag{19}$$

where $\nabla_{\boldsymbol{\xi}}\xi(\boldsymbol{\theta})$ is the directional derivative of the true score $\xi(\boldsymbol{\theta})$ in the direction $\boldsymbol{\xi}$. The EMCR function gives the average discrimination power of the observed score Y in the direction $\boldsymbol{\xi}$. They showed that vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi}=\sum_{j=1}^m\omega_i\boldsymbol{\beta}_j \tag{20}$$

where $\omega_i=cE\left[H_i^{'}(\alpha_j\boldsymbol{\beta}_j^t\boldsymbol{\theta}+\gamma_j)\,/\,\sqrt{Var(Y\mid\boldsymbol{\theta})}\right]$. $H_i(\cdot)$ represents the item response function. Clearly, the direction $\boldsymbol{\xi}$ in Equation (20) depends on the response function, and it is an average on the latent trait population. In this case, $\boldsymbol{\xi}^t\boldsymbol{\theta}$ is the composite that is best measured by the subtest. The reference direction $\boldsymbol{\xi}$ was called the direction of the subtest.

Wang (1985, 1986) constructed a unidimensional approximation to a multidimensional data matrix that he called the *reference composite trait*. He used the transformation $y=ln[p/(1-p)]$, the item logistic score, and rewrote the logistic MIRT model as

$$\boldsymbol{Y}=\boldsymbol{\theta}\boldsymbol{A}^t+\mathbf{1}_K\boldsymbol{\gamma}^t \tag{21}$$

where $\boldsymbol{\theta}$ is the matrix of the latent traits, $\boldsymbol{A}$ is the $K\times d$ matrix of the discrimination parameters in the MIRT model, K is the number of items, $\mathbf{1}_K$ is the K-vector of ones and $\boldsymbol{\gamma}$ is the vector associated with the difficulty. The objective function in this case is the trace of $\left(\boldsymbol{Y}-\widehat{\boldsymbol{Y}}\right)^t\left(\boldsymbol{Y}-\widehat{\boldsymbol{Y}}\right)$, where $\widehat{\boldsymbol{Y}}=\boldsymbol{G}\boldsymbol{H}^t+\mathbf{1}_K\boldsymbol{\gamma}^t$. Here $\boldsymbol{G}$ is the unidimensional latent trait in the approximate model and $\boldsymbol{H}$ is the vector of discrimination item parameters in that model. Observe that it is assumed that the difficult parameters do not change. Wang showed that in this case

$$\boldsymbol{G}=\boldsymbol{\theta}\boldsymbol{\omega}, \tag{22}$$

where $\boldsymbol{\omega}$ is the (unit-length) eigenvector associated with the largest eigenvalue of the matrix $\boldsymbol{A^t}\boldsymbol{A}$.

Theorem 3 states that all items in a compensatory MIRT model are essentially unidimensional. Then, the multidimensional nature of a MIRT model can only be attributed to the item directions $\boldsymbol{\beta_j}$. Corollary 3 states that when the directions of all items coincide, the model is a trivial extension of a UIRT model. These results encouraged us to derive a unidimensional synthetic ability in a different way than Yen, Wang, and Zhang and Stout.

We observed that if all $\boldsymbol{\beta_j}$'s are the same, and $\boldsymbol{\beta_j}=\boldsymbol{\beta}$, $j=1,\dots,K$, where K is the number of items in the test, then Equation (10) reduces to

$$P(X_{ij}=1\mid\boldsymbol{\theta_i},\alpha_j,\boldsymbol{\beta},b_j)=\frac{1}{1+e^{-\alpha_j(\boldsymbol{\beta^t\theta_i}-b_j)}} \tag{23}$$

