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# An Application of Semi-Markovian Models to the Ruin Problem

## Una aplicación de los modelos semi-markovianos al problema de la ruina

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**Abstract**

We consider the classical ruin problem due to Cramér and Lundberg and we generalize it. Ruin times of the considered models are studied and sufficient conditions to usual stochastic dominance between ruin times are established. In addition an algorithm to simulate processes verifying the conditions under consideration is proposed.

***Key words*:** Coupling, Markov chains, Semi-Markov process, Simulation, Stochastic ordering.

**Resumen**

Se considera el problema clásico de ruina de Cramér y Lundberg y se generaliza. Se estudian los tiempos hasta la ruina de los modelos considerados y se establecen condiciones suficientes para la dominancia estocástica en el sentido usual entre los tiempos de ruina. Por otro lado, se establecen algoritmos de simulación de los procesos bajo estudio y de obtención de estimadores para las probabilidades involucradas.

***Palabras clave*:** cadenas de Markov, dominancia estocástica, emparejamiento, proceso semi-markovianos, simulación.

# 1. Introduction

The main purpose of the Ruin Theory is to obtain exact formulas or approximations of ruin probabilities in different risk models, see Seal (1969), Gerber (1995) and Ramsay (1992). Some of the most popular approximations are due to Beekman (1969), in which a Gamma distribution is used to approximate the

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distribution of the claims, or the approximation due to De Vylder (1996), who approximates the ruin process using a simply process in which the ruin probability is an exponential type. A relatively recent approach to estimate the probability of ruin is presented by Goovaerts (1990), where bounds are established through the ordering of the risks. Another kind of approach arises from the use of nonparametric techniques such as resampling (see Frees 1986) or Monte Carlo simulation (see Beard, Pentikäinen & Pesonen 1984).

Many authors have studied the ruin problem, for example Reinhard (1984) and Asmussen (1989). Reinhard (1984) considers a class of risk models in which the frequency of claims and the quantities to be paid are influenced by an external Markovian process (or environmental process), Reinhard & Snoussi (2001, 2002) have analized the severity of ruin and the distribution of surplus prior to ruin in a discrete semi-Markovian risk model. For more information about risk theory see Beard et al. (1984), Latorre (1992) or Daykin (1994).

In what follows, times to ruin in certain risk models will be ordered without an explicit expression for the probability of ruin and without the use of approximations thereof, as was classically done by Ferreira & Pacheco (2005) and Ferreira & Pacheco (2007).

Many authors have studied these processes int he context of the Queuing Theory. However, they also have applicability for dynamic solvency models and survival analysis.

This paper is organized as follows: in Section 2; the classical Cramér and Lundberg risk model is described; in Section 3, the principal concepts and notation being used in the rest of the paper are defined; in Section 4, the generalized model is described and the principal results are shown; finally, in Section 5 algorithms of simulation of the processes considered in Section 4, will be proposed.

## 2. Classical ruin model

The Cramér-Lundberg's classical risk model has its origin in Filip-Lundberg's doctoral thesis in 1903. In this work, Lundbery studied the collective reinsurance problem and used compounded homogenous Poisson process. In 1930, Harald Cramér re-examined Lundberg's original ideas and formalize them in the stochastic processes context.

The original model is:

$$X(t) = X(0) + ct - \sum_{n=1}^{N_t} Y_n \tag{1}$$

with $c > 0, X(0) \geq 0$ and $X(0)$ Being the initial capital, $c$ the premium density, which is assumed to be constant, $Y_j$ the amount of the $j$-th claim and $N_t$ is an homogeneous Poisson process which represents the number of claims up to time $t$ (independent of the interval position and the history of the process). Claims $Y_j$ are supposed to positive independent random variables which are independent of the process $N_t$, with distribution $F$ such that $F(0) = 0$ and whose mean $\mu$ is finite.

If the arrival of the $n$-th claim is denoted by $S_n$, then:

$$N_t = \sup\left\{n \geq 1 : S_n \leq t\right\}, \quad t \geq 0$$

***Note* 1.** The number of claims that have occurred up to time $t$ can be approximated, in the Cramér and Lundberg's model, from other distribution functions.

The intervals between claims $T_k = S_k - S_{k-1}, k = 2, 3, \ldots$ are independent and identically distributed random variables with an exponential distribution with parameter $\lambda$ and finite mean and $T_1 = S_1$.

The aggregate claims until instant $t$ are given by the random variable

$$S(t) = \sum_{n=1}^{N_t} Y_n$$

known as compound Poisson. Its distribution is followed by

$$G_t(x) = P\left[S(t) \leq x\right] = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{(n)}(x)$$

with $x, t \geq 0$ and $F^{(n)}$ the $n$-th convolution of $F$ with $F^{(0)}$ the distribution function of the measure of Dirac in 0.

The time to ruin is defined as:

$$T = \inf\left\{t > 0 : X(t) \leq 0\right\} \tag{2}$$

where $\inf \emptyset = \infty$.

The probability of ruin in the interval $[0, t]$ or probability of ruin in a finite horizon is defined as:

$$\psi(u, t) = P[T \leq t | X(0) = u] \tag{3}$$

and the probability of ruin in an infinite horizon or simply probability of ruin is:

$$\psi(u) = \lim_{t \to \infty} \psi(u, t) = \lim_{t \to \infty} P[T \leq t | X(0) = u] = P[T < \infty | X(0) = u] \tag{4}$$

***Note* 2.** In this case, the probability in an infinite horizon is usually approximated by the Normal-Power.

**Definition 1.** The basic Cramér-Lundberg's process is described as

$$X(t) = X(0) + (1 + \upsilon)\lambda\mu t - S(t)$$

where $\lambda\mu t = E\left[S(t)\right]$ and $\upsilon = \frac{c}{\lambda\mu} - 1 > 0$ is referred to as "solvency or safety margin", in order to guarantee survival (defined as the set of free capital whose purpose is to address those risks that may threaten the solvency of the company, the latter being the capacity to face obligations).

## 3. Preliminaries

In this section we introduce notation that is used throughout the paper and we set up some definitions. The introduced definitions are general and they can be found in several texts, for example in Müller & Stoyan (2002), Shaked (2007) or Almaraz (2009) among others.

We let the following sets $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$ and $\mathbb{R} = (-\infty, \infty)$.

