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Optimization of Spearman's Rho

Optimización de Rho de Spearman

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Abstract

This paper proposes an approximation method to achieve optimum possible values of Spearman's rho for a special class of copulas.

***Key words*:** Approximation, Copula, Kendall's Tau, Spearman's Rho.

Resumen

El artículo propone un método de aproximación para alcanzar los valores óptimos posibles del coeficiente rho de Spearman para algunas clases especiales de cópulas.

***Palabras clave*:** aproximación, cópula, tau de Kendall, rho de Spearman.

1. Introduction

A study of dependence by using copulas has been getting more attention in the areas of finance, actuarial science, biomedical studies and engineering because a copula does not require a normal distribution and independent, identical distribution assumptions. Furthermore, the invariance property of copula has been attractive in the area of finance. But most copulas including the Archimedean copula family are symmetric functions so that fitting these copulas to asymmetric data is not appropriate. Recently, Liebscher (2008) and Durante (2009) have studied several methods for the construction of asymmetric multivariate copulas. Amblard & Girard (2009) have proposed a generalized FGM copula family and

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discussed the range of Spearman's rho. Rodríguez-Lallena & Úbeda-Flores (2004) and Kim, Sungur, Choi & Heo (2011) investigated new classes of bivariate copulas and studied different measures of association.

The two most important non-parametric measures of association between two random variables are Spearman's rho (ρ) and Kendall's tau (τ). In this paper we study bivariate copulas with copula densities of the form $c(u,v) = 1+g(u)h(v)$. We approximate these copulas through a two-parameter family of copulas, in general asymmetric in nature, and show that the Spearman's rho values of these copulas have a range of $(-\frac{3}{4}, \frac{3}{4})$. We also extend these results to more general case where $c(u,v)$ takes the form as $1 + \sum_{i=1}^{n} g_i(u)h_i(v)$.

2. Definition and Preliminary

In this section we recall some definitions and results that are necessary to understand a (bivariate) copula. A copula is a multivariate distribution function defined on $\mathbb{I}^n$, where $\mathbb{I} := [0,1]$, with uniformly distributed marginals. In this paper, we focus on bivariate (two-dimensional, $n = 2$) copulas.

Definition 1. A bivariate copula is a function $C : \mathbb{I}^2 \to \mathbb{I}$, which satisfies the following properties:

(P1) $C(0,v) \ = \ C(u,0) \ = \ 0, \qquad \forall u,v \in \mathbb{I}$

(P2) $C(1,u) \ = \ C(u,1) \ = \ u, \qquad \forall u \in \mathbb{I}$

(P3) C is 2-increasing, i.e., $\forall u_1,u_2,v_1,v_2 \in \mathbb{I}$ with $u_1 \leq u_2, v_1 \leq v_2$,

$$C(u_2,v_2) + C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) \geq 0.$$

The importance of copulas has been growing because of their applications in several fields of research. Their relevance primarily comes from Sklar's Theorem (see (Sklar 1959) and (Sklar 1973) for details): *If X and Y are two continuous random variables with joint distribution function H and marginal distribution functions F and G, respectively, then there exists a unique copula C such that $H(x,y) = C(F(x),G(y))$ for all $(x,y) \in \mathbb{R}^2$ and conversely, given a copula C and two univariate distribution functions F and G, the function H defined above is a joint distribution function with margins F and G.* Sklar's theorem clarifies the role that copulas play in the relationship between multivariate distribution functions and their univariate margins. A proof of this theorem can be found in (Schweizer & Sklar 1983).

Definition 2. Suppose X and Y are two random variables with marginal distribution functions F and G, respectively. Then *Spearman's rho* is the ordinary (Pearson) correlation coefficient of the transformed random variables $F(X)$ and $G(Y)$, while *Kendall's tau* is the difference between the probability of concordance $Pr[(X1{-}X2)(Y1{-}Y2) > 0]$ and the probability of discordance $Pr[(X1{-}X2)(Y1{-}Y2) < 0]$ for two independent pairs $(X1,Y1)$ and $(X2,Y2)$ of observations drawn from the distribution.