that is a trivial extension of an UIRT model, where each one of the items measures the same composite of the abilities given by $\boldsymbol{\beta}^t\boldsymbol{\theta}_i$. This observation suggests looking for a vector $\boldsymbol{\beta}$ that summarizes the $\boldsymbol{\beta}_j$'s. Since these vectors are all unitary, they are in the unitary hypersphere of $\mathbb{R}^d$. Also, we can assume that the components of the vectors $\boldsymbol{\beta_j}$ are all non-negative, then all the vectors are in the same hyper-quadrant. Therefore, it is reasonable to expect that the vector that summarizes all the $\boldsymbol{\beta_j}$'s is the same hyper-quadrant of the unitary hypersphere. This leads us to search the vector $\boldsymbol{\beta}$ by optimizing the objective function given by

$$h(\beta_1,\dots,\beta_d)=\sum_{l=1}^{d}\sum_{k=1}^{K}(\beta_{kl}^2-\beta_l^2)^2 \tag{24}$$

whose solution is the unitary vector given by

$$\beta_l=\sqrt{\left(\frac{1}{K}\sum_{j=1}^{K}\beta_{kl}^2\right)} \tag{25}$$

We will denote the solution vector in this case as $\boldsymbol{\beta}_h$.

Alternatively, it is also reasonable to optimize the objective function

$$g(\beta_1,\dots,\beta_d)=\sum_{l=1}^{d}\sum_{k=1}^{K}(\beta_{kl}-\beta_l)^2 \tag{26}$$

whose solution, considering a unitary vector is given by

$$\beta_l = \frac{\sum_{k=1}^{K} \beta_{kl}}{||\sum_{k=1}^{K} \beta_{kl}||} \quad l = 1, \dots, d \tag{27}$$

The solution vector in this case will be denoted as $\boldsymbol{\beta}_g$.

We finish this Section with an approach about the role of the latent trait correlation matrix. It is usual to assume that the abilities of the examinees in a test constitute a sample drawn from a normal d-dimensional distribution $N(0, \boldsymbol{\Sigma})$. The marginal EM estimation is based on this assumption (Bock & Aitkin 1981).

To obtain an identifiable model, most of the programs written to estimate MIRT models assume that $\boldsymbol{\Sigma} = \boldsymbol{I}_d$, where $\boldsymbol{I}_d$ is the identity matrix. Examples are TESTFACT (Wilson et al. 1987) and recently the ltm package (Rizopoulos 2006). Those are examples of programs that use this assumption. In general, this is not a realistic situation. Software NOHARM (Fraser 1988) estimates the item parameters and the correlation matrix, but it does not estimate the latent abilities. Bégin & Glass (2001) and De la Torre & Patz (2005) proposed MCMC algorithms that simultaneously estimate the item parameters, the latent abilities and the matrix $\boldsymbol{\Sigma}$. In this work, we assume only that the diagonal elements are all 1. This assumption defines a common scale along the canonical axis of the ability space. Ackerman (1989) stated that, in the case where the matrix $\boldsymbol{\Sigma}$ is not the identity, the difficulty and the dimensionality can be confused.

The usual assumption that the correlation matrix is the identity probably resulting the problem mentioned by Ackerman. Let's assume that $\boldsymbol{\theta}$, the latent ability of the examinees, is a sample from a normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$. Then $\boldsymbol{\Sigma}$ has the stochastic representation $\boldsymbol{\theta} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Upsilon}$, where $\boldsymbol{\Upsilon}$ has a multivariate normal standard distribution, and $\boldsymbol{\Sigma}^{\mathbf{1/2}}$ is the squared root of $\boldsymbol{\Sigma}$. Then, we have that

$$\boldsymbol{\beta}^t\boldsymbol{\theta} = \left(\boldsymbol{\Sigma}^{1/2}\boldsymbol{\beta}\right)^t \boldsymbol{\Upsilon} \tag{28}$$

Hence, when in the estimation process it is assumed that the correlation matrix is the identity matrix, the direction of each item is estimated in a transformed space determined by $\boldsymbol{\Sigma}^{1/2}$. Equation (28) shows a procedure to compute the reference direction when the correlation matrix is available.

It is clear that if $\boldsymbol{\theta}$ has a multivariate normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$, any composite $\boldsymbol{\beta}^t\boldsymbol{\theta}$ has a different scale, since $Var(\boldsymbol{\beta}^t\boldsymbol{\theta}) = \boldsymbol{\beta}^t\boldsymbol{\Sigma}\boldsymbol{\beta}$. In this case, the reference direction must be computed from the transformed vectors $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\beta}$, and the synthetic ability must be computed using the transformed ability $\boldsymbol{\Upsilon} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\theta}$.

