

# The Beta-Gompertz Distribution

## La distribución Beta-Gompertz

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### Abstract

In this paper, we introduce a new four-parameter generalized version of the Gompertz model which is called Beta-Gompertz (BG) distribution. It includes some well-known lifetime distributions such as Beta-exponential and generalized Gompertz distributions as special sub-models. This new distribution is quite flexible and can be used effectively in modeling survival data and reliability problems. It can have a decreasing, increasing, and bathtub-shaped failure rate function depending on its parameters. Some mathematical properties of the new distribution, such as closed-form expressions for the density, cumulative distribution, hazard rate function, the  $k$ th order moment, moment generating function, Shannon entropy, and the quantile measure are provided. We discuss maximum likelihood estimation of the BG parameters from one observed sample and derive the observed Fisher's information matrix. A simulation study is performed in order to investigate the properties of the proposed estimator. At the end, in order to show the BG distribution flexibility, an application using a real data set is presented.

**Key words:** Beta generator, Gompertz distribution, Maximum likelihood estimation.

### Resumen

En este artículo, se introduce una versión generalizada en cuatro parámetros de la distribución de Gompertz denominada como la distribución Beta-Gompertz (BG). Esta incluye algunas distribuciones de duración de vida bien conocidas como la Beta exponencial y distribuciones Gompertz generalizadas como casos especiales. Esta nueva distribución es flexible y puede ser usada de manera efectiva en datos de sobrevida y problemas de confiabilidad. Su función de tasa de falla puede ser decreciente, creciente o en forma de bañera

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dependiendo de sus parámetros. Algunas propiedades matemáticas de la distribución como expresiones en forma cerrada para la densidad, función de distribución, función de riesgo, momentos k-ésimos, función generadora de momentos, entropía de Shannon y cuantiles son presentados. Se discute la estimación máximo verosímil de los parámetros desconocidos del nuevo modelo para la muestra completa y se obtiene una expresión para la matriz de información. Con el fin de mostrar la flexibilidad de esta distribución, se presenta una aplicación con datos reales. Al final, un estudio de simulación es desarrollado.

**Palabras clave:** distribución de Gompertz, estimación máximo verosímil, función Beta.

## 1. Introduction

The Gompertz (G) distribution is a flexible distribution that can be skewed to the right and to the left. This distribution is a generalization of the exponential (E) distribution and is commonly used in many applied problems, particularly in lifetime data analysis (Johnson, Kotz & Balakrishnan 1995, p. 25). The G distribution is considered for the analysis of survival, in some sciences such as gerontology (Brown & Forbes 1974), computer (Ohishi, Okamura & Dohi 2009), biology (Economos 1982), and marketing science (Bemmaor & Glady 2012). The hazard rate function (hrf) of G distribution is an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers (Willemse & Koppelaar 2000). The G distribution with parameters  $\theta > 0$  and  $\gamma > 0$  has the cumulative distribution function (cdf)

$$G(x) = 1 - e^{-\frac{\theta}{\gamma}(e^{\gamma x} - 1)}, \quad x \geq 0, \quad \beta > 0, \quad \gamma > 0 \quad (1)$$

and the probability density function (pdf)

$$g(x) = \theta e^{\gamma x} e^{-\frac{\theta}{\gamma}(e^{\gamma x} - 1)} \quad (2)$$

This case is denoted by  $X \sim G(\theta, \gamma)$ .

Recently, a generalization based on the idea of Gupta & Kundu (1999) was proposed by El-Gohary & Al-Otaibi (2013).

This new distribution is known as generalized Gompertz (GG) distribution which includes the E, generalized exponential (GE), and G distributions (El-Gohary & Al-Otaibi 2013).

In this paper, we introduce a new generalization of G distribution which results of the application of the G distribution to the Beta generator proposed by Eugene, Lee & Famoye (2002), called the Beta-Gompertz (BG) distribution.

Several generalized distributions have been proposed under this methodology: beta-Normal distribution (Eugene et al. 2002), Beta-Gumbel distribution (Nadarajah & Kotz 2004), Beta-Weibull distribution (Famoye, Lee & Olumolade 2005), Beta-exponential (BE) distribution, (Nadarajah & Kotz 2006), Beta-Pareto

distribution (Akinsete, Famoye & Lee 2008), Beta-modified Weibull distribution (Silva & Cordeiro 2010), Beta-generalized normal distribution (Cintra & Nascimento 2012). The BG distribution includes some well-known distributions: E distribution, GE distribution (Gupta & Kundu 1999), BE distribution (Nadarajah & Kotz 2006), G distribution, GG distribution (El-Gohary & Al-Otaibi 2013).

This paper is organized as follows: In Section 2, we define the density and failure rate functions and outline some special cases of the BG distribution. In Sections 3 we provide some extensions and properties of the cdf, pdf,  $k$ th moment and moment generating function of the BG distribution. Furthermore, in these sections, we derive corresponding expressions for the order statistics, Shannon entropy and quantile measure. In Section 4, we discuss maximum likelihood estimation of the BG parameters from one observed sample and derive the observed Fisher's information matrix.

A simulation study is performed in Section 5. Finally, an application of the BG using a real data set is presented in Section 6.

## 2. The BG Distribution

In this section, we introduce the four-parameter BG distribution. The idea of this distribution rises from the following general class: If  $G$  denotes the cdf of a random variable then a generalized class of distributions can be defined by

$$F(x) = I_{G(x)}(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \alpha, \beta > 0 \quad (3)$$

where  $I_y(\alpha, \beta) = \frac{B_y(\alpha, \beta)}{B(\alpha, \beta)}$  is the incomplete beta function ratio and  $B_y(\alpha, \beta) = \int_0^y t^{\alpha-1} (1-t)^{\beta-1} dt$  is the incomplete beta function.

Consider that  $g(x) = \frac{dG(x)}{dx}$  is the density of the baseline distribution. Then the probability density function corresponding to (3) can be written in the form

$$f(x) = \frac{g(x)}{B(\alpha, \beta)} [G(x)]^{\alpha-1} [1 - G(x)]^{\beta-1} \quad (4)$$

We now introduce the BG distribution by taking  $G(x)$  in (3) to the cdf in (1) of the G distribution. Hence, the pdf of BG can be written as

$$f(x) = \frac{\theta e^{\gamma x} e^{-\frac{\beta\theta}{\gamma}(e^{\gamma x}-1)}}{B(\alpha, \beta)} [1 - e^{-\frac{\theta}{\gamma}(e^{\gamma x}-1)}]^{\alpha-1} \quad (5)$$

and we use the notation  $X \sim BG(\theta, \gamma, \alpha, \beta)$ .

