Revista Colombiana de Estadística
July 2015, Volume 38, Issue 2, pp. 353 to 370
DOI: http://dx.doi.org/10.15446/rce.v38n2.51665

Alpha-Skew Generalized t Distribution

Distribución t generalizada alfa sesgada

Sukru Acitas[1,a], Birdal Senoglu[2,b], Olcay Arslan[2,c]

[1]Department of Statistics, Faculty of Science, Anadolu University, Eskisehir, Turkey

[2]Department of Statistics, Faculty of Science, Ankara University, Ankara, Turkey

Abstract

The alpha-skew normal (ASN) distribution has been proposed recently in the literature by using standard normal distribution and a skewing approach. Although ASN distribution is able to model both skew and bimodal data, it is shortcoming when data has thinner or thicker tails than normal. Therefore, we propose an alpha-skew generalized t (ASGT) by using the generalized t (GT) distribution and a new skewing procedure. From this point of view, ASGT can be seen as an alternative skew version of GT distribution. However, ASGT differs from the previous skew versions of GT distribution since it is able to model bimodal data sest as well as it nests most commonly used density functions. In this paper, moments and maximum likelihood estimation of the parameters of ASGT distribution are given. Skewness and kurtosis measures are derived based on the first four noncentral moments. The cumulative distribution function (cdf) of ASGT distribution is also obtained. In the application part of the study, two real life problems taken from the literature are modeled by using ASGT distribution.

***Key words*:** Bimodality, Kurtosis, Maximum Likelihood Estimation, Modeling, Skewness.

Resumen

La distribución normal alfa-sesgada (ASN por sus siglas en inglés) ha sido propuesta recientemente en la literatura mediante el uso de una distribución normal estándar y procedimientos de sesgo. Aunque la distribución ASN es capaz de modelar tanto datos sesgados y bimodales, no es recomendada cuando los datos tienen colas más livianas o pesadas que la distribución normal. Por lo tanto, se propone una distribución t alfa-sesgada generalizada

[a]Professor. E-mail: sacitas@anadolu.edu.tr
[b]Professor. E-mail: senoglu@science.ankara.edu.tr
[c]Professor. E-mail: oarslan@ankara.edu.tr

(ASGT por sus siglas en inglés) mediante el uso de la distribución t generalizada (GT por sus siglas en inglés) y un nuevo procedimiento de sesgo. Bajo este punto de vista, la distribución ASGT se puede ver como una alternativa sesgada de la distribución GT. Sin embargo, ASGT difiere de previas versiones sesgadas de la distribución GT puesto que es capaz de modelar datos bimodales y agrupa funciones de densidad más comúnmente usadas. En este artículo, los momentos y la estimación máximo verosímil de los parámetros de la distribución ASGT son derivadas. Medidas del sesgo y la curtosis son derivadas con base a los primeros cuatro momentos no centrales. La función de distribución acumulada (cdf por sus siglas en inglés) de la distribución ASGT es también obtenida. En la parte de aplicación del estudio, dos problemas reales tomados de la literatura son modelados usando la distribución ASGT.

Palabras clave: bimodalidad, curtosis, estimación máximo verosímil, modelamiento, sesgo.

1. Introduction

Traditionally, normality assumption is made in most of the statistical procedures. However, in real life problems, nonnormal distributions for modeling data sets having skewness and/or kurtosis are more prevalent, see for example Tiku, Islam & Selcuk (2001) and Celik, Senoglu & Arslan (2015). Therefore, there has been enormous interest in the construction of the alternative distributions to normal distribution.

Generalized t (GT) is one of these alternative distributions which is proposed by McDonald & Newey (1988). The probability density function (pdf) of the GT distribution is given as follows

$$f_{GT}(z;p,q) = \frac{p}{2q^{1/p}B(1/p,q)}\left(1+\frac{|z|^p}{q}\right)^{-(q+1/p)}, \quad -\infty < z < \infty \tag{1}$$

where $p > 0$ and $q > 0$ are the shape parameters and $B(\cdot,\cdot)$ denotes the beta function. The pdf given equation (1) is symmetric about 0. Therefore, odd moments of GT distribution are zero. On the other hand, $n-$th moment is calculated by using the following formula

$$E(Z^n) = \frac{q^{n/p}\Gamma\left(\frac{n+1}{p}\right)\Gamma\left(q-\frac{n}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}, \quad pq > n. \tag{2}$$

where n is an even number.