In terms of dependence properties, Spearman's rho is a measure of average quadrant dependence, while Kendall's tau is a measure of average likelihood ratio dependence (see Nelsen 2006, for details). If X and Y are two continuous random variables with copula C, then Spearman's rho and Kendall's tau of X and Y are given by,

$$\rho = 12 \int\int_{\mathbb{I}^2} C(u,v)\ du\ dv - 3 \tag{1}$$

$$\tau = 4 \int\int_{\mathbb{I}^2} C(u,v)\ dC(u,v) - 1 \tag{2}$$

Definition 3. A copula C is called absolutely continuous if, when considered as a joint distribution function, $C(u,v)$ has a joint density function given by $c(u,v) := \frac{\partial^2 C}{\partial u \partial v}$ and in that case $dC(u,v) = \frac{\partial^2 C}{\partial u \partial v}\ du\ dv$.

Denoting $c(u,v)-1$ as $h(u,v)$, the following theorem gives a characterization of absolutely continuous copulas (see De la Peña, Ibragimov & Sharakhmetov 2006).

Theorem 1. *A function $C : \mathbb{I}^2 \to \mathbb{I}$ is an absolutely continuous bivariate copula if and only if there exists a function $h : \mathbb{I}^2 \to \mathbb{I}$, satisfying the following conditions,*

1. *Integrability:* $\int\int_{\mathbb{I}^2} |h(x,y)| dx\ dy < \infty$,
2. *Degeneracy:* $\int_{\mathbb{I}} h(x,\xi) d\xi = \int_{\mathbb{I}} h(\xi,y) d\xi = 0\ \ \forall x,y \in \mathbb{I}$,
3. *Positive Definiteness:* $h(x,y) \geq -1\ \ \forall (x,y) \in \mathbb{I}^2$,

and such that

$$C(u,v) = \int_0^v \int_0^u (1 + h(x,y))\ dx\ dy.$$

A copula C is called *symmetric* if $C(u,v) = C(v,u)$ for all $u,v \in \mathbb{I}$, otherwise *asymmetric*.

Let us denote the independent copulas as $\Pi(u,v) := uv$.

3. Optimization of Rho

We will assume that the function h, mentioned in Theorem 1, has the form $h(u,v) = \varphi(u)\psi(v)$, where φ and ψ are continuous real-valued functions on $\mathbb{I}$. Therefore h is continuous on $\mathbb{I}^2$. For non-triviality, we assume h is not identically equal to zero. Then from Theorem 1, integrability of h becomes obvious, degeneracy of h implies $\int_{\mathbb{I}} \varphi(x)\ dx = \int_{\mathbb{I}} \psi(x)\ dx = 0$, and positive definiteness simplifies to $\min\limits_{(u,v)\in\mathbb{I}^2} \varphi(u)\psi(v) \geq -1$ and hence

$$C(u,v) = \Pi(u,v) + \int_0^u \varphi(s)\ ds \int_0^v \psi(t)\ dt$$

forms a copula.

Notice that in this special case, ρ can be simplified into the following form,

$$\rho = 12 \int_0^1 \int_0^u \varphi(s) ds\, du \int_0^1 \int_0^v \psi(t) dt\, dv.$$

This suggests that optimizing ρ is equivalent to optimizing both $\int_0^1 \int_0^u \varphi(s)ds\, du$ and $\int_0^1 \int_0^v \psi(t)dt\, dv$.

Define $G(u) := \int_0^u \varphi(s)ds$ and $H(v) := \int_0^v \psi(t)dt$. Then for some positive $\alpha_1, \alpha_2, \beta_1, \beta_2$, the optimization problems become,

$$\begin{aligned} &\text{max/min} && I_1 := \int_0^1 G(u)\, du && \text{max/min} && I_2 := \int_0^1 H(v)\, dv \\ &\text{subject to} && G(0) = G(1) = 0 && \text{subject to} && H(0) = H(1) = 0 \\ & && -\alpha_1 \leq G'(u) \leq \beta_1, && && -\alpha_2 \leq H'(v) \leq \beta_2. \end{aligned}$$

Although it apparently looks like these two optimization problems can be solved independently, they are related by the fact that $G'(u)H'(v) = \varphi(u)\psi(v) \geq -1$ for all $(u, v) \in \mathbb{I}^2$. This implies $\max\{\alpha_1\beta_2, \alpha_2\beta_1\} \leq 1$. For the optimal possibility, we choose, $\beta_2 = (\alpha_1)^{-1}$ and $\alpha_2 = (\beta_1)^{-1}$. This is evident from the fact that both I_1 and I_2 can be positive or negative, $\rho_{\max}$ will occur either if both I_1 and I_2 are maximum or if both are minimum and $\rho_{\min}$ will occur if one of I_1 and I_2 is maximum and the other is minimum.