**Definition 2.** Given two random variables $X$ and $Y$ taking values in a countable ordered state space $I$, then $Y$ is stochastically smaller than $X$ in the usual sense, and it is denoted as $Y \leq_{st} X$, if $P(Y \leq i) \geq P(X \leq i)$ for all $i \in I$.

**Definition 3.** A subset $U$ of $\mathbb{R}^n$ is regarded to be as increasing if $y \in U$ when $y \geq x$ and $x \in U$.

**Definition 4.** Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors such that $P[\boldsymbol{X} \in U] \leq P[\boldsymbol{Y} \in U]$ for all the increasing subsets $U \subseteq \mathbb{R}^n$. Then $\boldsymbol{X}$ is stochastically smaller than $\boldsymbol{Y}$ in the usual sense and it is denoted as $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$.

**Definition 5.** Let $X = \{X(t), t \in T\}$ and $Y = \{Y(t), t \in T\}$, be two stochastic processes with state space $I \subseteq \mathbb{R}$ and time parameter space $T$ (usually $T = [0, \infty)$ or $T = \mathbb{N}_+$). Suppose that, for all choices of an integer $m$ and $t_1 < t_2 < \cdots < t_m$ en $T$, it happens that:

$$(X(t_1), X(t_2), \ldots, X(t_m)) \leq_{st} (Y(t_1), Y(t_2), \ldots, Y(t_m))$$

Then $X = \{X(t), t \in T\}$ is said to be stochastically smaller than $Y = \{Y(t), t \in T\}$ in the usual sense and it is denoted as $X = \{X(t), t \in T\} \leq_{st} Y = \{Y(t), t \in T\}$.

**Definition 6.** A finite measure matrix is a matrix with non-negative entries whose lines are finite measure vectors.

**Definition 7.** Let $I$ and $J$ be two countable ordered sets and let $A = (a_{ij})_{i \in I, j \in J}$ and $B = (b_{ij})_{i \in I, j \in J}$ be two finite measure matrix with common indices on $I \times J$. Then the matrix $A$ is said to be smaller than $B$ in the Kalmykov sense, and it is denoted as $A \leq_K B$, if and only if:

$$\sum_{m \geq n} a_{im} \leq \sum_{m \geq n} b_{jm}, \forall i \leq j \quad \forall n$$

Also the following concepts will be necessary.

**Definition 8.** The counting process $N = (N_t)_t$ is an homogeneous Poisson process with rate $\lambda > 0$ if:

1. $N_0 = 0$, almost sure.
2. $N$ has independent stationary increments.

3. $\forall \quad 0 \leq s < t < \infty$, $N_t - N_s \sim P(\lambda(t-s))$, that is,

$$P\left[N_t - N_s = k\right] = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}, \quad k \in \mathbb{N}$$

**Definition 9.** (Markov process (MP)). A stochastic process $\{X_t, t \in T\}$, is said to be a Markov process (or Markovian process) if

$$P[X_{t_{n+1}} = x_{n+1} \mid X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n] =$$

$$= P[X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n]$$

for each $n \in \mathbb{N}$ y $t_1 < t_2 < \cdots < t_n < t_{n+1}$.

This condition is known as the Markovian condition.

A Markovian process with finite state space is known as the Markov Chain and it can be in discrete time (DTMC) or continuous time (CTMC).

**Definition 10.** (Markovian Renewal process (MRP)). A bivariate process $(Z, S) = (Z_n, S_n)_{n \in \mathbb{N}}$ is a Markovian Renewal process with phase states (countable) $I$ and kernel $\boldsymbol{Q} = (\boldsymbol{Q}(t))_{t \in \mathbb{R}_+}$ where $\boldsymbol{Q}(t) = (Q_{ij}(t))_{i,j \in I}$ is a family of sub-distribution functions such that $\sum_{j \in I} Q_{ij}(t)$ is a distribution function, for each $i \in I$, if it is a Markov process in $I \times \mathbb{R}_+$ such that $S_0 = 0$ and

$$Q_{ij}(t) = P[Z_{n+1} = j, S_{n+1} - S_n \leq t \mid Z_n = i, S_n = s]$$

for each $n \in \mathbb{N}, i, j \in I$ and $s, t \in \mathbb{R}_+$

**Definition 11.** (Semi-Markovian process (SMP)) A process $W = (W_t)_{t \in \mathbb{R}_+}$ is a semi-Markovian process with state space $I$ and kernel $\boldsymbol{Q}$ (or admitting an embedded kernel $(\boldsymbol{P}, \boldsymbol{F})$) if

$$W_t = Z_n, \quad S_n \leq t < S_{n+1}$$

for some MRP $(Z, S)$ with phase space $I$ and kernel $\boldsymbol{Q}$ (embedded kernel $(\boldsymbol{P}, \boldsymbol{F})$)

## 4. Stochastic dominance of ruin times in semi-Markov modulated risk processes

Let us consider the following generalization of the classic model:

$$X(t) = X(0) + \int_0^t c_{J(s)} ds - \sum_{n=1}^{N_t} Y_n \tag{5}$$

where $c_j > 0$ for all $j$, and $X(0) \geq 0$.

Where $X(0)$ is a random variable that represents the initial capital; $J(s)$ a semi-Markovian process; $c_j$ the Premium density when the process $J(s)$ is in the state $j$; $Y_n$ the size of the $n$-th claim and $N_t$ a counting process associated to $J$ that represents the number of claims up to time $t$.

Let $(S_n, K_n)$ a Markovian sequence associated to the process $J$, where

$$S_n = \inf\{t \geq 0 : N_t \geq n\}, n \in \mathbb{N}$$

represents a sequence of events and

$$K_n = J(S_n), n \in \mathbb{N}$$

is an irreductible and discrete Markov chain with state space $I$, a countable subset of $\mathbb{R}$, transition matrix $P = (P_{ij})_{i,j \in I}$ and representing the state visited in the $n$-th tansistion, where

$$J_t = K_n, S_n \leq t < S_{n+1}$$

Let $H_n$ be the time between the $(n-1)$-th and the $n$-th claim:

$$H_n = S_n - S_{n-1}, n \geq 1 \tag{6}$$

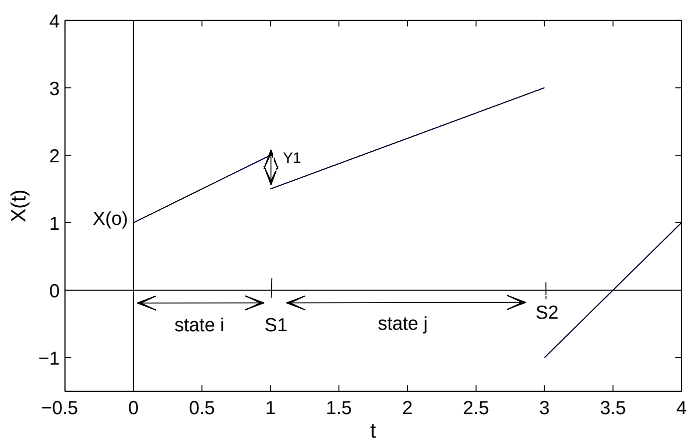


Figure 1: Path of the process $X(t)$.