**Theorem 1.** *Let  $f(x)$  be the pdf of the BG distribution. The limiting behavior of  $f$  for different values of its parameters is given below:*

- i. *If  $\alpha = 1$  then  $\lim_{x \rightarrow 0^+} f(x) = \theta\beta$*

- ii. If  $\alpha > 1$  then  $\lim_{x \rightarrow 0^+} f(x) = 0$ .
- iii. If  $0 < \alpha < 1$  then  $\lim_{x \rightarrow 0^+} f(x) = \infty$
- iv.  $\lim_{x \rightarrow \infty} f(x) = 0$

**Proof.** The proof of parts (i)-(iii) are obvious. For part (iv), we have

$$0 \leq [1 - e^{-\frac{\theta}{\gamma}(e^{\gamma x} - 1)}]^{\alpha-1} < 1 \Rightarrow 0 < f(x) < \frac{\theta e^{\gamma x} e^{-\frac{\beta\theta}{\gamma}(e^{\gamma x} - 1)}}{B(\alpha, \beta)}$$

It can be easily shown that

$$\lim_{x \rightarrow \infty} \theta e^{\gamma x} e^{-\frac{\beta\theta}{\gamma}(e^{\gamma x} - 1)} = 0.$$

and the proof is completed.  $\square$

The hrf of BG distribution is given by

$$h(x) = \frac{\theta e^{\gamma x} e^{-\frac{\beta\theta}{\gamma}(e^{\gamma x} - 1)}}{B(\alpha, \beta) - B_{G(x)}(\alpha, \beta)} [1 - e^{-\frac{\theta}{\gamma}(e^{\gamma x} - 1)}]^{\alpha-1} \quad (6)$$

Recently, it is observed (Gupta & Gupta 2007) that the reversed hrf plays an important role in the reliability analysis. The reversed hrf of the  $BG(\theta, \gamma, \alpha, \beta)$  is

$$r(x) = \frac{\theta e^{\gamma x} e^{-\frac{\beta\theta}{\gamma}(e^{\gamma x} - 1)}}{B_{G(x)}(\alpha, \beta)} [1 - e^{-\frac{\theta}{\gamma}(e^{\gamma x} - 1)}]^{\alpha-1} \quad (7)$$

Plots of pdf and hrf function of the BG distribution for different values of its parameters are given in Figure 1 and Figure 2, respectively.

Some well-known distributions are special cases of the BG distribution:

1. If  $\alpha = 1, \beta = 1, \gamma \rightarrow 0$ , then we get the E distribution.
2. If  $\beta = 1, \gamma \rightarrow 0$ , then we get the GE distribution which is introduced by Gupta & Kundu (1999)
3. If  $\beta = 1$ , then we get the GG distribution which is introduced by El-Gohary & Al-Otaibi (2013).
4. If  $\alpha = 1, \beta = 1$ , then we get the G distribution.
5. If  $\gamma \rightarrow 0$ , then we get the BE which is introduced by Nadarajah & Kotz (2006).

If the random variable  $X$  has BG distribution, then it has the following properties:

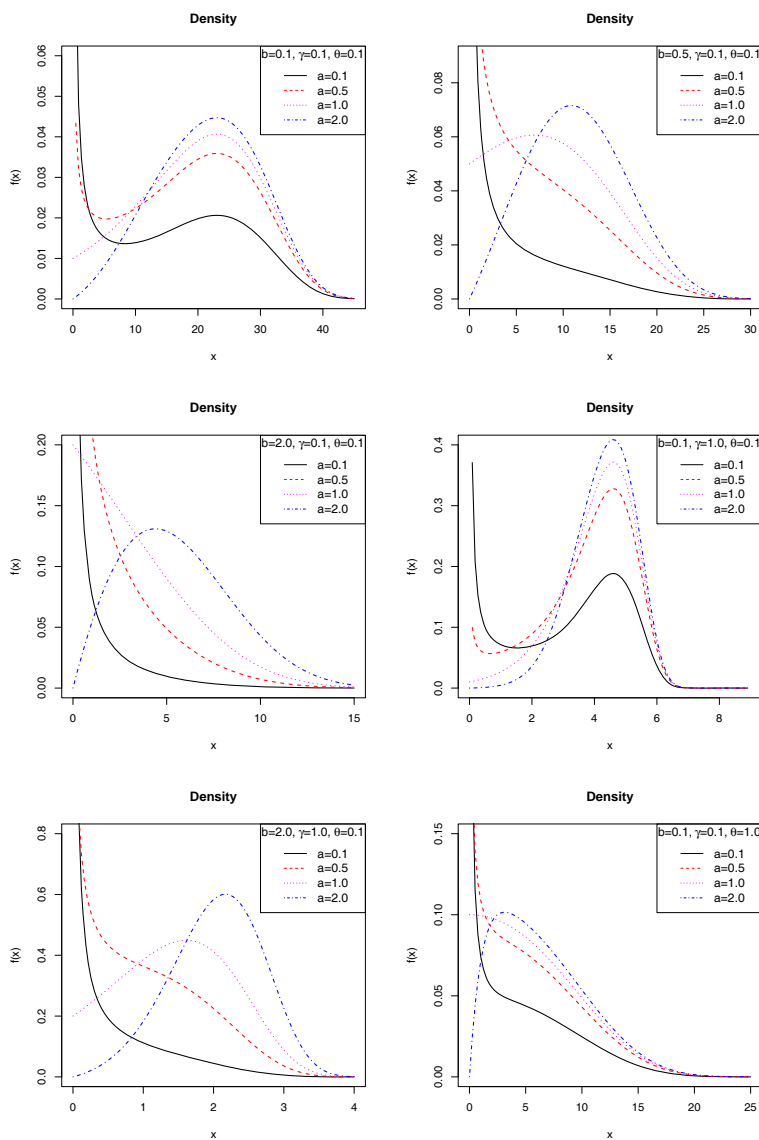


FIGURE 1: Plots of density functions of BG for different values of parameters.

1. the random variable

$$Y = 1 - e^{-\frac{\theta}{\gamma}(e^{\gamma X} - 1)}$$

satisfies the Beta distribution with parameters  $\alpha$  and  $\beta$ . Therefore,

$$T = \frac{\theta}{\gamma}(e^{\gamma X} - 1)$$

satisfies the BE distribution with parameters 1,  $\alpha$  and  $\beta$  ( $BE(1, \alpha, \beta)$ ).