Since G is continuous on $\mathbb{I}$, assuming $-\alpha_1 \leq G' \leq \beta_1$ whenever G' exists, geometrically, I_1 will be maximum if G has the form as in Figure 1 and will be minimum if G has the form as in Figure 2.

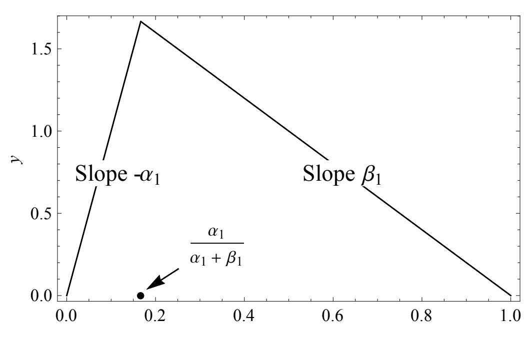

FIGURE 1: G, Maximizing I_1

One can easily prove that, in order to optimize I_1, β_1 must be equal to α_1. For convenience, now onwards we will write α for α_1, GM for the G that maximizes I_1 and Gm for the G that minimizes I_1. This suggests that if $GM(x) = -\alpha|x - 0.5| + 0.5\alpha$ and $Gm(x) = \alpha|x - 0.5| - 0.5\alpha$, then I_1 will be maximum and minimum, respectively. But in either case, G is not differentiable at $x = 0.5$, and hence φ is not continuous. To avoid this, we will approximate GM and Gm by smooth functions as follows: for arbitrarily small $\varepsilon_1 > 0$, define

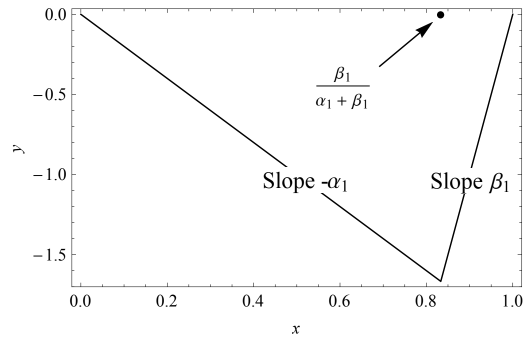

FIGURE 2: G, Minimizing I_1

$$\widetilde{GM}(x) = -\widetilde{Gm}(x) = \frac{\alpha}{2}\left(\sqrt{1+4\varepsilon_1^2} - \sqrt{(1-2x)^2+4\varepsilon_1^2}\right).$$

It is worth noting that $\sup\limits_{x\in\mathbb{I}}\left\{|\widetilde{GM}(x)-GM(x)|, |\widetilde{Gm}(x)-Gm(x)|\right\} \to 0$ as $\varepsilon_1 \to 0$ and $-\alpha \leq \widetilde{GM}'(x), \widetilde{Gm}'(x) \leq \alpha$. Figures 3 and 4 validate this fact.