GT is known to be a flexible family for modeling thicker or thinner tails since it nests most commonly used density functions. For example, when $p = 2$ and $q \to \infty$, GT reduces to normal distribution. Laplace distribution is obtained when $p = 1$ and $q \to \infty$. t distribution which has $2q$ degrees of freedom and scale parameter $\sigma = \sqrt{2}$ is also a special case of GT when $p = 2$. Another limiting case of GT is power exponential distribution when $q \to \infty$. GT is thick tailed when p and q are small and it is thin tailed when p and q are large. For further information, see McDonald & Newey (1988).

GT distribution is widely used in both theoretical and applied statistical literature. It is mainly considered and used in the context of partially adaptive estimation method, see for example McDonald & Nelson (1989), McDonald & Nelson (1993), Butler, McDonald, Nelson & White (1990), Kantar, Usta & Acitas (2011). Arslan & Genc (2003) use GT for robust modeling by considering GT as a scale mixture of power exponential and generalized gamma distribution. Multivariate form of GT is studied by Arslan (2004). GT distribution is proposed to be use as an alternative to normal distribution by Wang & Romagnoli (2005) to process data reconciliation and process monitoring. Nadarajah (2008) gives the explicit formulas for the cumulative distribution function (cdf) of the GT distribution. See also Choy & Chan (2008) and Fung & Seneta (2010) for other representations of GT distribution for the univariate and multivariate cases, respectively. Kasap, Senoglu, Arslan & Acitas (2011) derive modified maximum likelihood estimators of the location and the scale parameters of the GT distribution. See also Johnson, Kotz & Balakrishnan (2004), Lye & Martin (1993) and Theodossiou (1998) studies and the references therein for further details about the GT distribution.

However, nonnormal symmetric distributions such as GT are not flexible for modeling skew data sets. To overcome this difficulty, many skew distributions were proposed by various authors, see for example Genton (2004), Martínez-Flórez, Vergara-Cardozo & González (2013) and Pereira, Marques & da Costa (2012). Among these distributions, alpha-skew normal (ASN) is one of the most popular distribution for modeling unimodal or bimodal skew data sets in recent years which is proposed by Elal-Olivero (2010). The pdf of ASN distribution is given as follows

$$h_{ASN}(z) = \frac{(1-\alpha z)^2 + 1}{2 + \alpha^2}\phi(z), \quad -\infty < z < \infty \tag{3}$$

where $\phi(\cdot)$ is the pdf of the standard normal distribution and $\alpha \in \mathbb{R}$ is the parameter which controls both the skewness and the effect of unimodality.

When $\alpha = 0$, ASN reduces to the well known standard normal distribution. Elal-Olivero (2010) proves that ASN has at most two modes. Further, he obtains the moments, discusses maximum likelihood (ML) estimation and some other features of ASN distribution.

The main objective of this paper is to present a new skew generalized t distribution family as an extension of the symmetric GT distribution by using Elal-Olivero's skewing procedure. Thus, the proposed family is called an alpha-skew GT (ASGT) distribution, see Acitas, Senoglu & Arslan (2013). It should be noted that there are some other skew versions of GT distribution such as skew GT (SGT) given by Theodossiou (1998). As far as we know, ASGT is the first distribution used for modeling the bimodal and heavy/light tailed data.

Since ASGT distribution can be seen as a "hybrid" of GT and Elal-Olivero's approach, it inherits some distributional properties from both GT distribution and skewing procedure. For instance, ASGT distribution nests some distributions such as alpha-skew power exponential, alpha-skew student t, alpha-skew Laplace and ASN as special or limiting cases. ASGT takes this property from GT distribution. On the other hand, ASGT is able to model bimodal and skew data sets which is because of the skewing procedure.

We believe that these features of ASGT distribution make it attractive for the practitioners since data encountered in many applied studies may be skew and bimodal as well as having thinner or thicker tails than normal. ASGT distribution is able to accommodate these properties of the data.