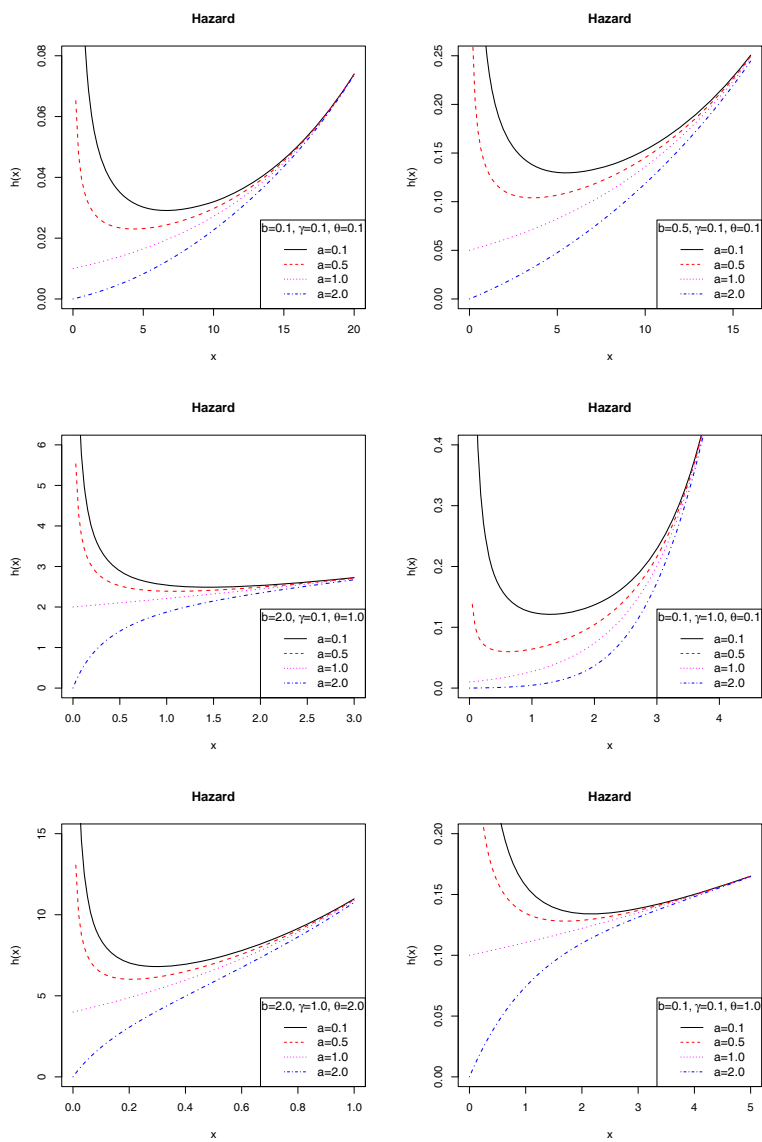


FIGURE 2: Plots of hrf of BG for different values of parameters.

2. If  $\alpha = i$  and  $\beta = n - i$ , where  $i$  and  $n$  are positive integer values, then the  $f(x)$  is the density function of  $i$ th order statistic of G distribution.
3. If  $V$  follows Beta distribution with parameters  $\alpha$  and  $\beta$ , then

$$X = G^{-1}(V) = \frac{1}{\gamma} \log \left( 1 - \frac{\gamma}{\theta} \log(1 - V) \right)$$

follows BG distribution. This result helps in simulating data from the BG distribution.

For checking the consistency of the simulating data set form BG distribution, the histogram for a generated data set with size 100 and the exact BG density with parameters  $\theta = 0.1$  and  $\gamma = 1.0$ ,  $\alpha = 0.1$ , and  $\beta = 0.1$ , are displayed in Figure 3 (left). Also, the empirical distribution function and the exact distribution function is given in Figure 3 (right).

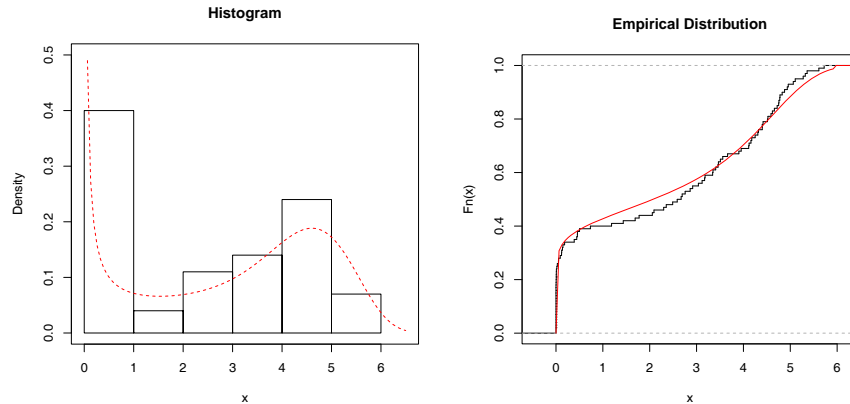


FIGURE 3: The histogram of a generated data set with size 100 and the exact GPS density (left) and the empirical distribution function and exact distribution function (right).

### 3. Some Extensions and Properties

Here, we present some representations of the cdf, pdf,  $k$ th moment and moment generating function of BG distribution. Also, we provide expressions for the order statistics, Shannon entropy and quantile measure of this distribution. The mathematical relation given below will be useful in this section. If  $\beta$  is a positive real non-integer and  $|z| < 1$ , then ( Gradshteyn & Ryzhik 2007, p. 25)

$$(1 - z)^{\beta-1} = \sum_{j=0}^{\infty} w_j z^j$$

and if  $\beta$  is a positive real integer, then the upper of the this summation stops at  $\beta - 1$ , where

$$w_j = \frac{(-1)^j \Gamma(\beta)}{\Gamma(\beta - j) \Gamma(j + 1)}$$

**Proposition 1.** We can express (3) as a mixture of distribution function of GG distributions as follows:

$$F(x) = \sum_{j=0}^{\infty} p_j [G(x)]^{\alpha+j} = \sum_{j=0}^{\infty} p_j G_j(x)$$

where  $p_j = \frac{(-1)^j \Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta-j) \Gamma(j+1) (\alpha+j)}$  and  $G_j(x) = (G(x))^{\alpha+j}$  is the distribution function of a random variable which has a GG distribution with parameters  $\theta$ ,  $\gamma$ , and  $\alpha + j$ . Also, we can write

$$\begin{aligned} G(x)^{\alpha+j} &= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha+j}{k} (1-G(x))^k \\ &= \sum_{r=0}^{\infty} \sum_{k=r}^{\infty} (-1)^{k+r} \binom{\alpha+j}{k} \binom{k}{r} G(x)^r \end{aligned} \quad (8)$$

and

$$F(x) = \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=r}^{\infty} p_j (-1)^{k+r} \binom{\alpha+j}{k} \binom{k}{r} G(x)^r = \sum_{r=0}^{\infty} b_r G(x)^r \quad (9)$$

where  $b_r = \sum_{j=0}^{\infty} \sum_{k=r}^{\infty} p_j (-1)^{k+r} \binom{\alpha+j}{k} \binom{k}{r}$

