

Estimation of Variance Components in Linear Mixed Models with Commutative Orthogonal Block Structure

Estimación de las componentes de varianza en modelos lineales mixtos con estructura de bloques ortogonal conmutativa

SANDRA S. FERREIRA^{1,a}, DÁRIO FERREIRA^{2,b}, CÉLIA NUNES^{3,c},
JOÃO T. MEXIA^{4,d}

¹DEPARTMENT OF MATHEMATICS AND CENTER OF MATHEMATICS, FACULTY OF SCIENCES,
UNIVERSITY OF BEIRA INTERIOR, COVILHÃ, PORTUGAL

²DEPARTMENT OF MATHEMATICS AND CENTER OF MATHEMATICS, FACULTY OF SCIENCES,
UNIVERSITY OF BEIRA INTERIOR, COVILHÃ, PORTUGAL

³DEPARTMENT OF MATHEMATICS AND CENTER OF MATHEMATICS, FACULTY OF SCIENCES,
UNIVERSITY OF BEIRA INTERIOR, COVILHÃ, PORTUGAL

⁴DEPARTMENT OF MATHEMATICS AND CENTER OF MATHEMATICS AND ITS APPLICATIONS,
FACULTY OF SCIENCE AND TECHNOLOGY, NEW UNIVERSITY OF LISBON, COVILHÃ, PORTUGAL

Abstract

Segregation and matching are techniques to estimate variance components in mixed models. A question arising is whether segregation can be applied in situations where matching does not apply. Our motivation for this research relies on the fact that we want an answer to that question and to explore this important class of models that can contribute to the development of mixed models. That is possible using the algebraic structure of mixed models. We present two examples showing that segregation can be applied in situations where matching does not apply.

Key words: Commutative Jordan algebra, Mixed model, Variance components.

^aProfessor. E-mail: sandraf@ubi.pt

^bProfessor. E-mail: dario@ubi.pt

^cProfessor. E-mail: celian@ubi.pt

^dProfessor. E-mail: jtm@fct.unl.pt

Resumen

La segregación y el emparejamiento son técnicas para estimar las componentes de varianza en modelos mixtos. Una pregunta que ha surgido es si la segregación puede ser aplicada en situaciones en las que el emparejamiento no es aplicable. Nuestra motivación para esta investigación se basa en el hecho de que se quiere una respuesta a esta pregunta y se quiere explorar esta importante clase de modelos con el fin de contribuir al desarrollo de los modelos mixtos. Esto es posible utilizando la estructura algebraica de los modelos mixtos con estructura de bloques ortogonal conmutativa. Se presentan dos ejemplos que muestran que la segregación puede ser aplicada en situaciones donde el emparejamiento no es aplicable.

Palabras clave: álgebra conmutativa Jordan, componentes de varianza, modelo mixto.

1. Introduction

Mixed models have orthogonal block structure, OBS, when their variance covariance matrices are orthogonal all the linear combinations of known pairwise projection matrices, POOPM, add up to \mathbf{I}_n with non negative coefficients. These models play an important role in design of experiments (Houtman & Speed 1983, Mejza 1992) and were introduced by Nelder (1965*a*, 1965*b*), continuing to play an important part in the theory of randomized block designs (see Caliński & Kageyama 2000, Caliński & Kageyama 2003).

A direct generalization of this class of models is that of models whose variance covariance matrices are linear combinations of known POOPM, we say these models to have generalized orthogonal block structure, GOBS. Moreover if the orthogonal projection matrix \mathbf{T} on the space spanned by the mean vectors commutes with these POOPM the model, (see Fonseca, Mexia & Zmysłony 2008) will have commutative orthogonal block structure, COBS. Then, (see Zmysłony 1978), its least square estimators, LSE, for estimable vectors will be best linear unbiased estimators, BLUE, whatever the variance components.