It should also be noted that these kinds of extensions have appeared in the statistical literature in recent years. For example, some extensions of Birnbaum-Saunders distribution are studided by Diaz-Garcia & Leiva-Sanchez (2005), Vilca-Labra & Leiva-Sánchez (2006), Gomez, Olivares-Pacheco & Bolfarine (2009), Castillo, Gomez & Bolfarine (2011) and Genc (2013). See also Arellano-Valle, Cortes & Gomez (2010) and Venegas, Rodríguez, Gomez, Olivares-Pacheco & Bolfarine (2012) for similar extensions of epsilon-skew-t distribution.

The rest of the paper is organized as follows. In section 2, we define ASGT distribution. Section 3 discusses the ML estimation of the parameters of ASGT distribution. Section 4 consists of two real life examples taken from the literature which are analyzed using ASGT distribution. The paper is finalized with a conclusion part.

2. Alpha-Skew Generalized t Distribution

In this section, we define ASGT distribution and give some basic properties of it.

Definition 1 (ASGT distribution)**.** Random variable Z is said to have ASGT distribution if it has the following pdf

$$g_{ASGT}(z;\alpha,p,q)=\frac{(1-\alpha z)^2+1}{2+\alpha^2 c(p,q)}f_{GT}(z;p,q),\qquad -\infty<z<\infty,\quad pq>2 \quad (4)$$

where

$$c(p,q)=\frac{q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(q\right)},$$

α is skewness and uni-bimodality parameter and $f_{GT}(z;p,q)$ is the pdf of GT distribution.

Random variable Z having $ASGT(\alpha,p,q)$ distribution is denoted shortly $Z\sim ASGT(\alpha,p,q)$.

When α tends to $\pm\infty$ in equation (4), the pdf becomes

$$g_{BGT}(z;p,q)=\frac{1}{c(p,q)}z^2 f_{GT}(z;p,q),\quad -\infty<z<\infty,\quad pq>2. \quad (5)$$

It is easy to see that the function given in equation (5) is also a pdf. Thus, this limiting case of ASGT is called a bimodal generalized t (BGT) distribution. BGT is symmetric and bimodal distribution. It can be seen as an extension of Elal-Olivero's bimodal normal distribution.

The following proposition can be proven as a result of Definition 1.

Proposition 1. *Let* $Z \sim ASGT(\alpha, p, q)$ *then*

(i) *if* $\alpha = 0$, $Z \sim GT(p, q)$,

(ii) $-Z \sim ASGT(-\alpha, p, q)$.

Proof. (i) is directly obtained from the definition of ASGT distribution. We prove (ii) below.

Let define $Y = -Z$ and $G_Y(y)$ denotes the cumulative distribution function (cdf) of Y. Then, one can easily see that

$$G_Y(y) = P(Y \leq y) = P(-Z \leq y) = P(Z \geq -y) = 1 - G_Z(-y) \tag{6}$$

where $G_Z(z)$ is cdf of Z. Taking derivative in equation (6) reveals

$$\begin{aligned} g_Y(y) &= (1 - G_Z(-y))' \\ &= g_Z(-y) \\ &= \frac{(1-\alpha(-y))^2 + 1}{2 + \alpha^2 c(p,q)} f_{GT}(-y; p, q). \end{aligned}$$

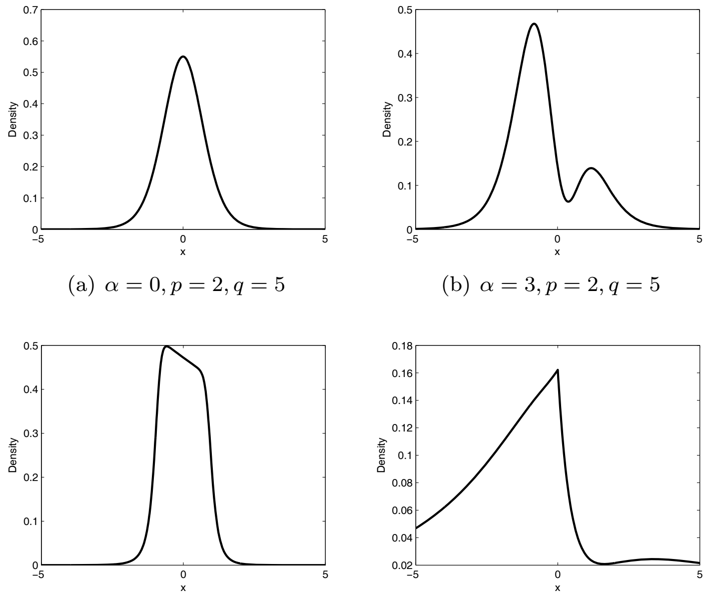
it is clear that $-Z \sim ASGT(-\alpha, p, q)$ since $f_{GT}(-y; p, q) = f_{GT}(y; p, q)$. $\square$