In classical literature, this graphical representation (Figure 1) is known as a surplus process of ruin or process (see Bowers, Gerber, Hickman, Jones & Nesbitt 1997).

In this way, the process may be written as:

$$X(t) = X(0) + \sum_{n=0}^{N_t - 1} c_{K_n} H_{n+1} + c_{K_{N_t}}(t - S_{N_t}) - \sum_{n=1}^{N_t} Y_n \tag{7}$$

where $c_j > 0, \quad \forall j \in I$.

The semi-Markovian dependence structure under consideration is of the following type:

$$P\left[H_{n+1} \leq x, Y_{n+1} \leq y, K_{n+1} = j | K_n = i, (H_r, Y_r, K_r), 0 \leq r \leq n\right] =$$

$$= P\left[H_1 \leq x, Y_1 \leq y, K_1 = j | K_0 = i\right] = Q_{ij}(x, y) \quad (8)$$

The sequence $Q = (Q_{ij})_{i,j \in I}$ is the kernel of this process.

The purpose of this paper is to establish sufficient conditions for the first order stochastic dominance between the times of ruin of two processes like described in (5).

## 4.1. Stochastic processes in which the amount of the claims depends on the environment

Let us consider the following structure of the kernel:

$$Q_{ij}(x, y) = p_{ij} F_{ij}(x) G_{ij}(y) \quad (9)$$

where

- $p_{ij} = Q_{ij}(\infty, \infty) = P[K_{n+1} = j | K_n = i], n \in \mathbb{N}_+, i, j \in I$
- $F_{ij}$ is the distribution function of $H_n | (K_{n-1} = i, K_n = j), n \in \mathbb{N}_+, i, j \in I$
- $G_{ij}$ is the distribution function of $Y_n | (K_n = i, K_{n+1} = j), n \in \mathbb{N}_+, i, j \in I$

This implies that $(Y_1, Y_2, \ldots)$ and $(H_1, H_2, \ldots)$ are conditionally independent given $(K_0, K_1, \ldots)$ That is, the are conditionally independent given the evolution of the process $J$.

The parametrization of this process is $(c, P, F, G)$, where $c = (c_i)_{i \in I}$, $P = (p_{ij})_{i,j \in I}$, $F = (F_{ij})_{i,j \in I}$ y $G = (G_{ij})_{i,j \in I}$.

We will denote the time to ruin of this process by:

$$T_{ab}^{(l)} = \inf\left\{t > 0 : X^{(l)}(t) \leq 0\right\} \mid \left(X^{(l)}(0) = b, K_0^{(l)} = a\right)$$

**Theorem 1.** *Let $X^{(1)} = (X^{(1)}(t))_{t \geq 0}$ and $X^{(2)} = (X^{(2)}(t))_{t \geq 0}$ be two stochastic processes with parametrizations $(c^{(1)}, P^{(1)}, F^{(1)}, G^{(1)})$ and $(c^{(2)}, P^{(2)}, F^{(2)}, G^{(2)})$ respectively, as described in (5) and (9). Let $J^{(1)}(0) \leq J^{(2)}(0)$ and $X^{(1)}(0) \leq X^{(2)}(0)$.*

*If*

$$c_i^{(1)} \leq c_k^{(2)}, \quad \forall i \leq k \quad (10)$$

$$P^{(1)} \leq_K P^{(2)} \quad (11)$$

$$F_{ij}^{(1)} \leq_{st} F_{kl}^{(2)}, \quad \forall i \leq k, j \leq l \tag{12}$$

$$G_{kl}^{(2)} \leq_{st} G_{ij}^{(1)}, \quad \forall i \leq k, j \leq l \tag{13}$$

*then* $T_{iu}^{(1)} \leq_{st} T_{jv}^{(2)} \quad \forall i \leq j, u \leq v$.

***Proof.*** Let $X^{(1)}$ and $X^{(2)}$ as in the statement. We must prove that $T_{iu}^{(1)} \leq_{st} T_{jv}^{(2)}$ for each $i \leq j, u \leq v$.

The Markovian renewal sequence associated to $J^{(l)}$, $l = 1, 2$ will be denoted as $\left(S_n^{(l)}, K_n^{(l)}\right)$.

It is defined:

$$T_{iu}^{*(l)} = \inf\left\{S_n^{(l)} : X_{S_n^{(l)}}^{(l)} \leq 0\right\} |(X_{S_0^{(l)}}^{(l)} = u, K_0^{(l)} = i), \quad l = 1, 2. \tag{14}$$

Note that the fact of $X^{(l)}$, $l = 1, 2$, being a non-decreasing sequence in $\left[S_n^{(l)}, S_{n+1}^{(l)}\right)$, denotes that:

$$T_{iu}^{(l)} = T_{iu}^{*(l)} \tag{15}$$

so it is enough to prove that $T_{iu}^{*(1)} \leq_{st} T_{jv}^{*(2)} \quad \forall u \leq v, i \leq j$.

Let

$$\left(\widetilde{X}_n^{(1)}, \widetilde{X}_n^{(2)}\right)$$

be a couple of

$$X_{S_n^{(1)}}^{(1)} | X_{S_0^{(1)}}^{(1)} = u \quad and \quad X_{S_n^{(2)}}^{(2)} | X_{S_0^{(2)}}^{(2)} = v$$

on a common product probability space

$$\Lambda = \Lambda_1 \times \Lambda_2 = (\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$$

such that

$$\widetilde{X}_n^{(1)}(\omega) \leq \widetilde{X}_n^{(2)}(\omega), \forall \omega \in \Omega$$

and

$$\widetilde{S}_n^1(\omega) \leq \widetilde{S}_n^2(\omega), \forall \omega \in \Omega$$

being $\widetilde{S}_n^{(l)}$, a copy of the process $S_n^{(l)}$, $l = 1, 2$.