In what follows, we will present techniques for the estimation of variance components in COBS. These techniques will be based on the algebraic structure of the models then being quite distinct from other techniques that require normality. Moreover it has interesting developments, namely these related to model segregation.

The next Section presents the algebraic structure of the models considering commutative Jordan algebras. Then we discuss, in section 3, the techniques for the estimation of variance components: Matching and segregation. Segregation displays the possibility of using the algebraic structure in estimation. Thus, in subsections 3.1 and 3.2, we present two models in which this technique has to be used to complete the structure based on estimation of variance components. Lastly, we present some final remarks.

2. Algebras and Models

Commutative Jordan Algebras, CJA, (of symmetric matrices) are linear spaces constituted by symmetric matrices that commute and containing the square of this matrices. Each CJA \mathcal{A} has a principal unique basis (see, Seely 1971), $pb(\mathcal{A})$, constituted by pairwise orthogonal projection matrices. Any orthogonal projection matrix belonging to \mathcal{A} will be the sum of matrices in $pb(\mathcal{A})$.

Moreover, given a family \mathbf{W} of symmetric matrices that commute, there is a minimal CJA $\mathcal{A}(\mathbf{W})$ containing \mathbf{W} (see, Fonseca et al. 2008).

Consider the model

$$\mathbf{Y} = \sum_{i=0}^w \mathbf{X}_i \beta_i \tag{1}$$

where β_0 is fixed and the β_1, \dots, β_w are independent, with null mean vectors and variance covariance matrices

$$\begin{cases} \boldsymbol{\mu} = \mathbf{X}_0 \beta_0 \\ \mathbf{V}(\boldsymbol{\theta}) = \sum_{i=1}^w \theta_i \mathbf{M}_i \end{cases} \tag{2}$$

with $\mathbf{M}_i = \mathbf{X}_i \mathbf{X}_i'$, $i = 1, \dots, w$. When the matrices in $\{\mathbf{T}, \mathbf{M}, \dots, \mathbf{M}_w\}$ commute we have the CJA $\mathcal{A}(\mathbf{W})$ with principal basis

$$\mathbf{Q} = \{\mathbf{Q}_1, \dots, \mathbf{Q}_m\}.$$

We can order the matrices in \mathbf{Q} to have $\mathbf{T} = \sum_{j=1}^z \mathbf{Q}_j$. Moreover

$$\mathbf{M}_i = \sum_{j=1}^m b_{i,j} \mathbf{Q}_j, i = 1, \dots, w,$$

so that

$$\mathbf{V}(\boldsymbol{\theta}) = \sum_{i=1}^w \theta_i \mathbf{M}_i = \sum_{j=1}^m \gamma_j \mathbf{Q}_j = \mathbf{V}(\boldsymbol{\gamma})$$

with $\gamma_j = \sum_{i=1}^w b_{i,j} \theta_i$, $j = 1, \dots, m$, thus the model will have COBS since its variance covariance matrices are linear combinations of known POOPM that commute with the $\mathbf{Q}_1, \dots, \mathbf{Q}_m$, belonging jointly to $\mathcal{A}(\mathbf{W})$.

3. Segregation and Matching

Since $R(\mathbf{Q}_j) \subseteq R(\mathbf{T})$, $j = 1, \dots, z$ we can estimate directly the $\gamma_{z+1}, \dots, \gamma_m$, for which we have the unbiased estimators

$$\tilde{\gamma}_j = \frac{\|\mathbf{Q}_j \mathbf{Y}\|^2}{r(\mathbf{Q}_j)}, j = z + 1, \dots, m. \tag{3}$$

Partitioning matrix $\mathbf{B} = [b_{i,j}]$ as $[\mathbf{B}_1 \ \mathbf{B}_2]$, where \mathbf{B}_1 has z columns, and taking $\boldsymbol{\gamma}_1 = (\gamma_1, \dots, \gamma_z)'$, $\boldsymbol{\gamma}_2 = (\gamma_{z+1}, \dots, \gamma_m)'$, and $\boldsymbol{\sigma}^2 = (\sigma_1^2, \dots, \sigma_w^2)'$, with $w \leq m - z$, we have

$$\boldsymbol{\gamma}_l = \mathbf{B}'_l \boldsymbol{\sigma}^2, l = 1, 2. \quad (4)$$