**Proposition 2.** We can express (5) as a mixture of density functions of a GG distribution as follows:

$$f(x) = \sum_{j=0}^{\infty} p_j (\alpha+j) g(x) [G(x)]^{\alpha+j-1} = \sum_{j=0}^{\infty} p_j g_j(x)$$

where  $g_j(x)$  is a density function of a random variable with a GG distribution and parameters  $\theta$ ,  $\gamma$ , and  $\alpha + j$ .

**Proposition 3.** The cdf can be expressed in terms of the hypergeometric function and the incomplete beta function ratio (see Cordeiro & Nadarajah 2011) in the following way:

$$F(x) = \frac{(G(x))^\alpha}{\alpha B(\alpha, \beta)} {}_2F_1(\alpha, 1-\beta; \alpha+1; G(x))$$

where  ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{((a)_k (b)_k)}{((c)_k k!)} z^k$  and  $(a)_k = a(a+1) \cdots (a+k-1)$

**Proposition 4.** The  $k$ th moment of the BG distribution can be expressed as a mixture of the  $k$ th moment of GG distributions as follows:

$$E(X^k) = \int_0^\infty x^k \sum_{j=0}^{\infty} p_j (\alpha+j) g(x) [G(x)]^{\alpha+j-1} = \sum_{j=0}^{\infty} p_j E(X_j^k) \quad (10)$$



where

$$E[X_j^k] = u_{jk} \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \binom{\alpha + j - 1}{i} \frac{(-1)^{i+r}}{\Gamma(r+1)} e^{\frac{\theta}{\gamma}(i+1)} \left[\frac{\theta}{\gamma}(i+1)\right]^r \left[\frac{-1}{\gamma(k+1)}\right]^{s+1}$$

$u_{jk} = (\alpha + j)\theta\Gamma(k + 1)$  and  $g_j(x)$  is the density function of a random variable  $X_j$  which has a GG distribution with parameters  $\theta$ ,  $\gamma$ , and  $\alpha + j$ .

**Proposition 5.** *The moment generating function of the BG distribution can be expressed as a mixture of moment generating function of GG distributions as follows:*

$$M_X(t) = \int_0^{\infty} e^{tx} \sum_{j=0}^{\infty} p_j(\alpha + j)g(x)[G(x)]^{\alpha+j-1} = \sum_{j=0}^{\infty} p_j M_{X_j}(t) \tag{11}$$

where

$$M_{X_j}(t) = \frac{(\alpha + j)\theta}{\gamma} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i \binom{\alpha + j - 1}{i} \binom{\frac{t}{\gamma}}{k} \frac{\Gamma(k + 1)}{\left[\frac{(i+1)\theta}{\gamma}\right]^{k+1}}$$

and  $g_j(x)$  is the density function of a random variable  $X_j$  which has a GG distribution with parameters  $\theta$ ,  $\gamma$ , and  $\alpha + j$ .

### 3.1. Order Statistics

Moments of order statistics play an important role in quality control testing and reliability. For example, if the reliability of an item is high, the duration of an all items fail life test can be too expensive in both time and money.

Therefore, a practitioner needs to predict the failure of future items based on the times of few early failures. These predictions are often based on moments of order statistics.

Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from  $BG(\theta, \gamma, \alpha, \beta)$ . Then the pdf and cdf of the  $i$ th order statistic, say  $X_{i:n}$ , are given by

$$f_{i:n}(x) = \frac{1}{B(i, n - i + 1)} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} f(x) F^{i+m-1}(x) \tag{12}$$

and

$$F_{i:n}(x) = \int_0^x f_{i:n}(t) dt = \frac{1}{B(i, n - i + 1)} \sum_{m=0}^{n-i} \frac{(-1)^m}{m+i} \binom{n-i}{m} F^{i+m}(x) \tag{13}$$

respectively, where  $F^{i+m}(x) = \left(\sum_{r=0}^{\infty} b_r G(x)^r\right)^{i+m}$ . Here and henceforth, we use an equation by Gradshteyn & Ryzhik (2007), page 17, for a power series raised to a positive integer  $n$

$$\left(\sum_{r=0}^{\infty} b_r u^r\right)^n = \sum_{r=0}^{\infty} c_{n,r} u^r \tag{14}$$

where the coefficients  $c_{n,r}$  (for  $r = 1, 2, \dots$ ) are easily determined from the recurrence equation

$$c_{n,r} = (r b_0)^{-1} \sum_{m=1}^r [m(n+1) - r] b_m c_{n,r-m}, \quad (15)$$

where  $c_{n,0} = b_0^n$ . The coefficient  $c_{n,r}$  can be calculated from  $c_{n,0}, \dots, c_{n,r-1}$  and hence from the quantities  $b_0, \dots, b_r$ .