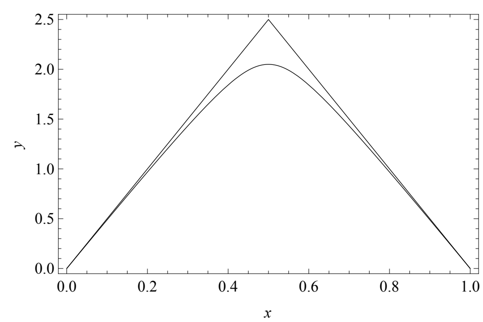

FIGURE 3: GM, $\widetilde{GM}$ for $\alpha = 5$, $\varepsilon_1 = 0.1$

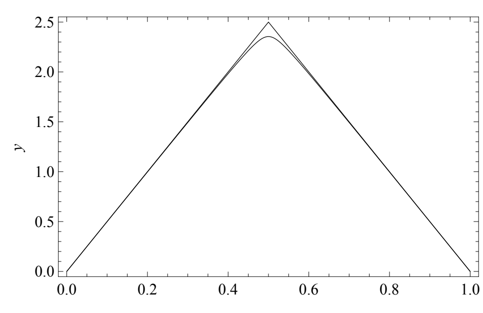

FIGURE 4: GM, $\widetilde{GM}$ for $\alpha = 5$, $\varepsilon_1 = 0.03$

We can similarly, for $\varepsilon_2 > 0$, optimize I_2 by approximating maximum and minimum of H by the following functions

$$\widetilde{HM}(x) = -\widetilde{Hm}(x) = \frac{1}{2\alpha}\left(\sqrt{1+4\varepsilon_2^2} - \sqrt{(1-2x)^2+4\varepsilon_2^2}\right).$$

Hence optimum values of ρ will be obtained by approximating h by the following functions,

$$\begin{aligned} h^{\varepsilon}_{\max}(x,y) &= \widetilde{GM}'(x)\widetilde{HM}'(y) = \widetilde{Gm}'(x)\widetilde{Hm}'(y) \\ &= \frac{(1-2x)(1-2y)}{\sqrt{(1-2x)^2+4\varepsilon_1^2}\sqrt{(1-2y)^2+4\varepsilon_2^2}}, \end{aligned}$$

$$\begin{aligned} h^{\varepsilon}_{\min}(x,y) &= \widetilde{GM}'(x)\widetilde{Hm}'(y) = \widetilde{Gm}'(x)\widetilde{HM}'(y) \\ &= -\frac{(1-2x)(1-2y)}{\sqrt{(1-2x)^2+4\varepsilon_1^2}\sqrt{(1-2y)^2+4\varepsilon_2^2}}, \end{aligned}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Notice that each of $h^{\varepsilon}_{\max}$ and $h^{\varepsilon}_{\min}$ will generate a copula as it satisfies all the hypothesis of Theorem 1 and the corresponding copulas are given by,

$$\begin{aligned} C^{\varepsilon}_{\max}(u,v) &= \Pi(u,v) \\ &+ \frac{1}{4}\left(\sqrt{1+4\varepsilon_1^2} - \sqrt{(1-2u)^2+4\varepsilon_1^2}\right)\left(\sqrt{1+4\varepsilon_2^2} - \sqrt{(1-2v)^2+4\varepsilon_2^2}\right), \\ C^{\varepsilon}_{\min}(u,v) &= \Pi(u,v) \\ &- \frac{1}{4}\left(\sqrt{1+4\varepsilon_1^2} - \sqrt{(1-2u)^2+4\varepsilon_1^2}\right)\left(\sqrt{1+4\varepsilon_2^2} - \sqrt{(1-2v)^2+4\varepsilon_2^2}\right). \end{aligned}$$

Figures 5 and 6 show the asymmetric behavior of these copulas.

Then corresponding Spearman's rho and Kendall's tau are given by,

$$\begin{aligned} \rho^{\varepsilon}_{\max} &= \frac{3}{4}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2\coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right] \\ \rho^{\varepsilon}_{\min} &= -\frac{3}{4}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2\coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right] \\ \tau^{\varepsilon}_{\max} &= \frac{1}{2}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2\coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right] \\ \tau^{\varepsilon}_{\min} &= -\frac{1}{2}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2\coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right] \end{aligned}$$

The optimal values of ρ and corresponding τ will be obtained by letting $(\varepsilon_1, \varepsilon_2) \to (0,0)$. Table 1 shows how the values of ρ approach the optimal values as $(\varepsilon_1, \varepsilon_2) \to (0,0)$ and it is clear that $-0.75 \leq \rho \leq 0.75$ and $-0.5 \leq \tau \leq 0.5$.