5. Simulation study

Two simulations were developed to evaluate and compare the synthetic indices $\boldsymbol{\beta}_h^t\boldsymbol{\theta}$ and $\boldsymbol{\beta}_g^t\boldsymbol{\theta}$. This indices are compared with the synthetic indices $\boldsymbol{\xi}^t\boldsymbol{\theta}$ and $\boldsymbol{\omega}^t\boldsymbol{\theta}$, where $\boldsymbol{\xi}$ is the reference direction obtained by Zhang and Stout and $\boldsymbol{\omega}$ is the reference direction obtained by Wang.

5.1. Comparison of the reference directions

Conceptually, the construction of the reference direction of Wang and the reference directions proposed in this paper are very similar. The construction of the reference direction proposed by Zhang and Stout is different.

The vector $\boldsymbol{\xi}$ is the direction in which the total score Y has maximum discriminating power (Zhang & Stout 1999). The vector $\boldsymbol{\omega}$ maximizes the projection of the direction of the items along of it. The vector $\boldsymbol{\beta}_h$ essentially minimizes the angle between this reference direction and the direction of the items. The vector $\boldsymbol{\beta}_g$ minimizes the distance between this reference direction and the direction of the items as points of the latent space. However, all the directions are very similar as we show in this Section.

To review this fact, we generate a set of 60 vector directions in the 3-dimensional latent space. We generate 4 clusters, each one with fifteen directions. To do that, we fixed four directions: $\boldsymbol{b}_1 = (1.0, 1.0, 1.0)^t$, $\boldsymbol{b}_2 = (1.0, 0.2, 0.1)^t$, $\boldsymbol{b}_3 = (0.3, 1.0, 0.1)^t$ and $\boldsymbol{b}_4 = (0.25, 0.25, 1.0)^t$. Then, we generate the vectors of each cluster by adding random noise to each component of the vectors $\boldsymbol{b}$. The noise is smaller in cluster 1 and is augmented progressively until cluster 4.

In a second step, we compute the reference directions $\boldsymbol{\omega}$, $\boldsymbol{\beta}_h$ and $\boldsymbol{\beta}_g$, from all the item directions and from the item directions in each cluster. Additionally, we simulate values of MDISC and MDIFF parameters to generate all the item parameters for 60 items, and then we also computed the reference direction $\boldsymbol{\xi}$ from all the items and from the items in each cluster. We used a logistic response function, and Equation (20).

We considered two different distributions for the latent ability. First, we assumed a 3-variate normal standard distribution and then a 3-variate normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1.0 & 0.3 & 0.6 \\ 0.3 & 1.0 & 0.4 \\ 0.6 & 0.4 & 1.0 \end{pmatrix}$$

Tables 1 and 2 show the results. In Table 1 columns 3, 4 and 5 correspond to the components of the reference directions for the first distribution of the latent abilities and columns 7, 8 and 9 are the components of the reference directions for the second distribution. Finally, we evaluate the synthetic indices comparing them with the original composites. We computed the quantity

$$\boldsymbol{\Delta}_v = \frac{1}{K_v} \sum_{j=1}^{Kv} E\left[\left|\boldsymbol{\beta}^t \boldsymbol{\theta}_v - \boldsymbol{\beta}_{vj}^t \boldsymbol{\theta}\right|\right] \tag{29}$$

where v is respective the cluster, and K_v the size of the cluster.

Table 2 shows the scalar product between the four reference directions.