The equations (12) and (13) can be written as

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \sum_{r=1}^{\infty} \frac{1}{m+i} (-1)^m r c_{i+m,r} g(x) G^{r-1}(x)$$

and

$$F_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \sum_{r=0}^{\infty} \frac{1}{m+i} (-1)^m c_{i+m,r} G^r(x)$$

Therefore, the  $s$ th moments of  $X_{i:n}$  follows as

$$\begin{aligned} E[X_{i:n}^s] &= \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \sum_{r=1}^{\infty} \frac{1}{m+i} (-1)^m r c_{i+m,r} \int_0^{+\infty} t^s g(t) G^{r-1}(t) dt \\ &= \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \sum_{r=1}^{\infty} \frac{1}{m+i} (-1)^m r c_{i+m,r} \\ &\quad \times \theta \Gamma(s+1) \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \binom{r-1}{i_1} \frac{(-1)^{i_1+i_2}}{\Gamma(i_2+1)} e^{\frac{\theta}{\gamma}(i_1+1)} \left[ \frac{\theta(i_1+1)}{\gamma} \right]^{i_2} \left[ \frac{-1}{\gamma(i_2+1)} \right]^{s+1} \end{aligned}$$

### 3.2. Quantile Measure

The quantile function of the BG distribution is given by

$$Q(u) = \frac{1}{\gamma} \log(1 - \frac{\gamma}{\theta} \log(1 - Q_{\alpha,\beta}(u)))$$

where  $Q_{\alpha,\beta}(u)$  is the  $u$ th quantile of Beta distribution with parameters  $\alpha$  and  $\beta$ . The effects of the shape parameters  $\alpha$  and  $\beta$  on the skewness and kurtosis can be considered based on quantile measures. The Bowley skewness (Kenney & Keeping 1962) is one of the earliest skewness measures defined by

$$\mathcal{B} = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

This adds robustness to the measure, since only the middle two quartiles are considered and the other two quartiles are ignored. The Moors kurtosis (Moors 1988) is defined as

$$\mathcal{M} = \frac{Q(\frac{3}{8}) - Q(\frac{1}{8}) + Q(\frac{7}{8}) - Q(\frac{5}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}$$

Clearly,  $\mathcal{M} > 0$  and there is a good concordance with the classical kurtosis measures for some distributions. These measures are less sensitive to outliers and they exist even for distributions without moments. For the standard normal distribution, these measures are 0 (Bowley) and 1.2331 (Moors).

In Figures 4 and 5, we plot the measures  $\mathcal{B}$  and  $\mathcal{M}$  for some parameter values. These plots indicate that both measures  $\mathcal{B}$  and  $\mathcal{M}$  depend on all shape parameters.

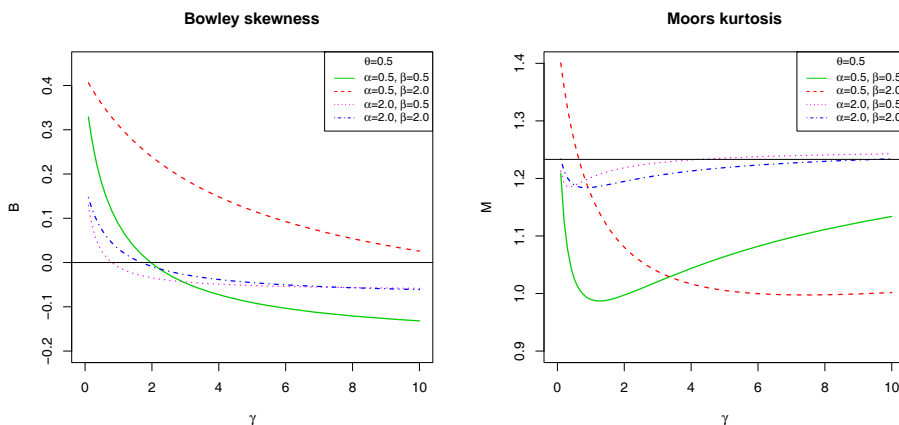


FIGURE 4: The Bowley skewness (left) and Moors kurtosis (right) coefficients for the BG distribution as a function of  $\gamma$ .

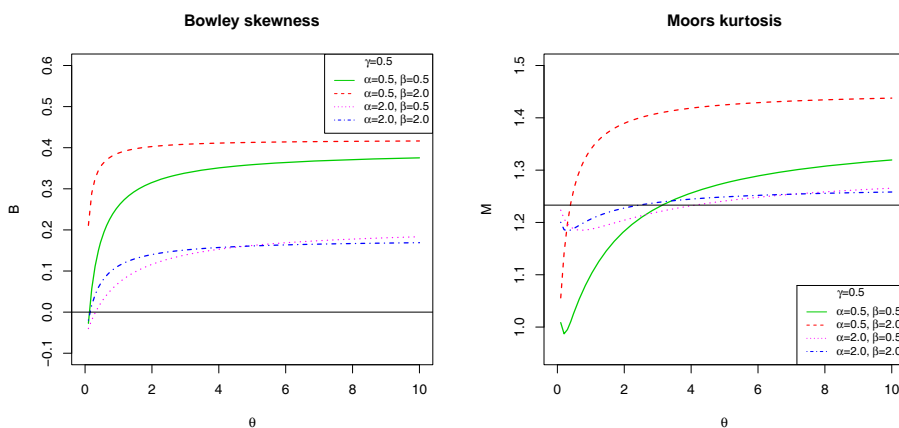


FIGURE 5: The Bowley skewness (left) and Moors kurtosis (right) coefficients for the BG distribution as a function of  $\theta$ .

### 3.3. Shannon and Rényi Entropy

If  $X$  is a non-negative continuous random variable with pdf  $f(x)$ , then Shannon's entropy of  $X$  is defined by Shannon (1948) as

$$H(f) = E[-\log f(X)] = - \int_0^{+\infty} f(x) \log(f(x)) dx$$

and this is usually referred to as the continuous entropy (or differential entropy). An explicit expression of Shannon entropy for BG distribution is obtained as

$$\begin{aligned} H(f) &= \log\left(\frac{B(\alpha, \beta)}{\theta}\right) - \frac{\theta\beta}{\gamma} - \gamma E(X) \\ &+ \frac{\theta\beta}{\gamma} M_X(\gamma) + (\alpha - 1)[\psi(\alpha + \beta) - \psi(\alpha)] \end{aligned} \quad (16)$$

where  $\psi(\cdot)$  is a digamma function.