When the column vectors of \mathbf{B}'_2 are linearly independent we have

$$\boldsymbol{\sigma}^2 = (\mathbf{B}'_2)^+ \boldsymbol{\gamma}_2, \quad (5)$$

as well as

$$\boldsymbol{\gamma}_1 = \mathbf{B}'_1 (\mathbf{B}'_2)^+ \boldsymbol{\gamma}_2, \quad (6)$$

allowing the estimation of $\boldsymbol{\sigma}^2$ and $\boldsymbol{\gamma}_1$, through $\boldsymbol{\gamma}_2$. It may be noted that if the matrices $\mathbf{Q}_1, \dots, \mathbf{Q}_m$ can be ordered in such a way that the transition matrix is

$$\mathbf{B} = \begin{bmatrix} B_{1,1} & 0 \\ B_{2,1} & B_{2,2} \end{bmatrix},$$

with $B_{1,1}$ a $z \times z$ matrix, the model is said to be segregated, see Ferreira, Ferreira & Mexia (2007) and Ferreira, Ferreira, Nunes & Mexia (2010). It can be pointed out that, in that case, sub-matrices $B_{1,1}$ and $B_{2,2}$ are regular.

When \mathbf{B}_1 is a sub-matrix of \mathbf{B}_2 , \mathbf{B}'_1 will be a sub-matrix of \mathbf{B}'_2 and so $\boldsymbol{\gamma}_1$ will be a sub-vector of $\boldsymbol{\gamma}_2$, see Mexia, Vaquinhas, Fonseca & Zmyslony (2010). In this case the match have between the components of $\boldsymbol{\gamma}_1$ and some components of $\boldsymbol{\gamma}_2$. When this happens we say that the model has matching. Thus $\boldsymbol{\gamma}_1$ and

$$\boldsymbol{\gamma} = [\boldsymbol{\gamma}'_1 \quad \boldsymbol{\gamma}'_2]',$$

can be directly estimated from $\boldsymbol{\gamma}_2$. If the row vectors of \mathbf{B} are linearly independent, we have

$$\boldsymbol{\sigma}^2 = (\mathbf{B}')^+ \boldsymbol{\gamma}, \quad (7)$$

and we can also estimate $\boldsymbol{\sigma}^2$. Requiring the row vectors of \mathbf{B} to be linearly independent is less restrictive than requiring the row vectors of \mathbf{B}_2 to be linearly independent.

Below we introduce two examples which show that segregation can be applied in situations where matching does not apply.

3.1. Segregation without Matching: Stair Nesting

We choose to present an example with stair nesting instead of the usual nesting because stair nesting designs are unbalanced and use fewer observations than the balanced case, and in addition, the degrees of freedom for all factors are more evenly distributed, as was shown by Fernandes, Mexia, Ramos & Carvalho (2011). Cox & Solomon (2003) suggested that having u factors, we will have u steps where each step corresponds to one factor of the model.

In order to describe the branching in such models, we can consider $u + 1$ steps. The first step, with index 0, has $a_0 = c_0 = u$ branches, one per factor. In the second step, with index 1, we have $c_1 = a(1) + u - 1$ branches, $a(1)$ the number of “active” levels for the first factor and $u - 1$ the number of the remaining factors. We point out that the branch for the first factors concerns its “active” levels. For the third step, with index 2, we have $c(2) = a(1) + a(2) + u - 2$, where $a(1)$ represents the number of “active” levels for the first two factors resulting from the branching for the first factor; $a(2)$ is the number levels for the second factor and $u - 2$, the number of the remaining factors. In this way, for the $(i + 1)$ -th step, with index i , we have $c(i) = \sum_{h=1}^i a(h) + u - i, i = 3, \dots, u$ branches. $a(1), \dots, a(i)$ are the number of “active” levels for the first i factors and $u - i$ the number of remaining factors. These designs are also studied in Fernandes, Ramos & Mexia (2010) and some results of nesting may be seen, for example, in Bailey (2004).