Table 1: Reference directions for each cluster. Columns 3, 4 and 5 are the components of the reference directions for the distribution $N(\mathbf{0}, \boldsymbol{I})$ and columns 7, 8 and 9 for the distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$.

cluster	vector	comp.1	comp.2	comp.3	Δ	comp.1	comp.2	comp.3	Δ
all	ξ	0.5614	0.6035	0.5663	0.4236	0.4029	0.5666	0.7188	0.2805
	ω	0.5887	0.6071	0.5337	0.4269	0.4254	0.5679	0.7047	0.2784
	β_h	0.5825	0.5843	0.5651	0.4237	0.4544	0.5758	0.6797	0.2822
	β_g	0.5870	0.6025	0.5408	0.4260	0.4187	0.5686	0.7081	0.2787
1	ξ	0.5637	0.5730	0.5949	0.0472	0.4335	0.5509	0.7131	0.0267
	ω	0.5618	0.5735	0.5962	0.0470	0.4324	0.5512	0.7136	0.0266
	β_h	0.5621	0.5731	0.5964	0.0470	0.4328	0.5512	0.7133	0.0267
	β_g	0.5870	0.6025	0.5408	0.0689	0.4187	0.5686	0.7081	0.0303
2	ξ	0.9667	0.2079	0.1495	0.0687	0.7367	0.3718	0.5649	0.0380
	ω	0.9675	0.2157	0.1318	0.0694	0.7388	0.3786	0.5576	0.0385
	β_h	0.9631	0.2206	0.1542	0.0700	0.7380	0.3797	0.5578	0.0386
	β_g	0.9675	0.2157	0.1317	0.0694	0.7388	0.3786	0.5576	0.0385
3	ξ	0.2500	0.9605	0.1225	0.0994	0.2193	0.8837	0.4134	0.0718
	ω	0.2488	0.9619	0.1133	0.0986	0.2195	0.8863	0.4078	0.0716
	β_h	0.2634	0.9534	0.1475	0.1046	0.2312	0.8827	0.4092	0.0711
	β_g	0.2488	0.9619	0.1135	0.0986	0.2195	0.8863	0.4078	0.0716
4	ξ	0.1317	0.2341	0.9632	0.1677	0.1301	0.2750	0.9526	0.1595
	ω	0.1571	0.2412	0.9577	0.1683	0.1533	0.2846	0.9463	0.1591
	β_h	0.2102	0.2825	0.9360	0.1757	0.2008	0.3148	0.9277	0.1628
	β_g	0.1581	0.2412	0.9575	0.1683	0.1537	0.2843	0.9464	0.1592

Table 2: Scalar product between the reference vectors.

cluster	$<\boldsymbol{\xi}.\boldsymbol{\beta}_h>$	$<\boldsymbol{\omega}.\boldsymbol{\beta}_h>$	$<\boldsymbol{\beta}_g.\boldsymbol{\beta}_h>$	$<\boldsymbol{\omega}.\boldsymbol{\beta}_g>$	$<\boldsymbol{\xi}.\boldsymbol{\beta}_g>$	$<\boldsymbol{\xi}.\boldsymbol{\omega}>$
all	0.9979	0.9992	0.9989	1.0000	0.9998	0.9997
1	1.0000	1.0000	0.9997	0.9997	0.9997	1.0000
2	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
3	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000
4	0.9964	0.9982	0.9983	1.0000	0.9997	0.9997
mean	0.9988	0.9995	0.9994	0.9999	0.9998	0.9999

5.2. Comparison of the synthetic ability $\boldsymbol{\beta}_h^t\boldsymbol{\theta}$ with the unidimensional ability of a UIRT model

To evaluate $\boldsymbol{\beta}_h^t\boldsymbol{\theta}$ as a synthetic index of the latent trait vector, we used the following strategy. It is reasonable to expect that the synthetic index of the ability is a good unidimensional summary of the ability vector. Then, if a multidimensional data set is fitted with a unidimensional model, the unidimensional estimative of the ability parameter must be also an estimative of the synthetic index.