The Rényi entropy of order  $\lambda$  is defined as

$$H_\lambda(f) = \frac{1}{1-\lambda} \log \int_{-\infty}^{+\infty} f^\lambda(x) dx, \quad \forall \lambda > 0 \quad (\lambda \neq 1) \quad (17)$$

where

$$H(X) = \lim_{\lambda \rightarrow 1} H_\lambda(X) = - \int_{-\infty}^{+\infty} f(x) \log f(x) dx$$

is the Shannon entropy, if both integrals exist. Finally, an explicit expression of Rényi entropy for BG distribution is obtained as

$$\begin{aligned} H_\lambda(f) &= -\log(\theta) + \frac{\lambda}{\lambda-1} \log(B(\alpha, \beta)) + \frac{1}{1-\lambda} \left[ \log(B(\alpha, (\beta-1)\lambda + 1)) \right. \\ &\left. + \log \left( \sum_{j=1}^{\infty} \sum_{k=0}^j (-1)^k \binom{\lambda-1}{j} \binom{j}{k} \left(\frac{\gamma}{\theta}\right)^j \frac{\Gamma(j+1)}{(j+1)^{k-1+(\beta-1)\lambda}} \right) \right] \end{aligned} \quad (18)$$

## 4. Estimation and Inference

In this section, we determine the maximum-likelihood estimates (MLEs) of the parameters of the BG distribution from a complete sample. Consider  $X_1, \dots, X_n$  is a random sample from BG distribution. The log-likelihood function for the vector of parameters  $\Theta = (\theta, \gamma, \alpha, \beta)$  can be written as

$$\begin{aligned} l_n &= l_n(\Theta) \\ &= n \log(\theta) - n \log(B(\alpha, \beta)) + n\gamma\bar{x} - \beta\theta \sum_{i=1}^n \log(t_i) \\ &+ (\alpha - 1) \sum_{i=1}^n \log(1 - t_i^\theta) \end{aligned} \quad (19)$$

where  $\bar{x} = n^{-1} \sum_{i=1}^n x_i$  and  $t_i = e^{-\frac{1}{\gamma}(e^{\gamma x_i} - 1)}$ . The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (19). The components of the score vector  $U(\Theta)$  are given by

$$\begin{aligned}
 U_\alpha(\Theta) &= \frac{\partial l_n}{\partial \alpha} = n\psi(\alpha + \beta) - n\psi(\alpha) + \sum_{i=1}^n \log(1 - t_i^\theta) \\
 U_\beta(\Theta) &= \frac{\partial l_n}{\partial \beta} = n\psi(\alpha + \beta) - n\psi(\beta) - \theta \sum_{i=1}^n \log(t_i) \\
 U_\theta(\Theta) &= \frac{\partial l_n}{\partial \theta} = \frac{n}{\theta} - \beta \sum_{i=1}^n \log(t_i) - (\alpha - 1) \sum_{i=1}^n \frac{t_i^\theta \log(t_i)}{1 - t_i^\theta} \\
 U_\gamma(\Theta) &= \frac{\partial l_n}{\partial \gamma} = n\bar{x} - \beta\theta \sum_{i=1}^n d_i - \theta(\alpha - 1) \sum_{i=1}^n \frac{d_i t_i^\theta}{1 - t_i^\theta}
 \end{aligned}$$

where  $\psi(\cdot)$  is the digamma function, and

$$d_i = \frac{\partial \log(t_i)}{\partial \gamma} = \frac{1}{\gamma}(-\log(t_i) + \gamma x_i \log(t_i) - x_i)$$

For interval estimation and hypothesis tests on the model parameters, we require the observed information matrix. The  $4 \times 4$  unit observed information matrix  $J = J_n(\Theta)$  is obtained as

$$J = - \begin{bmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\theta} & J_{\alpha\gamma} \\ J_{\alpha\beta} & J_{\beta\beta} & J_{\beta\theta} & J_{\beta\gamma} \\ J_{\alpha\theta} & J_{\beta\theta} & J_{\theta\theta} & J_{\theta\gamma} \\ J_{\alpha\gamma} & J_{\beta\gamma} & J_{\theta\gamma} & J_{\gamma\gamma} \end{bmatrix}$$

where the expressions for the elements of  $J$  are

$$\begin{aligned}
 J_{\alpha\alpha} &= \frac{\partial^2 l_n}{\partial \alpha^2} = n\psi'(\alpha + \beta) - n\psi'(\alpha), & J_{\alpha\beta} &= \frac{\partial^2 l_n}{\partial \alpha \partial \beta} = \frac{\partial^2 l_n}{\partial \beta \partial \alpha} = n\psi'(\alpha + \beta) \\
 J_{\alpha\theta} &= \frac{\partial^2 l_n}{\partial \alpha \partial \theta} = \frac{\partial^2 l_n}{\partial \theta \partial \alpha} = \sum_{i=1}^n \frac{t_i^\theta \log(t_i)}{1 - t_i^\theta}, & J_{\alpha\gamma} &= \frac{\partial^2 l_n}{\partial \alpha \partial \gamma} = \frac{\partial^2 l_n}{\partial \gamma \partial \alpha} = -\theta \sum_{i=1}^n \frac{d_i t_i^\theta}{1 - t_i^\theta}, \\
 J_{\beta\beta} &= \frac{\partial^2 l_n}{\partial \beta^2} = n\psi'(\alpha + \beta) - n\psi'(\beta), & J_{\beta\theta} &= \frac{\partial^2 l_n}{\partial \beta \partial \theta} = \frac{\partial^2 l_n}{\partial \theta \partial \beta} = -\sum_{i=1}^n \log(t_i) \\
 J_{\beta\gamma} &= \frac{\partial^2 l_n}{\partial \beta \partial \gamma} = \frac{\partial^2 l_n}{\partial \gamma \partial \beta} = -\theta \sum_{i=1}^n d_i, & J_{\theta\theta} &= \frac{\partial^2 l_n}{\partial \theta^2} = -\frac{n}{\theta^2} + \theta(\alpha - 1) \sum_{i=1}^n \frac{t_i^\theta (\log(t_i))^2}{(1 - t_i^\theta)^2} \\
 J_{\theta\gamma} &= \frac{\partial^2 l_n}{\partial \theta \partial \gamma} = \frac{\partial^2 l_n}{\partial \gamma \partial \theta} = -\beta \sum_{i=1}^n d_i - (\alpha - 1) \sum_{i=1}^n \frac{d_i t_i^\theta}{1 - t_i^\theta} \left( \theta \log(t_i) + 1 + \frac{\theta t_i^\theta \log(t_i)}{1 - t_i^\theta} \right) \\
 J_{\gamma\gamma} &= \frac{\partial^2 l_n}{\partial \gamma^2} = -\beta\theta \sum_{i=1}^n q_i - \theta(\alpha - 1) \sum_{i=1}^n \frac{t_i^\theta}{1 - t_i^\theta} (q_i + \theta d_i^2) - \theta^2(\alpha - 1) \sum_{i=1}^n \frac{d_i^2 t_i^{2\theta}}{1 - t_i^\theta}
 \end{aligned}$$

where  $q_i = \frac{\partial d_i}{\partial \gamma} = d_i(x_i - \frac{2}{\gamma}) + \frac{x_i}{\gamma} \log(t_i)$ .