The model for stair nesting designs is given by

$$\mathbf{Y} = \sum_{i=0}^u \mathbf{X}_i \boldsymbol{\beta}_i, \tag{8}$$

with

$$\begin{cases} \mathbf{X}_0 = D(\mathbf{1}_{a(1)}, \dots, \mathbf{1}_{a(i)}, \mathbf{1}_{a(i+1)}, \dots, \mathbf{1}_{a(u)}) \\ \vdots \\ \mathbf{X}_i = D(\mathbf{I}_{a(1)}, \dots, \mathbf{I}_{a(i)}, \mathbf{1}_{a(i+1)}, \dots, \mathbf{1}_{a(u)}), i = 1, \dots, u - 1 \\ \vdots \\ \mathbf{X}_u = D(\mathbf{I}_{a(1)}, \dots, \mathbf{I}_{a(i)}, \mathbf{I}_{a(i+1)}, \dots, \mathbf{I}_{a(u)}) \end{cases} \tag{9}$$

where $D(\mathbf{A}_1, \dots, \mathbf{A}_u)$ is the block diagonal matrix with principal blocks $\mathbf{A}_1, \dots, \mathbf{A}_u$ and $\mathbf{1}_{a(s)}$ is the vector with all $a(s)$ components equal to 1.

In this approach we will assume that $\boldsymbol{\beta}_0 = \mathbf{1}_u \boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is the general mean value and the vectors $\boldsymbol{\beta}_i, i = 1, \dots, u$, are independent normal with null mean vectors and variance-covariance matrix $\sigma_i^2 \mathbf{I}_{c(i)}, i = 1, \dots, u$, and

$$c(i) = \sum_{h=1}^i a(h) + u - i, i = 1, \dots, u$$

Hence \mathbf{Y} is normal distributed with mean vector $\boldsymbol{\mu} = \mathbf{1}_n \boldsymbol{\mu}$, and variance-covariance matrix $\mathbf{V} = \sum_{i=1}^u \sigma_i^2 \mathbf{M}_i$, where $\mathbf{M}_i = \mathbf{X}_i \mathbf{X}_i', i = 1, \dots, u$, we have

$$\begin{cases} \mathbf{M}_0 = D(\mathbf{J}_{a(1)}, \dots, \mathbf{J}_{a(i)}) \\ \vdots \\ \mathbf{M}_i = D(\mathbf{I}_{a(1)}, \dots, \mathbf{I}_{a(i)}, \mathbf{J}_{a(i+1)}, \dots, \mathbf{J}_{a(u)}), i = 1, \dots, u - 1 \\ \vdots \\ \mathbf{M}_u = D(\mathbf{I}_{a(1)}, \dots, \mathbf{I}_{a(i)}, \mathbf{I}_{a(i+1)}, \dots, \mathbf{I}_{a(u)}) \end{cases} \tag{10}$$

with $\mathbf{J}_s = \mathbf{1}_s \mathbf{1}'_s$. Now, the orthogonal projection matrix on $r(\mathbf{X}_0)$, will be \mathbf{T} given by

$$\mathbf{T} = D \left(\frac{1}{a(1)} \mathbf{J}_{a(1)}, \dots, \frac{1}{a(i)} \mathbf{J}_{a(i)}, \frac{1}{a(i+1)} \mathbf{J}_{a(i+1)}, \dots, \frac{1}{a(u)} \mathbf{J}_{a(u)} \right) \quad (11)$$