Figure 1 illustrates the shape of ASGT distribution for some selected values of the shape parameters. For example, when $\alpha = 0$, $p = 2$ and $q = 5$, the distribution is symmetric and unimodal, see Figure 1(a). On the other hand, Figure 1(b) demonstrates that if $\alpha = 3$, $p = 2$ and $q = 5$ ASGT becomes skew and bimodal. Figure 1(c) and 1(d) display interesting plots of ASGT distribution since the values of p and q are small.

It is clear from Figure 1 that ASGT distribution is flexible for modeling the data which may be skew and bimodal as well as having thinner or thicker tails than normal.

2.1. Special or Limiting Cases

ASGT distribution includes several distributions as special or limiting cases for specified values of the shape parameters α, p and q. However, the case $\alpha = 0$ corresponds to symmetric subdistributions which are available in literature (see for example McDonald and Newey, 1988), we therefore focus on the specified values of p and q. Thus, in this subsection, we give "alpha-skew" subdistributions that ASGT distribution nests.



(a) $\alpha = 0, p = 2, q = 5$ (b) $\alpha = 3, p = 2, q = 5$

(c) $\alpha = 0.1, p = 10, q = 0.5$ (d) $\alpha = 1, p = 1, q = 5$

FIGURE 1: The shape of the ASGT distribution for some selected values of the shape parameters α, p and q.

- ***Alpha-Skew Normal distribution***: In equation (4), if $p = 2$ and $q \to \infty$, we get the pdf of ASN distribution, see equation (3).
- ***Alpha-Skew Student's t distribution***: In equation (4), if $p = 2$, we get the pdf of *alpha-Skew Student's t* (ASST) distribution as follows

$$g_{ASST}(z; \alpha, q) = \frac{(1 - \alpha z)^2 + 1}{2 + \alpha^2 \left(\frac{q}{q-1}\right)} f_t(z; 2q), \quad -\infty < z < \infty, \quad q > 1 \tag{7}$$

where $f_t(z; 2q)$ denotes the pdf of student's t distribution with $2q$ degrees of freedom.

- ***Alpha-Skew Laplace distribution***: In equation (4), if $p = 1$ and $q \to \infty$, we get the pdf of *alpha-skew Laplace* (ASL) distribution as follows

$$g_{ASL}(z; \alpha, p) = \frac{(1 - \alpha z)^2 + 1}{2(1 + \alpha^2)} \frac{e^{-|z|}}{2}, \quad -\infty < z < \infty. \tag{8}$$

- ***Alpha-Skew Power Exponential distribution***: In equation (4), if $q \to \infty$, we get the pdf of *alpha-skew power exponential* (ASPE) distribution as follows

$$g_{ASPE}(z; \alpha, p) = \frac{(1 - \alpha z)^2 + 1}{2 + \alpha^2 \left(\frac{\Gamma(\frac{3}{p})}{\Gamma(\frac{1}{p})}\right)} \frac{p e^{-|z|^p}}{2\Gamma(\frac{1}{p})}, \quad -\infty < z < \infty. \tag{9}$$

It is clear that these distributions are alpha-skew versions of well known normal, student's t, Laplace and power exponential distributions, respectively.

This subsection shows that ASGT family nests subdistributions can be unimodal/bimodal and symmetric/skew. Thus, ASGT distribution is flexible for modeling both skew and bimodal data which may also have thinner or thicker tails than normal. Since ASGT distribution is able to accommodate these properties of the data, we believe that it is an attractive distribution family for practitioners.