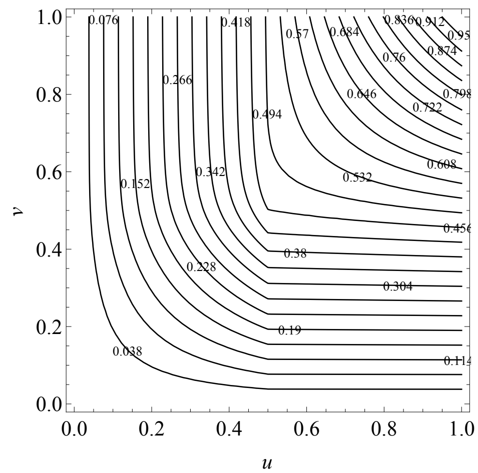

FIGURE 5: Contour plot of $C^{\varepsilon}_{\max}$ for$\varepsilon = (0.001, 0.1)$

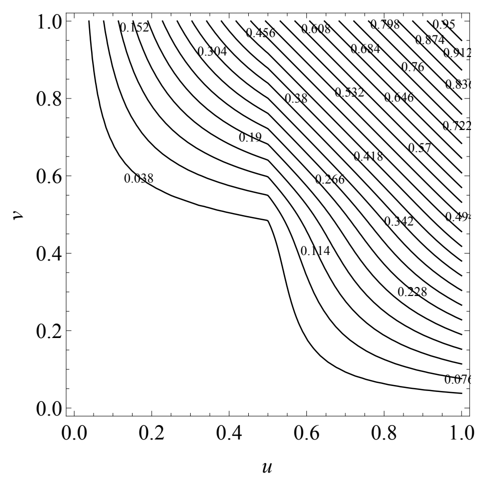

FIGURE 6: Contour plot of $C^{\varepsilon}_{\min}$ for$\varepsilon = (0.001, 0.1)$

TABLE 1: ρ and τ values as ε changes.

$\varepsilon = (\varepsilon_1, \varepsilon_2)$	$\rho^{\varepsilon}_{\max}$	$\rho^{\varepsilon}_{\min}$	$\tau^{\varepsilon}_{\max}$	$\tau^{\varepsilon}_{\min}$
(1, 1)	0.0726437	-0.0726437	0.0484292	-0.0484292
(0.1, 0.1)	0.644923	-0.644923	0.429949	-0.429949
(0.01, 0.01)	0.747539	-0.747539	0.498359	-0.498359
(0.001, 0.001)	0.749962	-0.749962	0.499974	-0.499974
(0.0001, 0.0001)	0.749999	-0.749999	0.5	-0.5

3.1. General Case: $h(u, v) = \sum_{i=1}^{n} \varphi_i(u)\psi_i(v)$

The above optimization method can be generalized to the case when the function h has the form $h(u, v) = \sum_{i=1}^{n} \varphi_i(u)\psi_i(v)$, where, as before, φ_i, ψ_i are continuous real valued functions on $\mathbb{I}$. Then h is automatically integrable. The degeneracy of h, from Theorem 1, implies

$$\sum_{i=1}^{n} A_i \varphi_i(x) \;=\; 0 \;=\; \sum_{i=1}^{n} B_i \psi_i(x) \quad \forall x \in \mathbb{I}, \tag{3}$$

where $A_i = \int_0^1 \psi_i(\xi)\, d\xi$ and $B_i = \int_0^1 \varphi_i(\xi)\, d\xi$, $i = 1, 2, \ldots, n$. Since φ_i and ψ_i are arbitrary, we have from equation (3)

$$\int_0^1 \varphi_i(\xi)\, d\xi = 0 = \int_0^1 \psi_i(\xi)\, d\xi \quad \text{ for } i = 1, 2, \ldots, n.$$

Positive definiteness of h implies that

$$\min_{(u,v)\in\mathbb{I}^2} \sum_{i=1}^{n} \varphi_i(u)\psi_i(v) \ge -1.$$

In this case, the copula and the corresponding Spearman's rho will take the following forms,

$$\begin{aligned} C(u,v) &= \Pi(u,v) + \sum_{i=1}^{n} \int_0^u \varphi_i(s)\, ds \int_0^v \psi_i(t)\, dt, \\ \rho &= 12 \sum_{i=1}^{n} \int_0^1 G_i(u)\, du \int_0^1 H_i(v)\, dv, \end{aligned}$$

where $G_i(u) = \int_0^u \varphi_i(s) ds$ and $H_i(v) = \int_0^v \psi_i(t) dt$, $i = 1, 2, \ldots, n$. Hence optimization of ρ leads towards the problems of optimizing the following quantities,

$$I_1^{(i)} := \int_0^1 G_i(u)\, du,\; I_2^{(i)} := \int_0^1 H_i(v)\, dv,\; i = 1, 2, \ldots, n.$$