Moreover, with $\mathbf{K}_{a(i)} = \mathbf{I}_{a(i)} - \frac{1}{a(i)} \mathbf{J}_{a(i)}$ and $\mathbf{0}_{a(i)}$ the null $a(i) \times a(i)$ matrix, $i = 1, \dots, u$, taking

$$\begin{cases} \mathbf{Q}_i = D(\mathbf{0}_{a(1)}, \dots, \frac{1}{a(i)} \mathbf{J}_{a(i)}, \dots, \mathbf{0}_{a(u)}), & i = 1, \dots, u \\ \mathbf{Q}_{i+u} = D(\mathbf{0}_{a(1)}, \dots, \mathbf{K}_{a(i)}, \dots, \mathbf{0}_{a(u)}), & i = 1, \dots, u \end{cases} \quad (12)$$

we will have

$$\begin{cases} \mathbf{T} = \sum_{j=1}^u \mathbf{Q}_j \\ \mathbf{M}_i = \sum_{j=1}^i (\mathbf{Q}_j + \mathbf{Q}_{j+u}) + \sum_{j=i+1}^u a(j) \mathbf{Q}_j, & i = 1, \dots, u-1. \\ \mathbf{M}_u = \sum_{j=1}^u (\mathbf{Q}_j + \mathbf{Q}_{j+u}) \end{cases} \quad (13)$$

So we have

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2],$$

with

$$\mathbf{B}_1 = \begin{bmatrix} 1 & a(2) & \dots & a(u) \\ 1 & 1 & \dots & a(u) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a(u) \\ 1 & 1 & \dots & 1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 0 \\ 1 & 1 & \dots & 1 \end{bmatrix},$$

so we have segregation but we do not have matching.

Let us consider an example where $u = 3$, $a(1) = 3$, $a(2) = 2$ and $a(3) = 3$ “active” levels and the number of observations in the design is $n = 3 + 2 + 3 = 8$. So, we have $g(1) = 2$, $g(2) = 1$ and $g(3) = 2$ degrees of freedom for the first, second, and third factors, respectively. The design is shown in Figure 1.

The random effects model for stair nesting can be summarized as

$$\mathbf{Y} = \sum_{i=0}^3 \mathbf{X}_i \beta_i \quad (14)$$

where $a(1) = 3$, $a(2) = 2$ and $a(3) = 3$ are the levels for the 3 factors that nest. We make the same assumptions on the random effects as we did in the section 3.1,

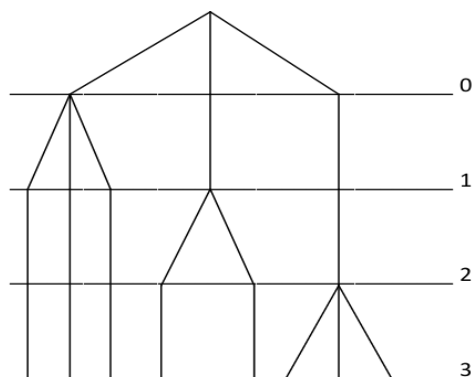


FIGURE 1: Stair nested design.

where

$$\left\{ \begin{array}{l} \mathbf{X}_0 = D(\mathbf{1}_3, \mathbf{1}_2, \mathbf{1}_3) \\ \mathbf{X}_1 = D(\mathbf{I}_3, \mathbf{1}_2, \mathbf{1}_3) \\ \mathbf{X}_2 = D(\mathbf{I}_3, \mathbf{I}_2, \mathbf{1}_3) \\ \mathbf{X}_3 = D(\mathbf{I}_3, \mathbf{I}_2, \mathbf{I}_3) \end{array} \right. \quad (15)$$

From formula (13) we obtain

$$\left\{ \begin{array}{l} \mathbf{M}_1 = D(\mathbf{I}_3, \mathbf{J}_2, \mathbf{J}_3) \\ \mathbf{M}_2 = D(\mathbf{I}_3, \mathbf{I}_2, \mathbf{J}_3) \\ \mathbf{M}_3 = D(\mathbf{I}_3, \mathbf{I}_2, \mathbf{I}_3) \end{array} \right. \quad (16)$$