In this Section, we evaluate the synthetic index $\boldsymbol{\beta}_h^t\boldsymbol{\theta}$ in forty simulated examples. For clarity, the subscript h will be omitted. All examples are based on 2-dimensional models. One hundred item parameters were simulated as follows. First, the MDISC (the α_j's) parameters were generated from a uniform distribution in the range $[.4, 2]$. Second, the parameters b_j were generated from a normal distribution $N(0, 1)$. Third, the angles that determine the direction of the vectors

$\boldsymbol{\beta_j}$ were generated from a uniform distribution in the range $[5, 50]$. The MDISC parameters were generated in the range $[.4, 2]$ because, this is the usual range of this parameter in real tests. Different prior distributions are assumed for these parameters as a log-normal or a non-informative positive flat distribution (Sheng 2008). We used the last option. The range of angles was chosen to yield a more disperse set of angles as possible. In the simulation of the previous Section, the simulated angles were less dispersed in each cluster.

A sample of 4000 examinees was drawn from the normal bivariate distribution $N(0, \boldsymbol{I_d})$. To examine the impact of the correlation between the $\boldsymbol{\theta}$'s, we respectively introduced correlations of $0, .3, .6$ and $.9$. In all cases, the diagonal elements were 1, thus $\boldsymbol{\Sigma}$ is always a correlation matrix. Also, in all cases a normal standard distribution is assumed for the ability vectors in the estimation process.

Finally, for each correlation matrix a set of binary responses were generated as follows: for each ability vector and each parameter set, the probability of a correct response was computed using Equation (8). Then, a random number u was obtained, from the uniform distribution in the range $[0, 1]$. If the probability of correct response was greater or equal than u the value 1 was assigned to the response. Otherwise, the 0 value was assigned (Kromrey et al. 1999).

We fitted 10 unidimensional models for each set of responses using the ltm package (Rizopoulos 2006). First, we took the first 10 items; then, we took the first 20 items and so until all items were taken. Table 3 shows the main results.

A number of statistical indices were calculated at the simulate sample level to evaluate the synthetic index $\boldsymbol{\beta^t\theta}$. Let $\boldsymbol{\beta_k}$, $k = 1, \dots, 40$ be the vector $\boldsymbol{\beta}$ in each one of the 40 simulations. Let $\widehat{\theta}_i$ be the estimation of the ability parameter obtained, when the multidimensional data was fitted with the unidimensional model. The bias index can be expressed as

$$bias_k = \frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\beta_k^t\theta_i} - \widehat{\theta}_i\right) \quad (k = 1, \dots, 40) \tag{30}$$

The error index included is the mean absolute error (mae) defined as

$$mae_k = \frac{1}{N}\sum_{i=1}^{N}\left|\boldsymbol{\beta_k^t\theta_i} - \widehat{\theta}_i\right| \quad (k = 1, \dots, 40) \tag{31}$$

To evaluate the precision of the mae index, we included the standard deviation sd_k of values $\left|\boldsymbol{\beta_k^t\theta_i} - \widehat{\theta}_i\right|$. A fidelity index was computed, the Pearson product-moment rho correlation, denoted by ρ. Additionally, we computed the least squares (LS) - fitting between the values $\boldsymbol{\beta_k^t\theta_i}$ and $\widehat{\theta}_i$. We took the synthetic index as the explanatory variable. The c-values were the coefficients and the R^2-values the corresponding R^2 statistics of the fitting in each simulation.

Also, we compared the estimations $\widehat{\theta}_i$ with $(\theta_1 + \theta_2)/2$. The indices mae^1 and c^1 were computed by replacing the values $\boldsymbol{\beta_k^t\theta_i}$ with the values $(\theta_1 + \theta_2)/2$ in the previous respective indices.

TABLE 3: Statistical indices to evaluate the synthetic index $\boldsymbol{\beta}^t\boldsymbol{\theta}$. The value p is the number of items, σ is the correlation between the θ's, β_1 and β_2 are the components of vector $\boldsymbol{\beta}$, γ_1 and γ_2 are the minimum and maximum angles of the vectors $\boldsymbol{\beta_j}$ with respect to the horizontal in each simulation.