To do that, the following independent sequences of independent uniform random variables on the interval $(0, 1)$ will be used: $(U_n)_{n \in \mathbb{N}_+}$ in $\Lambda_1$, $(V_n)_{n \in \mathbb{N}_+}$ and $(W_n)_{n \in \mathbb{N}_+}$ in $\Lambda_2$.

Let $\widetilde{K}_0^{(1)}(\omega_1) = i$ and $\widetilde{K}_0^{(2)}(\omega_1) = j$.

In detail, for $l = 1, 2$:

$$\widetilde{K}_n^{(l)}(\omega_1) = \left[P_{\widetilde{K}_{n-1}^{(l)},\cdot}^{(l)}\right]^{-1}(U_n(\omega_1))), \quad n \in \mathbb{N}_+, \omega_1 \in \Omega_1 \tag{16}$$

$$\widetilde{H}_n^{(l)}(\omega) = \left[F^{(l)}_{(\widetilde{K}^{(l)}_{n-1}(\omega_1),\widetilde{K}^{(l)}_{n}(\omega_1))}\right]^{-1}(V_n(\omega_2)), \quad n \in \mathbb{N}_+, \omega = (\omega_1, \omega_2) \in \Omega \tag{17}$$

$$\widetilde{Y}_n^{(l)}(\omega) = \left[G^{(l)}_{(\widetilde{K}^{(l)}_{n-1}(\omega_1),\widetilde{K}^{(l)}_{n}(\omega_1))}\right]^{-1}(W_n(\omega_2)), \quad n \in \mathbb{N}, \omega = (\omega_1, \omega_2) \in \Omega \tag{18}$$

Let $\widetilde{X}_0^{(1)} = u$ and $\widetilde{X}_0^{(2)} = v$ and:

$$\widetilde{X}_n^{(l)}(\omega) = \widetilde{X}_0^{(l)} + \sum_{m=0}^{n-1} c^{(l)}_{\widetilde{K}^{(l)}_m} \widetilde{H}^{(l)}_{m+1}(\omega) - \sum_{m=1}^{n} \widetilde{Y}^{(l)}_m(\omega), \tag{19}$$

$n \in \mathbb{N}$, $l = 1, 2$, $\omega = (\omega_1, \omega_2) \in \Omega$, be the embedded Markov Process of $(X^{(l)}_{S^{(l)}_n})_{n\geq 0}$, $l = 1, 2$.

Using (11), (12) and (13) we have respectively by construction:

$$\widetilde{K}_n^{(1)}(\omega_1) \leq \widetilde{K}_n^{(2)}(\omega_1), \quad \forall \omega_1 \in \Omega_1, n \in \mathbb{N} \tag{20}$$

$$\widetilde{H}_n^{(1)}(\omega) \leq \widetilde{H}_n^{(2)}(\omega), \quad \forall \omega = (\omega_1, \omega_2) \in \Omega, n \in \mathbb{N} \tag{21}$$

and

$$\widetilde{Y}_n^{(1)}(\omega) \geq \widetilde{Y}_n^{(2)}(\omega), \quad \forall \omega = (\omega_1, \omega_2) \in \Omega, n \in \mathbb{N} \tag{22}$$

On the other hand, using (10), (21) and (22), we have that:

$$\sum_{m=1}^{n} \widetilde{Y}_m^{(1)}(\omega) \geq \sum_{m=1}^{n} \widetilde{Y}_m^{(2)}(\omega), \quad \forall \omega = (\omega_1, \omega_2) \in \Omega, n \in \mathbb{N} \tag{23}$$

$$\widetilde{S}_n^{(1)}(\omega) = \sum_{m=1}^{n} \widetilde{H}_m^{(1)}(\omega) \leq \sum_{m=1}^{n} \widetilde{H}_m^{(2)}(\omega) = \widetilde{S}_n^{(2)}(\omega), \quad \forall \omega = (\omega_1, \omega_2) \in \Omega, n \in \mathbb{N} \tag{24}$$

$$\sum_{m=0}^{n-1} c^{(1)}_{\widetilde{K}^{(1)}_m} \widetilde{H}^{(1)}_{m+1}(\omega) \leq \sum_{m=0}^{n-1} c^{(2)}_{\widetilde{K}^{(2)}_m} \widetilde{H}^{(2)}_{m+1}(\omega), \quad \forall \omega = (\omega_1, \omega_2) \in \Omega, n \in \mathbb{N} \tag{25}$$

thus, leading to

$$\widetilde{X}_n^{(1)}(\omega) \leq \widetilde{X}_n^{(2)}(\omega) \quad \forall \omega = (\omega_1, \omega_2) \in \Omega, n \in \mathbb{N} \tag{26}$$

If denoted by:

$$\widetilde{T}_{iu}^{*(l)} = \inf\left\{\widetilde{S}_n^{(l)} : \widetilde{X}_n^{(l)} \leq 0\right\} \mid (\widetilde{X}_0 = u, \widetilde{K}_0^{(l)} = i), \quad l = 1, 2 \tag{27}$$

we have that

$$\widetilde{T}_{iu}^{*(1)}(\omega) \leq \widetilde{T}_{jv}^{*(2)}(\omega), \quad \forall i \leq j, u \leq v, \omega \in \Omega \tag{28}$$

intended to prove. □

In the special case in which the environment $J$ is Markovian, particulary a continuous time Markov chain (CTMC), the parametrization of the process (5) is $(c, P, q, G)$, where $P = (p_{ij})_{i,j\in I}$, $q$ is the vector of transition rates from states of $J$ and $G = (G_{ij})_{i,j\in I}$. In this case, the previous theorem has an immediate application, it is enough to see that the condition $F_{ij}^{(1)} \leq_{st} F_{kl}^{(2)}, \quad \forall i \leq k, j \leq l$ is translated in a condition to the vectors of transition rates from the states $q_k^{(2)} \leq q_i^{(1)} \quad \forall i \leq k$, that is, the distribution function of $H_n^{(l)} | \left(K_{n-1}^{(l)} = i, K_n^{(l)} = j\right), l = 1, 2$, which was denoted as $F_{ij}^{(l)}$, $l = 1, 2$ has in this case the following expression: $F_{ij}^{(l)}(x) = q_i^{(l)} e^{-q_i^{(l)} x}$ for $l = 1, 2$.[1]

In this way, the following two processes will be considered $X^{(1)}$ and $X^{(2)}$ as describes in (5) with parametrizations $(c^{(1)}, P^{(1)}, q^{(1)}, G^{(1)})$ and $(c^{(2)}, P^{(2)}, q^{(2)}, G^{(2)})$ respectively and then, the result for this particular case is described in the next corollary.