Considering $m = 6, z = 3$, we have the pairwise orthogonal projection matrices

$$\left\{ \begin{array}{l} \mathbf{Q}_1 = \{\frac{1}{3}\mathbf{J}_3, \mathbf{0}_2, \mathbf{0}_3\} \\ \mathbf{Q}_2 = \{\mathbf{0}_3, \frac{1}{2}\mathbf{J}_2, \mathbf{0}_3\} \\ \mathbf{Q}_3 = \{\mathbf{0}_3, \mathbf{0}_2, \frac{1}{3}\mathbf{J}_3\} \\ \mathbf{Q}_4 = \{\mathbf{K}_3, \mathbf{0}_2, \mathbf{0}_3\} \\ \mathbf{Q}_5 = \{\mathbf{0}_3, \mathbf{K}_2, \mathbf{0}_3\} \\ \mathbf{Q}_6 = \{\mathbf{0}_3, \mathbf{0}_2, \mathbf{K}_3\} \end{array} \right.$$

and the matrices

$$\begin{cases} \mathbf{M}_1 = \mathbf{Q}_1 + a(2)\mathbf{Q}_2 + a(3)\mathbf{Q}_3 + \mathbf{Q}_4 \\ \mathbf{M}_2 = \mathbf{Q}_1 + \mathbf{Q}_2 + a(3)\mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5 \\ \mathbf{M}_3 = \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5 + \mathbf{Q}_6 \end{cases}$$

It follows readily that

$$\mathbf{B} = \begin{bmatrix} 1 & a(2) & a(3) & 1 & 0 & 0 \\ 1 & 1 & a(3) & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

considering

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2]$$

where

$$\mathbf{B}_1 = \begin{bmatrix} 1 & a(2) & a(3) \\ 1 & 1 & a(3) \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

3.2. Segregation without Matching: Crossing

Let there be a first factor that crosses with a second that nests a third. The factors will have a , b and c levels, respectively. The first and the third factors have random effects and the second has fixed effects.

The mean vector will then be

$$\boldsymbol{\mu} = (\mathbf{1}_a \otimes \mathbf{1}_b \otimes \mathbf{1}_c) \boldsymbol{\mu} + (\mathbf{1}_a \otimes \mathbf{I}_b \otimes \mathbf{1}_c) \boldsymbol{\beta}(2)$$

where $\boldsymbol{\beta}(2)$ is the fixed vector of the effects for the second factor and \otimes represent the Kronecker matrix product.

The random effects part of the model will be

$$\begin{aligned} & (\mathbf{I}_a \otimes \mathbf{1}_b \otimes \mathbf{1}_c) \boldsymbol{\beta}(1) + (\mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{1}_c) \boldsymbol{\beta}(1, 2) + (\mathbf{1}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_c) \boldsymbol{\beta}(3) + \\ & + (\mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_c) \boldsymbol{\beta}(1, 3), \end{aligned}$$

where $\boldsymbol{\beta}(1)$, $\boldsymbol{\beta}(1, 2)$, $\boldsymbol{\beta}(3)$ and $\boldsymbol{\beta}(1, 3)$ correspond to the effects of the first factor, to the interactions of the first and second factors, to the effects of the third factor and to the interactions between the first and the third factors. As usual, we assume these vectors to be independent, homoscedastic and represent the corresponding

variance components by $\sigma^2(1)$, $\sigma^2(1,2)$, $\sigma^2(3)$ and $\sigma^2(1,3)$. So the variance-covariance matrix will be given by

$$\mathbf{V} = \sigma^2(1) \mathbf{I}_a \otimes \mathbf{J}_b \otimes \mathbf{J}_c + \sigma^2(1,2) \mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{J}_c + \sigma^2(3) \mathbf{J}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_c + \sigma^2(1,3) \mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_c.$$