Then for some positive constants α_i, β_i, k_i, with $\sum_{i=1}^{n} k_i \le 1$, the optimization problems for $i = 1, 2, \ldots, n$, become,

$$\begin{array}{lllll} \max/\min & I_1^{(i)} & \qquad & \max/\min & I_2^{(i)} \\ \text{subject to} & G_i(0) = G_i(1) = 0 & & \text{subject to} & H_i(0) = H_i(1) = 0 \\ & -\alpha_i \le G_i'(u) \le \beta_i, & & & -k_i/\beta_i \le H_i'(v) \le k_i/\alpha_i. \end{array}$$

Again, as before, the optimal values will occur if $\alpha_i = \beta_i$. Since, for every i, G_i and H_i have similar forms as G and H of special case, by a similar approximation method, as mentioned in the special case, we obtain,

$$\rho_{\max}^{\varepsilon} = \frac{3}{4}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right]\sum_{j=1}^{n} k_j$$

$$\rho_{\min}^{\varepsilon} = -\frac{3}{4}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right]\sum_{j=1}^{n} k_j$$

$$\tau_{\max}^{\varepsilon} = \frac{1}{2}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right]\sum_{j=1}^{n} k_j$$

$$\tau_{\min}^{\varepsilon} = -\frac{1}{2}\prod_{i=1}^{2}\left[\sqrt{1+4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1}\left(\sqrt{1+4\varepsilon_i^2}\right)\right]\sum_{j=1}^{n} k_j$$

Since $\sum_{i=1}^{n} k_i \leq 1$, by taking $(\varepsilon_1, \varepsilon_2) \rightarrow (0,0)$ we have $-0.75 \leq \rho \leq 0.75$ and $-0.5 \leq \tau \leq 0.5$.

4. Conclusion

We proposed an optimization method to increase the range of Spearman's rho for a special class of copulas and by doing so we generated a two-parameter family of asymmetric copulas.

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References

Amblard, C. & Girard, S. (2009), 'A new extension of bivariate FGM copulas', *Metrika* **70**, 1–17.

De la Peña, V. H., Ibragimov, R. & Sharakhmetov, S. (2006), Characterizations of joint distributions, copulas, information, dependence and decoupling, with applications to time series, *in* J. Rojo, ed., 'Optimality: The second Erich L. Lehmann Symposium, IMS Lecture Notes - Monograph Series', Vol. 49, Institute of Mathematical Statistics, Beachwood, Ohio, pp. 183–209.

Durante, F. (2009), 'Construction of non-exchangeable bivariate distribution functions', *Statistical Papers* **50**(2), 383–391.

Kim, J.-M., Sungur, E. A., Choi, T. & Heo, T.-Y. (2011), 'Generalized bivariate copulas and their properties', *Model Assisted Statistics and Applications* **6**, 127–136.

Liebscher, E. (2008), 'Construction of asymmetric multivariate copulas', *Journal of Multivariate Analysis* **99**(10), 2234–2250.

Nelsen, R. B. (2006), *An Introduction to Copulas*, Springer, New York.

Rodríguez-Lallena, J. A. & Úbeda-Flores, M. (2004), 'A new class of bivariate copulas', *Statistics & Probability Letters* **66**(3), 315–325.

Schweizer, B. & Sklar, A. (1983), *Probabilistic Metric Spaces*, Elsevier, New York.

Sklar, A. (1959), 'Fonctions de répartition à n dimensions et leurs marges', *l'Institut de statistique de l'Université de Paris* **8**, 229–231.

Sklar, A. (1973), 'Random variables, joint distribution functions, and copulas', *Kybernetika* **9**(6), 449–460.