p	σ	β_1	β_2	γ_1	γ_2	bias	mae	sd	ρ	c	R^2	mae^1	c^1	mae^α
10	0.0	0.94	0.34	5.2	34.3	0.022	0.38	0.30	0.88	0.73	0.77	0.40	0.93	0.10
10	0.3	0.90	0.44	13.6	37.1	0.024	0.35	0.28	0.89	0.74	0.80	0.36	0.99	0.06
10	0.6	0.85	0.53	22.4	39.6	0.026	0.35	0.27	0.90	0.75	0.81	0.32	1.03	0.08
10	0.9	0.78	0.63	34.2	42.5	0.026	0.33	0.28	0.90	0.76	0.82	0.29	1.07	0.07
20	0.0	0.87	0.49	5.2	49.3	0.006	0.30	0.24	0.92	0.83	0.85	0.35	1.11	0.12
20	0.3	0.84	0.55	13.6	48.1	0.005	0.28	0.22	0.93	0.84	0.87	0.31	1.16	0.08
20	0.6	0.80	0.60	22.4	47.1	0.007	0.27	0.21	0.94	0.86	0.89	0.29	1.20	0.07
20	0.9	0.75	0.66	34.2	46.0	0.004	0.26	0.21	0.94	0.86	0.89	0.27	1.22	0.08
30	0.0	0.88	0.47	5.2	49.3	0.002	0.27	0.21	0.94	0.87	0.88	0.36	1.16	0.10
30	0.3	0.84	0.54	13.6	48.1	0.007	0.25	0.19	0.95	0.88	0.90	0.32	1.21	0.06
30	0.6	0.80	0.59	22.4	47.1	0.000	0.23	0.18	0.96	0.89	0.91	0.29	1.25	0.07
30	0.9	0.75	0.66	34.2	46.0	0.022	0.23	0.18	0.96	0.90	0.91	0.27	1.28	0.08
40	0.0	0.88	0.47	5.2	49.3	0.003	0.25	0.19	0.95	0.90	0.90	0.35	1.20	0.10
40	0.3	0.84	0.54	13.6	48.1	−0.008	0.23	0.18	0.96	0.91	0.92	0.31	1.25	0.07
40	0.6	0.80	0.59	22.4	47.1	−0.011	0.21	0.16	0.96	0.92	0.93	0.29	1.29	0.08
40	0.9	0.75	0.66	34.2	46.0	0.033	0.21	0.16	0.96	0.93	0.93	0.27	1.32	0.09
50	0.0	0.89	0.46	5.2	49.3	−0.002	0.22	0.17	0.96	0.92	0.92	0.35	1.24	0.10
50	0.3	0.85	0.53	13.6	48.1	−0.008	0.21	0.16	0.97	0.94	0.93	0.32	1.30	0.08
50	0.6	0.81	0.59	22.4	47.1	0.001	0.19	0.15	0.97	0.96	0.94	0.30	1.35	0.10
50	0.9	0.76	0.65	34.2	46.0	0.038	0.19	0.15	0.97	0.97	0.94	0.29	1.37	0.12
60	0.0	0.88	0.48	5.2	50.0	−0.007	0.20	0.16	0.97	0.95	0.93	0.35	1.28	0.11
60	0.3	0.84	0.55	13.6	48.7	−0.014	0.19	0.15	0.97	0.97	0.94	0.32	1.34	0.11
60	0.6	0.80	0.60	22.4	47.5	0.014	0.18	0.14	0.97	1.00	0.95	0.32	1.40	0.15
60	0.9	0.75	0.66	34.2	46.1	0.040	0.18	0.14	0.98	1.01	0.95	0.30	1.42	0.16
70	0.0	0.87	0.49	5.2	50.0	0.006	0.19	0.15	0.97	0.97	0.94	0.35	1.33	0.13
70	0.3	0.84	0.55	13.6	48.7	−0.034	0.18	0.14	0.98	1.00	0.95	0.33	1.38	0.14
70	0.6	0.80	0.60	22.4	47.5	−0.012	0.17	0.14	0.98	1.03	0.96	0.33	1.45	0.18
70	0.9	0.75	0.66	34.2	46.1	−0.001	0.17	0.14	0.98	1.05	0.96	0.33	1.48	0.22
80	0.0	0.88	0.48	5.2	50.0	0.020	0.18	0.14	0.97	0.99	0.95	0.36	1.35	0.14
80	0.3	0.84	0.54	13.6	48.7	−0.045	0.17	0.14	0.98	1.02	0.96	0.34	1.41	0.15
80	0.6	0.80	0.60	22.4	47.5	0.074	0.18	0.14	0.98	1.05	0.96	0.34	1.48	0.20
80	0.9	0.75	0.66	34.2	46.1	0.011	0.17	0.13	0.98	1.06	0.96	0.33	1.50	0.22
90	0.0	0.88	0.48	5.2	50.0	0.002	0.18	0.14	0.98	1.01	0.95	0.37	1.37	0.14
90	0.3	0.84	0.54	13.6	48.7	−0.040	0.17	0.14	0.98	1.04	0.96	0.35	1.44	0.17
90	0.6	0.80	0.60	22.4	47.5	0.076	0.18	0.14	0.98	1.07	0.96	0.35	1.50	0.21
90	0.9	0.75	0.66	34.2	46.1	−0.068	0.18	0.14	0.98	1.09	0.96	0.36	1.55	0.26
100	0.0	0.88	0.48	5.2	50.0	0.009	0.17	0.13	0.98	1.02	0.96	0.37	1.39	0.15
100	0.3	0.84	0.55	13.6	48.7	−0.054	0.17	0.14	0.98	1.07	0.96	0.36	1.47	0.19
100	0.6	0.80	0.60	22.4	47.5	0.079	0.18	0.15	0.98	1.10	0.96	0.37	1.54	0.25
100	0.9	0.75	0.66	34.2	46.1	−0.076	0.19	0.15	0.99	1.12	0.97	0.38	1.59	0.29