Here an important question may arise: For which values of α, ASGT distribution is bimodal? It should be noted that this problem has a demanding solution process and the solution also depends on the values of the shape parameters p and q. We therefore present a brief discussion here by considering relatively simple cases: (i) ASL distribution and (ii) ASN distribution. As mentioned above, ASGT reduces to ASL distribution when $p = 1$ and $q \to \infty$. After dense algebraic operations, we see that ASL has at most two modes and the transition from unimodality to bimodality occurs around $\alpha = \pm 1$. A similar argument is also given for ASN distribution, which is one of the limiting cases of ASGT distribution for $p = 2$ and $q \to \infty$, by Elal-Olivero (2010). He indicates that the transition from unimodality to bimodality occurs around $\alpha = \pm 1.34$ via a numerical method. It is clear from these examples that finding a transition point (either algebraically or numerically) is a very demanding job even for simple cases. For some other values of the p and q, the solution for the transition point is much more difficult.

2.2. Moments

In this subsection, *moments* of ASGT distribution are derived. As a result, the *skewness* and the *kurtosis* measures are given.

Here, it should be realized that the even and the odd moments of ASGT distribution can be obtained by using the moments of GT distribution. This is because of the fact that the pdf of ASGT distribution includes the pdf of GT distribution. The following proposition gives for both even and odd n values of $E(Z^n)$ based on this phenomena where $Z \sim ASGT(\alpha, p, q)$.

Proposition 2. *Let $Z \sim ASGT(\alpha, p, q)$ then for $k \in \mathbb{N}$*

$$\begin{aligned}\mu_{2k} = E(Z^{2k}) = \frac{1}{2+\alpha^2 c(p,q)} \Bigg[& \frac{2q^{2k/p}\Gamma(\frac{2k+1}{p})\Gamma(q-\frac{2k}{p})}{\Gamma(\frac{1}{p})\Gamma(q)} \\ & + \frac{\alpha^2 q^{(2k+2)/p}\Gamma(\frac{2k+3}{p})\Gamma(q-\frac{2k+2}{p})}{\Gamma(\frac{1}{p})\Gamma(q)} \Bigg], \ pq > 2k+2\end{aligned} \tag{10}$$

$$\mu_{2k-1} = E(Z^{2k-1}) = \frac{1}{2+\alpha^2 c(p,q)} \frac{-2\alpha q^{2k/p}\Gamma(\frac{2k+1}{p})\Gamma(q-\frac{2k}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}, \ pq > 2k. \tag{11}$$

Proof. Let us begin with $E(Z^{2k})$:

$$\begin{aligned}
E(Z^{2k}) &= \frac{1}{2+\alpha^2 c(p,q)} \int_{-\infty}^{\infty} z^{2k}[(1-\alpha z)^2+1] f_{GT}(z)dz \\
&= \frac{1}{2+\alpha^2 c(p,q)} \int_{-\infty}^{\infty} \left(2z^{2k} - 2\alpha z^{2k+1} + \alpha^2 z^{2k+2}\right) f_{GT}(z)dz \\
&= \frac{1}{2+\alpha^2 c(p,q)} \left[2E_{GT}(Z^{2k}) - 2\alpha E_{GT}(Z^{2k+1}) + \alpha^2 E_{GT}(Z^{2k+2})\right] \\
&= \frac{1}{2+\alpha^2 c(p,q)} \Bigg[2E_{GT}(Z^{2k}) + \alpha^2 E_{GT}(Z^{2k+2})\Bigg].
\end{aligned}$$

By incorporating equation (2) into the last equation, we obtain

$$\begin{aligned}
E(Z^{2k}) = \frac{1}{2+\alpha^2 c(p,q)} \Bigg[&\frac{2q^{2k/p}\Gamma(\frac{2k+1}{p})\Gamma(q-\frac{2k}{p})}{\Gamma(\frac{1}{p})\Gamma(q)} \\
&+ \frac{\alpha^2 q^{(2k+2)/p}\Gamma(\frac{2k+2}{p})\Gamma(q-\frac{2k+2}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\Bigg]
\end{aligned}$$

where $pq > 2k+2$ and $E_{GT}(\cdot)$ denotes the expectation based on GT distribution.