**Corollary 1.** *Let $X^{(1)} = (X_t^{(1)})_{t\geq 0}$ and $X^{(2)} = (X_t^{(2)})_{t\geq 0}$ be two stochastic processes with parameterizations $(c^{(1)}, P^{(1)}, q^{(1)}, G^{(1)})$ and $(c^{(2)}, P^{(2)}, q^{(2)}, G^{(2)})$ respectively, as described in (5), with environments $J^{(1)}$ and $J^{(2)}$ beings CTMCs with state space $I$, embedded transition probability matrices $P^{(1)}$ and $P^{(2)}$ and vectors of transition rates from states $q^{(1)}$ and $q^{(2)}$, respectively.*

*Let $J^{(1)}(0) \leq J^{(2)}(0)$, and $X^{(1)}(0) \leq X^{(2)}(0)$. If*

$$c_i^{(1)} \leq c_k^{(2)}, \quad \forall i \leq k \tag{29}$$

$$P^{(1)} \leq_K P^{(2)} \tag{30}$$

$$q_k^{(2)} \leq q_i^{(1)} \quad \forall i \leq k \tag{31}$$

$$G_{kl}^{(2)} \leq_{st} G_{ij}^{(1)}, \quad \forall i \leq k, j \leq l \tag{32}$$

*then $T_{iu}^{(1)} \leq_{st} T_{jv}^{(2)} \quad \forall i \leq j, u \leq v$.*

***Proof.*** It is a direct consequence of Theorem 1. □

For the particular case in which the processes have the same transition matrix P we may relax the conditions of the Theorem 1 to conditions involving only one pair of states $(i, j)$ such that $p_{ij} > 0$, in the following way.

**Corollary 2.** *Let $X^{(1)} = (X_t^{(1)})_{t\geq 0}$ and $X^{(2)} = (X_t^{(2)})_{t\geq 0}$ be two stochastic processes with parameterizations $(c^{(1)}, P, F^{(1)}, G^{(1)})$ and $(c^{(2)}, P, F^{(2)}, G^{(2)})$ respectively, as described in (5) and (9). If*

$$c_j^{(1)} \leq c_j^{(2)}, \quad \forall j \tag{33}$$

[1] Exponential distribution with intensity $q_i^{(l)}$ and the distribution decreases stochastically in the usual sense with this intensity

*and for each pair* $(i, j)$ *such that* $P_{ij} > 0$

$$F_{ij}^{(1)} \leq_{st} F_{ij}^{(2)} \tag{34}$$

$$G_{ij}^{(2)} \leq_{st} G_{ij}^{(1)} \tag{35}$$

*then*

$$T_{iu}^{(1)} \leq_{st} T_{iv}^{(2)} \quad \forall u \leq v.$$

***Proof*.** The result follows is derived from the construction used in the proof of Theorem 1 since this construction now leads to $\widetilde{K}_n^{(1)}(\omega_1) = \widetilde{K}_n^{(2)}(\omega_1)$, $\forall n \in \mathbb{N}, \omega_1 \in \Omega_1$. This fact allows to conclude (21) and (22) using (34) and (35) despite of (12) and (13). The rest of the proof is analogous. □

### 4.1.1. A simple application

In this section, a case from the counting process $N_t$, identified as a semi-Markovian process $J$ whose state space is $\{0, 1, 2 \ldots\}$ and whose transition probability matrix $P$ is deterministic, in which case the probability to go from state $n$ to $n + 1$ is 1, is studied.

Let $(S_n^{(l)})_{n \geq 0}$ be a stochastic process with

$$0 = S_0^{(l)} < S_1^{(l)} < \cdots$$

such that

$$H_n^{(l)} = S_n^{(l)} - S_{n-1}^{(l)}, \quad n \in \mathbb{N}_+, \quad l = 1, 2$$

are independent random variables with distribution function

$$F_n^{(l)}, l = 1, 2$$

and let

$$N_t^{(l)} = \sup \left\{ n \geq 0 : S_n^{(l)} \leq t \right\}, t \geq 0, l = 1, 2$$

the counting process.

Let $(Y_j^{(l)}), j \in \mathbb{N}, l = 1, 2$ be a sequence of independent random variables with distribution function $(G_j^{(l)}), j \in \mathbb{N}_+, l = 1, 2$. Let $H_n^{(l)}, G_n^{(l)}$ be independent $\forall n \in \mathbb{N}_+$.

Let us consider the process $(X^{(l)}(t))_{t \geq 0}$ with parametrization $(c^{(l)}, F^{(l)}, G^{(l)})$ con $l = 1, 2$ defined as follows:

$$X^{(l)}(t) = X^{(l)}(0) + c^{(l)} t - \sum_{j=1}^{N_t^{(l)}} Y_j^{(l)} \tag{36}$$

with $c^{(l)} > 0, X^{(l)}(0) \geq 0$, and it is defined

$$T_u^{(l)} = \inf \left\{ t \geq 0 : X^{(l)}(t) \leq 0 \right\} | X^{(l)}(0) = u \tag{37}$$

**Theorem 2.** *Let $X^{(1)} = (X^{(1)}(t))_{t\geq 0}$ and $X^{(2)} = (X^{(2)}(t))_{t\geq 0}$ be two stachastic processes with parametrizations $(c^{(1)}, F^{(1)}, G^{(1)})$ and $(c^{(2)}, F^{(2)}, G^{(2)})$ as described in (36), with $X^{(1)}(0) \leq X^{(2)}(0)$. If*

$$c^{(1)} \leq c^{(2)} \tag{38}$$

$$F_n^{(1)} \leq_{st} F_n^{(2)}, \quad \forall n \in \mathbb{N}_+ \tag{39}$$

$$G_n^{(2)} \leq_{st} G_n^{(1)}, \quad \forall n \in \mathbb{N}_+ \tag{40}$$

*then*

$$T_u^{(1)} \leq_{st} T_v^{(2)}, \quad \forall u \leq v$$

***Proof*.** Let $X^{(1)}$ and $X^{(2)}$ as stated. It must be proved that $T_u^{(1)} \leq_{st} T_u^{(2)}$ for all $u \leq v$.