Finally, in Table 3 we included the mae^{α} index for the α-parameters. This index was computed as

$$mae_k^{\alpha} = \frac{1}{p}\sum_{j=1}^{p} |\alpha_{jk} - \widehat{\alpha}_{jk}| \tag{32}$$

for each simulation k. The value $\widehat{\alpha}_{jk}$ is the slope parameter of the unidimensional model estimated in simulation k.

6. Discussion

Carroll & Levine (2007) and Levine (2003) proved that any MIRT model can be approximated by unidimensional models. However, their approximate models are non-parametric and the response functions are not necessarily monotone.

In this paper, we reviewed the main aspects concerning to synthesize the latent ability vector in compensatory MIRT models. We used composites, that are linear combinations of the latent trait vector.

Theorem 3 shows that each item j in a MIRT model is essentially unidimensional along the direction given by the vector $\boldsymbol{\beta_j}$. Item j measures the composite $\boldsymbol{\beta_j}^t\boldsymbol{\theta_i}$. Then, each item measures a different linear combination of the $\boldsymbol{\theta_i}$, unless all the vectors $\boldsymbol{\beta_j}$ have the same direction.

In realistic problems, where a test measures more than one latent trait, the components of the latent trait vector are correlated. However, Equation (28) shows that if the latent trait random vector $\boldsymbol{\theta}$ has multivariate normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$, then any composite $\boldsymbol{\beta}^t\boldsymbol{\theta}$ can be rewritten as $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\beta}^t\boldsymbol{\Upsilon}$, where $\boldsymbol{\Upsilon}$ has a normal standard distribution. This transformation has two important consequences. First, according to Corollary 1, the transformation induced by $\boldsymbol{\Sigma}^{\mathbf{1/2}}$ shrinks the direction vectors $\boldsymbol{\beta}_j$. Second, if a vector $\boldsymbol{\beta}$ is unitary, the composite $\boldsymbol{\beta}^t\boldsymbol{\Upsilon}$ is normalized, and any normalized composite has a normal standard distribution.

In Section 3, we stated that each item is essentially unidimensional along the direction of the item. In Corollary 3 we proved that if all the directions of the items coincide, the test is essentially unidimensional along the unique direction of the items.