$E(Z^{2k-1})$ can be obtained by following the same lines. Therefore, we omit the details for the sake of brevity. □

Based on these moments, we can obtain the *skewness* ($\sqrt{\beta_1}$) and the *kurtosis* (β_2) measures of ASGT distribution by using the following formulas:

$$\sqrt{\beta_1} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}}$$

and

$$\beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_2\mu_1^2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2},$$

respectively. Here,

$$\mu_1 = \frac{1}{2+\alpha^2 c(p,q)} \frac{-2\alpha q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}, \tag{12}$$

$$\mu_2 = \frac{1}{2+\alpha^2 c(p,q)} \left[\frac{2q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)} + \frac{\alpha^2 q^{4/p}\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}\right], \tag{13}$$

$$\mu_3 = \frac{1}{2+\alpha^2 c(p,q)} \frac{-2\alpha q^{4/p}\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}, \tag{14}$$

$$\mu_4 = \frac{1}{2+\alpha^2 c(p,q)} \left[\frac{2q^{4/p}\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)} + \frac{\alpha^2 q^{6/p}\Gamma\left(\frac{7}{p}\right)\Gamma\left(q-\frac{6}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}\right]. \tag{15}$$

In Table 1, we give the *skewness* ($\sqrt{\beta_1}$) and the *kurtosis* (β_2) values of ASGT distribution for some selected values of the shape parameters. It is obvious that Table 1 provides extra information for modeling performance of ASGT distribution.

TABLE 1: The *skewness* ($\sqrt{\beta_1}$) and the *kurtosis* (β_2) values of the ASGT distribution based on some selected values of the shape parameters α, p and q.

α	p	q	$\sqrt{\beta_1}$	β_2
0	2	5	0	4
0	10	2	0	1.9
0.1	10	0.5	0	3.5
1	4	8	0.4	2.7
1	10	2	0.6	2.5
3	2	5	0.6	3.9

2.3. Distribution Function

In this subsection, we derive the explicit formula for the cdf of ASGT distribution. It should be noted that the cdf of GT distribution is obtained by Nadarajah (2008). In Proposition 3, we obtain the cdf of ASGT distribution by following the similar lines given by Nadarajah (2008).

Proposition 3. *If $Z \sim ASGT(\alpha, p, q)$ then the cdf of random variable Z is given as*

$$G_{ASGT}(z) = \frac{1}{2+\alpha^2 c(p,q)}\left\{2F_{GT}(z) - 2\alpha G_1(z) + \alpha^2 G_2(z)\right\} \tag{16}$$

where

$$F_{GT}(z)=\begin{cases}\dfrac{1}{2}-\dfrac{1}{2}I_{\left(1-\frac{1}{1+(-z)^p/q}\right)}\left(\dfrac{1}{p},q\right), & z<0\\[2ex] \dfrac{1}{2}+\dfrac{1}{2}I_{\left(1-\frac{1}{1+z^p/q}\right)}\left(\dfrac{1}{p},q\right), & z\geq 0\end{cases}$$

$$G_1(z)=\frac{q^{1/p}\Gamma(\frac{2}{p})\Gamma(q-\frac{1}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\begin{cases}-\dfrac{1}{2}+\dfrac{1}{2}I_{\left(1-\frac{1}{1+(-z)^p/q}\right)}\left(\dfrac{2}{p},q-\dfrac{1}{p}\right), & z<0\\[2ex] -\dfrac{1}{2}+\dfrac{1}{2}I_{\left(1-\frac{1}{1+z^p/q}\right)}\left(\dfrac{2}{p},q-\dfrac{1}{p}\right), & z\geq 0\end{cases}$$

$$G_2(z)=\frac{q^{2/p}\Gamma(\frac{3}{p})\Gamma(q-\frac{2}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\begin{cases}\dfrac{1}{2}-\dfrac{1}{2}I_{\left(1-\frac{1}{1+(-z)^p/q}\right)}\left(\dfrac{3}{p},q-\dfrac{2}{p}\right), & z<0\\[2ex] \dfrac{1}{2}+\dfrac{1}{2}I_{\left(1-\frac{1}{1+z^p/q}\right)}\left(\dfrac{3}{p},q-\dfrac{2}{p}\right), & z\geq 0\end{cases}$$