It is defined:

$$T_u^{*(l)} = \inf\left\{S_n^{(l)} : X_{S_n^{(l)}}^{(l)} \leq 0\right\} | X_{S_0^{(l)}}^{(l)} = u, \quad l = 1, 2 \tag{41}$$

Note that the fact of $X^{(l)}$, $l = 1, 2$, being a non-decreasing sequence in $\left[S_n^{(l)}, S_{n+1}^{(l)}\right)$, gives that:

$$T_u^{(l)} = T_u^{*(l)} \tag{42}$$

therefore is enough to prove that $T_u^{*(1)} \leq_{st} T_v^{*(2)} \quad \forall u \leq v$.

For that, couplings will be build

$$\left(\widetilde{S}_n^{(1)}, \widetilde{S}_n^{(2)}\right) \quad \text{and} \quad \left(\widetilde{X}_n^{(1)}, \widetilde{X}_n^{(2)}\right)$$

of

$$\left(S_n^{(1)}, S_n^{(2)}\right) \quad \text{and} \quad \left(X_{S_n^{(1)}}^{(1)}, X_{S_n^{(2)}}^{(2)}\right)$$

given $\left(X_{S_n^{(1)}}^{(1)}, X_{S_n^{(2)}}^{(2)}\right) = (u, v)$ such that

$$\widetilde{X}_n^{(1)}(\omega) \leq \widetilde{X}_n^{(2)}(\omega), \forall \omega \in \Omega, \quad n \in \mathbb{N}$$

and

$$\widetilde{S}_n^{(1)}(\omega) \leq \widetilde{S}_n^{(2)}(\omega), \forall \omega \in \Omega, \quad n \in \mathbb{N}$$

To do that, independent sequences of independent and identically distributed $U(0,1)$ random variables will be used $(U_n)_{n\in\mathbb{N}_+}$ and $(V_n)_{n\in\mathbb{N}_+}$, defined on a common probability space $(\Omega, \mathcal{F}, P)$.

Let for $\omega \in \Omega$ and $l = 1, 2$:

$$\widetilde{H}_n^{(l)}(\omega) = \left[F_n^{(l)}\right]^{-1} (U_n(\omega))), \quad n \in \mathbb{N}_+ \tag{43}$$

$$\widetilde{Y}_n^{(l)}(\omega) = \left[G_n^{(l)}\right]^{-1} (V_n(\omega))), \quad n \in \mathbb{N}_+ \tag{44}$$

Let for $\omega \in \Omega$ and $l = 1, 2$:

$$\widetilde{X}_n^{(l)}(\omega) = \widetilde{X}_0^{(l)}(\omega) + c^{(l)} \sum_{m=1}^{n} \widetilde{H}_m^{(l)}(\omega) - \sum_{m=1}^{n} \widetilde{Y}_m^{(l)}(\omega), \quad n \in \mathbb{N} \tag{45}$$

with $\widetilde{X}_0^{(1)}(\omega) = u$ and $\widetilde{X}_0^{(2)}(\omega) = v$, be the embedded Markov process of $(X_t^{(l)})_{t\geq 0}$, coupling of $X^{(l)}_{S_n^{(l)}}$ $\quad l = 1, 2$.

Using (39) and (40) we have for construction that:

$$\widetilde{H}_n^{(1)}(\omega) \leq \widetilde{H}_n^{(2)}(\omega), \quad \forall \omega \in \Omega, n \in \mathbb{N} \tag{46}$$

$$\widetilde{Y}_n^{(1)}(\omega) \geq \widetilde{Y}_n^{(2)}(\omega), \quad \forall \omega \in \Omega, n \in \mathbb{N} \tag{47}$$

On the other hand, from (46) and (47):

$$\widetilde{S}_n^{(1)}(\omega) = \sum_{m=1}^{n} \widetilde{H}_m^{(1)}(\omega) \leq \sum_{m=1}^{n} \widetilde{H}_m^{(2)}(\omega) = \widetilde{S}_n^{(2)}(\omega), \quad \forall \omega \in \Omega, n \in \mathbb{N} \tag{48}$$

$$\sum_{m=1}^{n} \widetilde{Y}_m^{(1)}(\omega) \geq \sum_{m=1}^{n} \widetilde{Y}_m^{(2)}(\omega), \quad \forall \omega \in \Omega, n \in \mathbb{N} \tag{49}$$

which leads with condition (38) to:

$$\widetilde{X}_n^{(1)}(\omega) \leq \widetilde{X}_n^{(2)}(\omega) \quad \forall \omega \in \Omega, n \in \mathbb{N} \tag{50}$$

If we denote:

$$\widetilde{T}_u^{*(l)} = \inf \left\{ \widetilde{S}_n^{(l)} : \widetilde{X}_n^{(l)} \leq 0 \right\}, \quad l = 1, 2 \tag{51}$$

being $\widetilde{T}_u^{*(1)}$ and $\widetilde{T}_u^{*(2)}$ a coupling of $(T_u^{*(1)}, T_u^{*(2)})$ we have using (48) and (50) that

$$\widetilde{T}_u^{*(1)}(\omega) \leq \widetilde{T}_u^{*(2)}(\omega), \quad \forall \omega \in \Omega \tag{52}$$

as it was pretended. □

## 4.2. Comparisons of ruin probabilities

An algorithm which leads to simulate processes verifying the conditions of Theorem 1 will be described.