In this case the matrices in the principal basis will be

$$\begin{cases} \mathbf{Q}_1 = \frac{1}{a} \mathbf{J}_a \otimes \frac{1}{b} \mathbf{J}_b \otimes \frac{1}{c} \mathbf{J}_c \\ \mathbf{Q}_2 = \mathbf{K}_a \otimes \frac{1}{b} \mathbf{J}_b \otimes \frac{1}{c} \mathbf{J}_c \\ \mathbf{Q}_3 = \frac{1}{a} \mathbf{J}_a \otimes \mathbf{K}_b \otimes \frac{1}{c} \mathbf{J}_c \\ \mathbf{Q}_4 = \mathbf{K}_a \otimes \mathbf{K}_b \otimes \frac{1}{c} \mathbf{J}_c \\ \mathbf{Q}_5 = \frac{1}{a} \mathbf{J}_a \otimes \frac{1}{b} \mathbf{J}_b \otimes \mathbf{K}_c \\ \mathbf{Q}_6 = \mathbf{K}_a \otimes \frac{1}{b} \mathbf{J}_b \otimes \mathbf{K}_c \end{cases}$$

Moreover the orthogonal projection matrix on Ω will be

$$\mathbf{T} = \frac{1}{a} \mathbf{J}_a \otimes \mathbf{I}_b \otimes \frac{1}{c} \mathbf{J}_c = \mathbf{Q}_1 + \mathbf{Q}_3.$$

We will also have

$$\begin{cases} \mathbf{I}_a \otimes \mathbf{J}_b \otimes \mathbf{J}_c = bc\mathbf{Q}_1 + bc\mathbf{Q}_2 \\ \mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{J}_c = c\mathbf{Q}_1 + c\mathbf{Q}_2 + c\mathbf{Q}_3 + c\mathbf{Q}_4 \\ \mathbf{J}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_c = a\mathbf{Q}_1 + a\mathbf{Q}_3 + a\mathbf{Q}_5 \\ \mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_c = \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5 + \mathbf{Q}_6 \end{cases}$$

Therefore

$$\mathbf{V} = \sum_{j=1}^6 \gamma_j \mathbf{Q}_j,$$

with

$$\begin{cases} \gamma_1 = bc\sigma^2(1) + c\sigma^2(1,2) + a\sigma^2(3) + \sigma^2(1,3) \\ \gamma_2 = bc\sigma^2(1) + c\sigma^2(1,2) + \sigma^2(1,3) \\ \gamma_3 = c\sigma^2(1,2) + a\sigma^2(3) + \sigma^2(1,3) \\ \gamma_4 = c\sigma^2(1,2) + \sigma^2(1,3) \\ \gamma_5 = a\sigma^2(3) + \sigma^2(1,3) \\ \gamma_6 = \sigma^2(1,3) \end{cases}$$

Now γ_1 and γ_3 are different from all other canonical variance components so there is no matching. Despite this we have

$$\begin{cases} \sigma^2(1,3) = \gamma_6 \\ \sigma^2(3) = \frac{\gamma_5 - \gamma_6}{a} \\ \sigma^2(1,2) = \frac{\gamma_4 - \gamma_6}{c} \\ \sigma^2(1) = \frac{\gamma_2 - c\sigma^2(1,2) - \sigma^2(1,3)}{bc} = \frac{\gamma_2 - \gamma_4}{bc} \end{cases}$$

so all variance components either usual or canonic can be estimated.

4. Final Remarks

COBS models consider important cases. In the second example in Section 3 we presented an example of a balanced crossing which, (see Fonseca, Mexia & Zmysłony 2003, Fonseca, Mexia & Zmysłony 2007) can be extended to apply to all models with balanced cross nesting, thus including a wide variety of well behaved models.