The important issue about how to obtain a unidimensional synthetic index of the multidimensional latent trait was discussed in Section 4. Previous works of Yen, Wang and Zhang and Stout was reviewed. Wang and Zhang and Stout proposed two alternative synthetic indices, called respectively reference composite ($\boldsymbol{\omega}^t\boldsymbol{\theta}$) and the direction of the test ($\boldsymbol{\xi}^t\boldsymbol{\theta}$). We proposed two new synthetic indices: $\boldsymbol{\beta}_h^t\boldsymbol{\theta}$ and $\boldsymbol{\beta}_g^t\boldsymbol{\theta}$. Computing these alternative indices can be easier that Computing the previous indices, and they are more natural and easy to use by the experts.

Tables 1 and 2 of the first simulation study (Section 5.1) show that all the reference directions are very similar. This is not surprising, because although the constructions are different, the objective in all cases is the same: to obtain a synthetic index of the multidimensional latent trait. However, if we joint all the results, we can conclude additionally that each one of the reference composites is an approximation to the best composite that is measured by a subtest. This fact, is illustrated in Section 4, where we compared the theoretical synthetic index $\boldsymbol{\beta}_h^t\boldsymbol{\theta}$ with the unidimensional latent trait index obtained by fitting a multidimensional data set with a UIRT model.

7. Conclusions and future work

From a geometrical point of view, we showed in this paper how in tests that measure more than one latent trait the multidimensional latent trait vectors can be synthesized to obtain unidimensional measures of the examinees. The approach can be applied to subtests obtained from clusters of the items, or to the full test.

In the paper, nothing was stated about the item parameters that are estimated when a unidimensional model is used to fit a multidimensional data set. We showed that the correlation in the latent trait vector must be considered when a synthetic latent trait must be computed. In this case, a right computation implies to transform the direction of the items by a non orthogonal projection. But, in this scenery, the open question is: how must be modified the MDISC and MDIFF parameters to conserve approximately the same probability of response?. In other words, what is the relationship between the item parameters of the MIRT model and the item parameters of the UIRT when a unidimensional model is used to fit a multidimensional data set?.

Acknowledgements

This research is based on the doctoral dissertation in Statistics at the Universidad Nacional de Colombia of Álvaro Mauricio Montenegro Díaz, under the advice of Edilberto Cepeda. The research was supported by Universidad Nacional de Colombia through a study grant. The calculations were computed with R (Team 2008). The estimations were run with the ltm-package (Rizopoulos 2006) written for R. The code can be obtained by e-mailing the authors. The authors thank Professor Mark Reckase for sending the manuscripts of Min Wang to us, and thank the referees by their suggestions to improve the paper.

[Recibido: julio de 2009 — Aceptado: mayo de 2010]

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Appendix

For the concepts of n-dimensional geometry, see for example (Kendall 1961). Let $v_1, \dots, v_d$ be an ordered set of vectors in $\mathbb{R}^n, n \geq d$. The parallelotope [1] with sides $v_1, \dots, v_d$ is the convex hull created by this vectors. This parallelotope is denoted by $P(v_1, \dots, v_d)$. It is well known that the volume or content of $P(v_1, \dots, v_d)$ is

$$vol(v_1, \dots, v_d) = |V^t V|^{1/2} \tag{33}$$

where $V = (v_1, \dots, v_d)$, see for example (Mathai 1999). It is immediate that

$$vol(\lambda v_1, \dots, v_d) = \lambda \cdot vol(v_1, \dots, v_d) \tag{34}$$

Also, if S is region of $\mathbb{R}^n$ and Σ a $n \times n$ matrix, then

$$vol(\Sigma S) = |\Sigma| vol(S) \tag{35}$$

From Equation (35) it is straightforward that

$$vol(\Sigma v_1, \dots, \Sigma v_d) = |\Sigma| \cdot vol(v_1, \dots, v_d) \tag{36}$$

Lemma 3. *Let $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ be unitary vectors of $\mathbb{R}^n$, then*

$$vol^2(\boldsymbol{\beta_1}, \boldsymbol{\beta_2}) = 1 - \boldsymbol{\beta_1^t \beta_2} \tag{37}$$

Proof. It follows directly from Equation (33). □

[1] The parallelotope is the generalization of a parallelepiped to $\mathbb{R}^d$