**Input:** Independent sequences of independent random variables $U(0,1)$: $(U_n)_{n\in\mathbb{N}_+}$, $(V_n)_{n\in\mathbb{N}_+}$, $(W_n)_{n\in\mathbb{N}_+}$. Values $x^{(1)}$ y $x^{(2)}$, $f^{(1)}$ y $f^{(2)}$ with $x^{(1)} \leq x^{(2)}$, $f^{(1)} \leq f^{(2)}$.
$\widetilde{X}_0^{(1)} = x^{(1)}$
$\widetilde{X}_0^{(2)} = x^{(2)}$
$\widetilde{K}_0^{(1)} = f^{(1)}$
$\widetilde{K}_0^{(2)} = f^{(2)}$
**for** $n = 0, ..., N$ **do**
  **for** $l = 1, 2$ **do**

$$\widetilde{K}_{n+1}^{(l)} = \left[P^{(l)}_{\widetilde{K}_n^{(l)},\cdot}\right]^{-1}(U_{n+1})$$
$$\widetilde{H}_{n+1}^{(l)} = \left[F^{(l)}_{(\widetilde{K}_n^{(l)},\widetilde{K}_{n+1}^{(l)})}\right]^{-1}(V_{n+1})$$
$$\widetilde{Y}_{n+1}^{(l)} = \left[G^{(l)}_{(\widetilde{K}_n^{(l)},\widetilde{K}_{n+1}^{(l)})}\right]^{-1}(W_{n+1})$$
$$\widetilde{X}_n^{(l)} = \widetilde{X}_0^{(l)} + \sum_{m=0}^{n-1} c^{(l)}_{\widetilde{K}_m^{(l)}} \widetilde{H}_{m+1}^{(l)} - \sum_{m=1}^{n} \widetilde{Y}_m^{(l)}$$

  **end for**
**end for**
**Output:** Two sequences $\widetilde{X}^{(1)}$ and $\widetilde{X}^{(2)}$ such that $T_{iu}^{(1)} \leq_{st} T_{jv}^{(2)} \quad \forall i \leq j, u \leq v$

FIGURE 2: Simulation of sequences of random variables as described in (5), under conditions of Theorem 1.

Next algorithm consists of showing a method which allows to estimate the difference between the ruin probabilities in a given period $T$, of two processes which satisfy conditions of Theorem 1, that is, $\psi^{(1)}(u,T) - \psi^{(2)}(u,T)$ is wanted to be estimated. For simplicity, $p^{(l)}$ will denote the ruin probability of the process $l$ under the interval of consideration, so: $p^{(l)} = \psi^{(l)}(u,T)$, for $l = 1, 2$.

For the above purpose, $M$ replicas of each process will be simulated. Let $X_r^{(l)}$ be, for $l = 1, 2$ and $r = 1, \dots, M$, the $r$-th replica of the process $l$. Let

$$T_r^{(l)} = \inf\left\{t \geq 0 : X_r^{(l)}(t) \leq 0\right\}$$

be the time to ruin of the $r$-th replica $r$ of the process $l$, and $R_r^{(l)} = \mathbf{1}_{\left\{T_r^{(l)} \leq T\right\}}$ be a random variable which indicates if the process $X_r^{(l)}$ reaches ruin in the interval $[0,T]$.

The estimator of $p^{(l)}$, $l = 1, 2$, which will be denoted as $P^{(l)}$, is the proportion of replicas in which ruin has happened, that is:

$$P^{(l)} = \frac{\sum_{r=1}^{M} R_r^{(l)}}{M}$$

and the estimator for the difference of these probabilities is:

$$P = P^{(1)} - P^{(2)}$$

The method used in the proof of the Theorem 1 is based on independent simulations of random variables which gives a less variance for the estimator $P$:

$$Var(P) = \frac{P \cdot (1-P)}{M}$$

in relation to an independent simulation of the some.

For the algorithm, a control variable $I_r^{(l)}$ is required, which has value 1 when the simulation must go on or 0 in other case; $l$ denotes the process $l = 1, 2$ and $r = 1, \dots, M$ the number of the replica.

For $l = 1, 2$, $P^{(l)}$ represents the proportion of replicas in which ruin occurs in process $X^{(l)}$ until time $T$. The number of replicas of the process $X^{(l)}$ with $l = 1, 2$ in which ruin happens up to time $T$, has a Binomial (Bi) distribution, that is, $M \cdot P^{(l)}$ is $Bi(M, p^{(l)})$. On the other hand, $R_r^{(1)} - R_r^{(2)}$ has a Bernoulli (Be) distribution $Be(p^{(1)} - p^{(2)})$ and so, $M \cdot P$ is $Bi(M, P)$.

As it was mentioned, the method used in Theorem 1 gives an estimator with less variance than the estimator obtained with independent simulations of $P^{(1)}$ and $P^{(2)}$. In fact, let

$$V_d = \frac{(p^{(1)} - p^{(2)})(1 - (p^{(1)} - p^{(2)}))}{M}$$

and

$$V_i = \frac{p^{(1)}(1 - p^{(1)}) + p^{(2)}(1 - p^{(2)})}{M}$$

the variances of the estimator in the case of dependent and independent simulations, respectively, and let

$$E = \frac{\sqrt{V_i} - \sqrt{V_d}}{\sqrt{V_i}}$$

The following table shows a numerical example of the reduction that is obtained by applying a dependent simulation method. In each entry a the table there the following three values are displayed: $\sqrt{M \cdot V_d}$, $\sqrt{M \cdot V_i}$ and $E$ in percentage:

**Input:** Independent sequences of independent random variables identically distributed as $U(0,1)$: $(U_{r,n})_{n\in\mathbb{N}_+}$, $(V_{r,n})_{n\in\mathbb{N}_+}$
**for** $r = 1, \dots, M$ **do**
  $(W_{r,n})_{n\in\mathbb{N}_+}$, $r = 1, \dots, M$. Values $T$, $x$, $f^{(1)}$ and $f^{(2)}$ with $f^{(1)} \leq f^{(2)}$.
  $\widetilde{X}^{(1)}_{r,0} = x$, $\widetilde{X}^{(2)}_{r,0} = x$
  $\widetilde{K}^{(1)}_{r,0} = f^{(1)}$, $\widetilde{K}^{(2)}_{r,0} = f^{(2)}$
  $I^{(1)}_r = 1$, $I^{(2)}_r = 1$, $n_1 = 0$, $n_2 = 0$
  **while** $\max\left\{I^{(1)}_r, I^{(2)}_r\right\} = 1$ **do**
    **for** $l = 1, 2$ **do**
      **if** $I^{(l)}_r = 1$ **then**