## 5. Simulation Studies

In this section, we performed a simulation study in order to investigate the proposed estimator of parameters based on the proposed MLE method. We generate 10,000 data sets with size  $n$  from the BG distribution with parameters  $a$ ,  $b$ ,  $\theta$ , and  $\gamma$ , and compute the MLE's of the parameters. We assess the accuracy of the approximation of the standard error of the MLE's determined through the Fisher information matrix and variance of the estimated parameters. Table 1 show the results for the BG distribution. From these results, we can conclude that:

- i. the differences between the average estimates and the true values are almost small,
- ii. the MLE's converge to true value in all cases when the sample size increases,
- iii. the standard errors of the MLEs decrease when the sample size increases.

From these simulations, we can conclude that estimation of the parameters using the MLE are satisfactory.

## 6. Application of BG to a Real Data Set

In this section, we perform an application to real data and demonstrate the superiority of BG distribution as compared to some of its sub-models. The data have been obtained from Aarset (1987), and widely reported in some literatures (for example see Silva & Cordeiro 2010). It represents the lifetimes of 50 devices, and also, possess a bathtub-shaped failure rate property. The numerical evaluations were implemented using R software (`nlminb` function).

Based on some goodness-of-fit measures, the performance of the BG distribution is quantified and compared with others due to five literature distributions: E, GE, BE, G, and GG, distributions. The MLE's of the unknown parameters (standard errors in parentheses) for these distributions are given in Table 2. Also, the values of the log-likelihood functions ( $-\log(L)$ ), the Kolmogorov Smirnov (KS) test statistic with its p-value, the statistics AIC (Akaike Information Criterion), the statistics AICC (Akaike Information Criterion with correction) and BIC (Bayesian Information Criterion) are calculated for the six distributions in order to verify which distribution fits better to these data. All the computations were done using the R software.

The BG distribution yields the highest value of the log-likelihood function and smallest values of the AIC, AICC and BIC statistics. From the values of these statistics, we can conclude that the BG model is better than the other distributions to fit these data. The plots of the densities (together with the data histogram) and cumulative distribution functions (with empirical distribution function) are given in Figure 6. It is evident that the BG model provides a better fit than the other models. In particular, the histogram of data shows that the BG model provides an excellent fit to these data.

TABLE 1: The simulated MLE's and mean of the standard errors for BG distribution based on information matrix and variance of MLE's.

| $n$ | parameters |         |          |          | estimations    |               |                | simulated      |                      |                     | Information matrix   |                      |                      |                     |                      |                      |
|-----|------------|---------|----------|----------|----------------|---------------|----------------|----------------|----------------------|---------------------|----------------------|----------------------|----------------------|---------------------|----------------------|----------------------|
|     | $\alpha$   | $\beta$ | $\theta$ | $\gamma$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\gamma}$ | $s.e.(\hat{\alpha})$ | $s.e.(\hat{\beta})$ | $s.e.(\hat{\theta})$ | $s.e.(\hat{\gamma})$ | $s.e.(\hat{\alpha})$ | $s.e.(\hat{\beta})$ | $s.e.(\hat{\theta})$ | $s.e.(\hat{\gamma})$ |
| 30  | 0.5        | 0.5     | 0.5      | 0.5      | 0.515          | 0.739         | 0.555          | 0.616          | 0.171                | 0.646               | 0.818                | 0.291                | 0.111                | 0.183               | 0.117                | 0.072                |
|     | 0.5        | 0.5     | 0.5      | 1.0      | 0.516          | 0.734         | 0.605          | 1.164          | 0.180                | 0.609               | 0.800                | 0.472                | 0.123                | 0.168               | 0.137                | 0.089                |
|     | 0.5        | 0.5     | 1.0      | 1.0      | 0.517          | 0.858         | 1.195          | 1.245          | 0.166                | 0.604               | 0.845                | 0.583                | 0.117                | 0.191               | 0.154                | 0.132                |
|     | 0.5        | 2.0     | 0.5      | 1.0      | 0.509          | 1.741         | 0.902          | 1.368          | 0.144                | 0.824               | 0.388                | 0.839                | 0.111                | 0.306               | 0.129                | 0.222                |
|     | 0.5        | 2.0     | 1.0      | 1.0      | 0.497          | 1.830         | 1.441          | 1.478          | 0.130                | 0.877               | 0.575                | 1.100                | 0.105                | 0.354               | 0.217                | 0.352                |
| 50  | 2.0        | 2.0     | 0.5      | 0.5      | 2.068          | 1.916         | 0.811          | 0.826          | 0.972                | 0.744               | 0.881                | 0.544                | 0.333                | 0.147               | 0.217                | 0.094                |
|     | 0.5        | 0.5     | 0.5      | 0.5      | 0.506          | 0.612         | 0.526          | 0.567          | 0.125                | 0.600               | 0.900                | 0.214                | 0.088                | 0.143               | 0.111                | 0.054                |
|     | 0.5        | 0.5     | 0.5      | 1.0      | 0.509          | 0.729         | 0.556          | 1.092          | 0.131                | 0.633               | 0.864                | 0.342                | 0.085                | 0.137               | 0.096                | 0.077                |
|     | 0.5        | 0.5     | 1.0      | 1.0      | 0.512          | 0.829         | 1.104          | 1.149          | 0.126                | 0.598               | 0.941                | 0.421                | 0.086                | 0.147               | 0.123                | 0.104                |
|     | 0.5        | 2.0     | 0.5      | 1.0      | 0.507          | 1.896         | 0.888          | 1.183          | 0.111                | 0.795               | 0.345                | 0.631                | 0.081                | 0.244               | 0.082                | 0.179                |
| 100 | 0.5        | 2.0     | 1.0      | 1.0      | 0.501          | 1.932         | 1.202          | 1.202          | 0.101                | 0.856               | 0.515                | 0.873                | 0.080                | 0.288               | 0.147                | 0.282                |
|     | 2.0        | 2.0     | 0.5      | 0.5      | 2.201          | 1.991         | 0.678          | 0.659          | 0.845                | 0.734               | 0.856                | 0.350                | 0.372                | 0.114               | 0.150                | 0.060                |
|     | 0.5        | 0.5     | 0.5      | 0.5      | 0.498          | 0.520         | 0.511          | 0.532          | 0.086                | 0.617               | 0.948                | 0.149                | 0.060                | 0.101               | 0.076                | 0.039                |
|     | 0.5        | 0.5     | 0.5      | 1.0      | 0.498          | 0.531         | 0.537          | 1.039          | 0.090                | 0.655               | 0.917                | 0.241                | 0.061                | 0.098               | 0.073                | 0.054                |
|     | 0.5        | 0.5     | 1.0      | 1.0      | 0.503          | 0.540         | 0.984          | 1.088          | 0.086                | 0.634               | 1.033                | 0.295                | 0.059                | 0.107               | 0.085                | 0.076                |
|     | 0.5        | 2.0     | 0.5      | 1.0      | 0.504          | 1.929         | 0.492          | 1.062          | 0.076                | 0.722               | 0.349                | 0.425                | 0.055                | 0.172               | 0.046                | 0.133                |
|     | 0.5        | 2.0     | 1.0      | 1.0      | 0.502          | 1.917         | 1.259          | 1.163          | 0.072                | 0.814               | 0.479                | 0.608                | 0.055                | 0.212               | 0.084                | 0.209                |
|     | 2.0        | 2.0     | 0.5      | 0.5      | 2.195          | 2.062         | 0.550          | 0.590          | 0.648                | 0.738               | 0.791                | 0.229                | 0.261                | 0.083               | 0.060                | 0.044                |