The first example in section 3, that of stair nesting, displays a different model also with COBS. Besides the algebraic structure enables us to obtain unbiased estimators without normality. The LSE for estimable vectors are BLUE, whatever the variance components.

Acknowledgements

This work was partially supported by the center of Mathematics, University of Beira Interior, under the project PEst-OE/MAT/UI0212/2011.

We thank the anonymous referees and the Editor for useful comments and suggestions on a previous version of the paper, which helped to improve substantially the initial manuscript.

[Recibido: octubre de 2012 — Aceptado: septiembre de 2013]

References

- Bailey, R. A. (2004), *Association Schemes: Designed Experiments, Algebra and Combinatorics*, Cambridge University Press, Cambridge.
- Caliński, T. & Kageyama, S. (2000), *Block Designs: A Randomization Approach Vol. I: Analysis*, Springer-Verlag, New York.
- Caliński, T. & Kageyama, S. (2003), *Block Designs: A Randomization Approach Vol. II: Analysis*, Springer-Verlag, New York.
- Cox, D. & Solomon, P. (2003), *Components of Variance*, Chapman and Hall, New York.
- Fernandes, C., Mexia, J., Ramos, P. & Carvalho, F. (2011), ‘Models with stair nesting’, *AIP Conference Proceedings - Numerical Analysis and Applied Mathematics* **1389**, 1627–1630.
- Fernandes, C., Ramos, P. & Mexia, J. (2010), ‘Algebraic structure of step nesting designs’, *Discussiones Mathematicae. Probability and Statistics* **30**, 221–235.
- Ferreira, S. S., Ferreira, D. & Mexia, J. T. (2007), ‘Cross additivity in balanced cross nesting models’, *Journal of Statistical Theory and Practice* (3), 377–392.

- Ferreira, S. S., Ferreira, D., Nunes, C. & Mexia, J. T. (2010), 'Nesting segregated mixed models', *Journal of Statistical Theory and Practice* **4**(2), 233–242.
- Fonseca, M., Mexia, J. T. & Zmysłony, R. (2003), 'Estimators and tests for variance components in cross nested orthogonal models', *Discussiones Mathematicae - Probability and Statistics* **23**(3), 175–201.
- Fonseca, M., Mexia, J. T. & Zmysłony, R. (2007), 'Jordan algebras generating pivot variables and orthogonal normal models', *Journal of Interdisciplinary Mathematics* (10), 305–326.
- Fonseca, M., Mexia, J. T. & Zmysłony, R. (2008), 'Inference in normal models with commutative orthogonal block structure', *Acta et Commentationes Universitatis Tartuensis de Mathematica* (12), 3–16.
- Houtman, A. & Speed, T. (1983), 'Balance in designed experiments with orthogonal block structure', *Annals of Statistics* **11**(4), 1069–1085.
- Mejza, S. (1992), 'On some aspects of general balance in designed experiments', *Statistica* **52**, 263–278.
- Mexia, J. T., Vaquinhas, R., Fonseca, M. & Zmysłony, R. (2010), 'COBS: Segregation, Matching, Crossing and Nesting', *Latest Trends on Applied Mathematics, Simulation, Modeling, 4th International Conference on Applied Mathematics, Simulation, Modelling (ASM'10)* pp. 249–255.
- Nelder, J. (1965a), 'The analysis of randomized experiments with orthogonal block structure. I. Block structure and the null analysis of variance', *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **283**(1393), 147–162.
- Nelder, J. (1965b), 'The analysis of randomized experiments with orthogonal block structure. II. Treatment structure and the general analysis of variance', *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **273**(1393), 163–178.
- Seely, J. (1971), 'Quadratic subspaces and completeness', *The Annals of Mathematical Statistics* **42**, 710–721.
- Zmysłony, R. (1978), 'A characterization of best linear unbiased estimators in the general linear model', *Mathematical Statistics and Probability Theory* **2**, 365–373.