and $I_y(a,b)$ *denotes the incomplete beta function.*

***Proof*.** The cdf of ASGT is obtained as shown below.

$$\begin{aligned}G_{ASGT}(z)&=\int_{-\infty}^{z}g_{ASGT}(t)dt\\&=\int_{-\infty}^{z}\frac{(1-\alpha t)^2+1}{2+\alpha^2c(p,q)}f_{GT}(t;p,q)dt\\&=\frac{1}{2+\alpha^2c(p,q)}\int_{-\infty}^{z}(2-2\alpha t+\alpha t^2)f_{GT}(t;p,q)dt\\&=\frac{2}{2+\alpha^2c(p,q)}\int_{-\infty}^{z}f_{GT}(t;p,q)dt-\frac{2\alpha}{2+\alpha^2c(p,q)}\int_{-\infty}^{z}tf_{GT}(t;p,q)dt\\&+\frac{\alpha^2}{2+\alpha^2c(p,q)}\int_{-\infty}^{z}t^2f_{GT}(t;p,q)dt.\end{aligned}$$

Let us define

$$F_{GT}(z)=\int_{-\infty}^{z}f_{GT}(t;p,q)dt,\quad G_1(z)=\int_{-\infty}^{z}tf_{GT}(t;p,q)dt$$

and

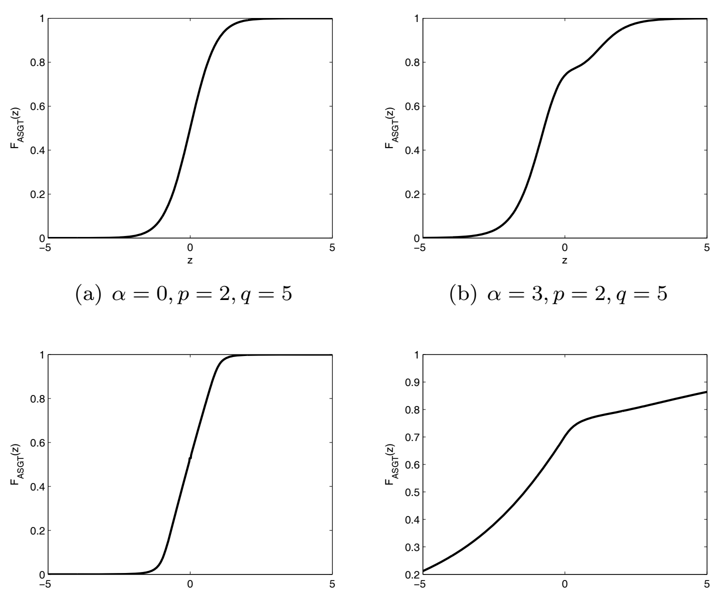
$$G_2(z) = \int_{-\infty}^{z} t^2 f_{GT}(t; p, q) dt.$$

for the sake of simplicity. The main idea for obtaining $F_{GT}(z)$, $G_1(z)$ and $G_2(z)$ is

$$u = 1 - \frac{1}{1 + \frac{t^p}{q}}, \quad t > 0$$

transformation, see Nadarajah (2008). Since similar lines are followed as in Nadarajah (2008), we do not give the details here for the sake of brevity. However, the complete proof can be provided upon request. □

In Figure 2, the plots of the cdf of ASGT distribution for some selected values of the shape parameters α, p and q are given. The selected values of the shape parameters are the same as those given in Figure 1.



(a) $\alpha = 0, p = 2, q = 5$ (b) $\alpha = 3, p = 2, q = 5$

(c) $\alpha = 0.1, p = 10, q = 0.5$ (d) $\alpha = 1, p = 1, q = 5$

FIGURE 2: The plots of the cdf of ASGT distribution for some selected values of the shape parameters α, p and q.