For this data set, we perform the Likelihood Ratio Test (LRT) for testing the following hypotheses:

1.  $H_0$ : E distribution vs.  $H_1$ : BG distribution
2.  $H_0$ : GE distribution vs.  $H_1$ : BG distribution
3.  $H_0$ : BE distribution vs.  $H_1$ : BG distribution
4.  $H_0$ : G distribution vs.  $H_1$ : BG distribution, or equivalently  $H_0: (\alpha, \beta) = (1, 1)$  vs.  $H_1: (\alpha, \beta) \neq (1, 1)$
5.  $H_0$ : GG distribution vs.  $H_1$ : BG distribution, or equivalently  $H_0: \beta = 1$  vs.  $H_1: \beta \neq 1$ .

TABLE 2: Parameter estimates (with std.), K-S statistic,  $p$ -value for K-S, AIC, AICC, BIC, LRT statistic and  $p$ -value of LRT for the data set.

| Distribution   | E        | GE       | BE       | G        | GG       | BG       |
|----------------|----------|----------|----------|----------|----------|----------|
| $\hat{\alpha}$ | —        | 0.9021   | 0.5236   | —        | 0.2625   | 0.2158   |
| (std.)         | —        | (0.1349) | (0.1714) | —        | (0.0395) | (0.0392) |
| $\hat{\beta}$  | —        | —        | 0.0847   | —        | —        | 0.2467   |
| (std.)         | —        | —        | (0.0828) | —        | —        | (0.0448) |
| $\hat{\theta}$ | 0.0219   | 0.0212   | 0.2352   | 0.0097   | 0.0001   | 0.0003   |
| (std.)         | (0.0031) | (0.0036) | (0.2111) | (0.0029) | (0.0001) | (0.0001) |
| $\hat{\gamma}$ | —        | —        | —        | 0.0203   | 0.0828   | 0.0882   |
| (std.)         | —        | —        | —        | (0.0058) | (0.0031) | (0.0030) |
| $-\log(L)$     | 241.0896 | 240.3855 | 238.1201 | 235.3308 | 222.2441 | 220.6714 |
| K-S            | 0.1911   | 0.1940   | 0.1902   | 0.1696   | 0.1409   | 0.1322   |
| p-value (K-S)  | 0.0519   | 0.0514   | 0.0538   | 0.1123   | 0.2739   | 0.3456   |
| AIC            | 484.1792 | 484.7710 | 482.2400 | 474.6617 | 450.4881 | 449.3437 |
| AICC           | 484.2625 | 485.0264 | 482.7617 | 475.1834 | 451.0099 | 450.2326 |
| BIC            | 486.0912 | 488.5951 | 487.9760 | 482.3977 | 456.2242 | 456.9918 |
| LRT            | 40.8355  | 39.4273  | 34.8962  | 29.3179  | 3.1444   | —        |
| p-value (LRT)  | 0.0000   | 0.0000   | 0.0000   | 0.0001   | 0.0762   | —        |

Values of the LRT statistic and its corresponding p-value for each hypotheses are given in Table 2. From these results, we can conclude that the null hypotheses are rejected in all situations, and therefore, the BG distribution is an adequate model.

**Note 1.** El-Gohary & Al-Otaibi (2013) found the following estimations for the parameters of th GG distribution:

$$\hat{a} = 0.421, \quad \hat{\theta} = 0.00143, \quad \hat{\gamma} = 0.044.$$

Based on these estimations, the log-likelihood function is equal to  $-224.1274$ . But we found the following estimations for the parameters of GG distribution:

$$\hat{a} = 0.2625, \quad \hat{\theta} = 0.0001, \quad \hat{\gamma} = 0.0828.$$

Based on these estimations, the log-likelihood function is equal to  $-222.2441$ . Therefore, the estimations of El-Gohary & Al-Otaibi (2013) for GG distribution is not the MLE.



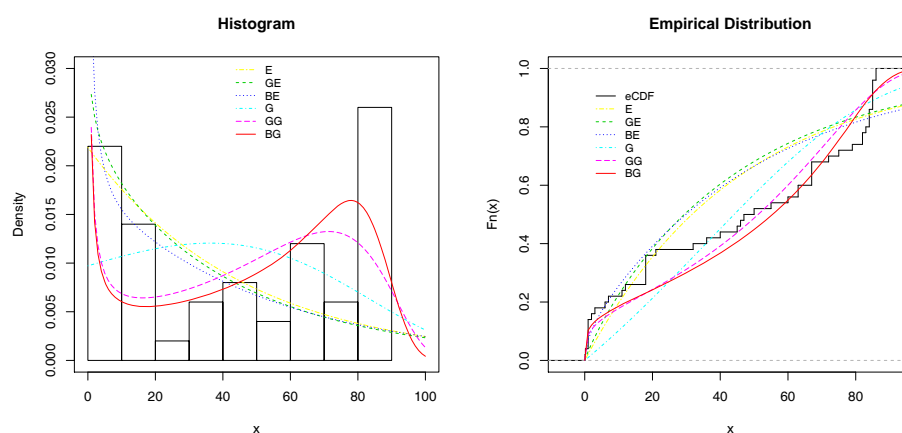


FIGURE 6: Plots (density and distribution) of fitted E, GE, BE, G, GG and BG distributions for the data set.

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