$$\widetilde{K}^{(l)}_{r,n_l+1} = \left[P^{(l)}_{\widetilde{K}^{l}_{r,n_l},\cdot}\right]^{-1}(U_{r,n_l+1})$$
$$\widetilde{H}^{(l)}_{r,n_l+1} = \left[F^{(l)}_{(\widetilde{K}^{(l)}_{r,n_l},\widetilde{K}^{(l)}_{r,n_l+1})}\right]^{-1}(V_{r,n_l+1})$$
$$\widetilde{Y}^{(l)}_{r,n_l+1} = \left[G^{(l)}_{(\widetilde{K}^{(l)}_{r,n_l},\widetilde{K}^{(l)}_{r,n_l+1})}\right]^{-1}(W_{r,n_l+1})$$
$$\widetilde{S}^{(l)}_{r,n_l+1} = \widetilde{S}^{(l)}_{r,n_l} + \widetilde{H}^{(l)}_{r,n_l+1}$$
$$\widetilde{X}^{(l)}_{r,n_l+1} = \widetilde{X}^{(l)}_{r,n_l} + c^{(l)}_{\widetilde{K}^{(l)}_{r,n_l}}\widetilde{H}^{(l)}_{r,n_l+1} - \widetilde{Y}^{(l)}_{r,n_l+1}$$

      $n_l = n_l + 1$
      **end if**
      **if** $\widetilde{S}^{(l)}_{r,n_l} \leq T$ and $\widetilde{X}^{(l)}_{r,n_l} \leq 0$ **then** $R^{(l)}_r = 1$ **end if**
      **if** $\widetilde{S}^{(l)}_{r,n_l} > T$ or $R^{(l)}_r = 1$ **then** $I^{(l)}_r = 0$ **end if**
    **end for**
  **end while**
**end for**
**for** $l = 1, 2$ **do**
$P^{(l)} = \frac{\sum_{r=1}^{M} R^{(l)}_r}{M}$
**end for**
$P = P^{(1)} - P^{(2)}$
$\hat{V}_d = \frac{P*(1-P)}{M}$
**Output:** Estimator $P$ of the difference between ruin probabilities of the two processes under consideration and its approximate variance $\hat{V}_d$

FIGURE 3: Algorithm to estimate the difference of ruin probabilities during a given time $T$, of two processes which satisfy conditions of Theorem 1.

Table 1: Reduction obtained by applying the dependent simulation method.

| $p^{(2)}$ \ $p^{(1)}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| | 0.000 | 0.300 | 0.400 | 0.458 | 0.490 | 0.500 | 0.490 | 0.458 | 0.400 |
| 0.1 | 0.424 | 0.500 | 0.5477 | 0.5745 | 0.5831 | 0.5745 | 0.5477 | 0.500 | 0.4243 |
| | 100% | 40.0% | 27.0% | 20.2% | 16.0% | 13.0% | 10.6% | 8.3% | 5.7% |
| | - | 0.000 | 0.300 | 0.400 | 0.458 | 0.490 | 0.500 | 0.490 | 0.458 |
| 0.2 | - | 0.566 | 0.608 | 0.632 | 0.640 | 0.632 | 0.608 | 0.566 | 0.500 |
| | - | 100% | 50.7% | 36.7% | 28.4% | 22.5% | 17.8% | 13.4% | 8.3% |
| | - | - | 0.000 | 0.300 | 0.400 | 0.458 | 0.490 | 0.500 | 0.490 |
| 0.3 | - | - | 0.648 | 0.671 | 0.678 | 0.671 | 0.648 | 0.6083 | 0.548 |
| | - | - | 100% | 55.3% | 41.0% | 31.7% | 24.4% | 17.8% | 10.6% |
| | - | - | - | 0.000 | 0.300 | 0.400 | 0.458 | 0.490 | 0.500 |
| 0.4 | - | - | - | 0.693 | 0.700 | 0.693 | 0.671 | 0.632 | 0.575 |
| | - | - | - | 100% | 57.1% | 42.3% | 31.7% | 22.5% | 13.0% |
| | - | - | - | - | 0.000 | 0.300 | 0.400 | 0.458 | 0.490 |
| 0.5 | - | - | - | - | 0.707 | 0.700 | 0.678 | 0.640 | 0.583 |
| | - | - | - | - | 100% | 57.1% | 41.0% | 28.4% | 16.0% |
| | - | - | - | - | - | 0.000 | 0.300 | 0.400 | 0.458 |
| 0.6 | - | - | - | - | - | 0.693 | 0.671 | 0.632 | 0.574 |
| | - | - | - | - | - | 100% | 55.3% | 36.7% | 20.2% |
| | - | - | - | - | - | - | 0.000 | 0.300 | 0.400 |
| 0.7 | - | - | - | - | - | - | 0.648 | 0.608 | 0.548 |
| | - | - | - | - | - | - | 100% | 50.7% | 27.0% |
| | - | - | - | - | - | - | - | 0.000 | 0.300 |
| 0.8 | - | - | - | - | - | - | - | 0.566 | 0.500 |
| | - | - | - | - | - | - | - | 100% | 40.0% |
| | - | - | - | - | - | - | - | - | 0.000 |
| 0.9 | - | - | - | - | - | - | - | - | 0.424 |
| | - | - | - | - | - | - | - | - | 100% |

As it can be seen, values of $\sqrt{V_i}$ are higher than the correspondents $\sqrt{V_d}$ obtained with dependent simulation as in the proof of Theorem 1. In the particular case in which $p^{(1)} = p^{(2)}$, this method gives a big reduction, because the described method present a value $V_d = 0$, while values in the independent case are strictly positive.

With this method confidence intervals with lower amplitude can be built:

$$IC(1-\alpha) = \left( P \pm \Phi(1-\alpha/2) \cdot \sqrt{\frac{P*(1-P)}{M}} \right)$$

# 5. Conclusions

The problem of ruin was addressed from a different perspective to the traditional. Instead of setting expressions or quotations for the ruin probability of a particular model for the selection of each other, times to ruin have been ranked. This will allow to make a selection without knowing explicitly the expression of the probability of ruin or an approximation thereof.

On the other hand, simulation algorithms have been proposed for these processes and statistical inference methods to estimate differences between the probability of ruin of the models have been considered.

This paper is a reference tool which can be used to determine the actual level of risk assumed by insurers (sufficiency of financial resources, reserves and capital).

The problems of the minimum solvency margin and the probability of survival of the reserves can be approached from the perspective proposed, since it allows to model stochastic processes at groups, taking into account those risks that may occur at the group level and not necessarily at the level of companies considered individually.

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