2.4. Location-Scale Case

Let $Z \sim ASGT(\alpha, p, q)$, $\mu \in \mathbb{R}$ and $\sigma > 0$ denote the location and the scale parameters, respectively. If we define random variable X as $X = \mu + \sigma Z$, then the corresponding pdf is obtained as follows

$$g(x;\mu,\sigma,\alpha,p,q)=\frac{1}{\sigma}\frac{\left(1-\alpha\left(\frac{x-\mu}{\sigma}\right)\right)^2+1}{2+\alpha^2c(p,q)}\frac{p}{2q^{1/p}B(1/p,q)}\left(1+\frac{\left|\frac{x-\mu}{\sigma}\right|^p}{q}\right)^{-(q+1/p)}. \tag{17}$$

Random variable X having the distribution with the density given in equation (17) is shown as $X\sim ASGT(\mu,\sigma,\alpha,p,q)$.

3. Maximum-Likelihood Estimation

Let $x_1,x_2,\dots,x_n$ be a random sample from $ASGT(\mu,\sigma,\alpha,p,q)$ distribution. The log-likelihood function can be written as follows

$$\begin{aligned}\ell(\mu,\sigma,\alpha,p,q)\propto -n\log\sigma+n\log p-\frac{n}{p}\log q-n\log B\left(\frac{1}{p},q\right)-\\ n\log(2+\alpha^2c(p,q))+\sum_{i=1}^{n}\log\left[(1-\alpha z_i)^2+1\right]-\\ \left(\frac{pq+1}{p}\right)\sum_{i=1}^{n}\log\left(1+\frac{|z_i|^p}{q}\right).\end{aligned} \tag{18}$$

where $z_i=(x_i-\mu)/\sigma$, $i=1,2,\dots,n$.

After taking partial derivatives of log-likelihood function with respect to the parameters of interest and setting them equal to zero, we obtain the following equations:

$$\frac{\partial\ell}{\partial\mu}=\frac{2\alpha}{\sigma}\sum_{i=1}^{n}\frac{(1-\alpha z_i)}{(1-\alpha z_i)^2+1}+\left(\frac{pq+1}{q\sigma}\right)\sum_{i=1}^{n}\frac{|z_i|^{p-1}sgn(z_i)}{1+\frac{1}{q}|z_i|^p}=0, \tag{19}$$

$$\frac{\partial\ell}{\partial\sigma}=-\frac{n}{\sigma}+\frac{2\alpha}{\sigma}\sum_{i=1}^{n}\frac{(1-\alpha z_i)z_i}{(1-\alpha z_i)^2+1}\left(\frac{pq+1}{q\sigma}\right)\sum_{i=1}^{n}\frac{|z_i|^p}{1+\frac{1}{q}|z_i|^p}=0, \tag{20}$$

$$\frac{\partial\ell}{\partial\alpha}=\sum_{i=1}^{n}\frac{2(1-\alpha z_i)z_i}{(1-\alpha z_i)^2+1}+\frac{2n\alpha c(p,q)}{2+\alpha^2c(p,q)}=0, \tag{21}$$

$$\begin{aligned}\frac{\partial\ell}{\partial p}=\frac{n}{p}-\frac{n}{p^2}\gamma\left(q+\frac{1}{p}\right)+\frac{n}{p^2}\gamma\left(\frac{1}{p}\right)+\frac{n}{p^2}\log q-\frac{n\alpha^2c'_p(p,q)}{2+\alpha^2c(p,q)}\\ +\frac{1}{p^2}\sum_{i=1}^{n}\log\left(1+\frac{|z_i|^p}{q}\right)-\frac{pq+1}{pq}\sum_{i=1}^{n}\frac{|z_i|^p\log|z_i|}{1+\frac{|z_i|^p}{q}}=0,\end{aligned} \tag{22}$$

$$\frac{\partial\ell}{\partial q}=n\gamma\left(q+\frac{1}{p}\right)-n\gamma\left(q\right)-\frac{n}{pq}-\sum_{i=1}^{n}\log\left(1+\frac{|z_i|^p}{q}\right) \tag{23}$$

$$-\frac{n\alpha^2c'_q(p,q)}{2+\alpha^2c(p,q)}+\frac{pq+1}{pq^2}\sum_{i=1}^{n}\frac{|z_i|^p}{1+\frac{|z_i|^p}{q}}=0 \tag{24}$$

where

$$c'_p(p,q) = \frac{c(p,q)}{p^2}\left[-2\log q - 3\gamma\left(\frac{3}{p}\right) + 2\gamma\left(q-\frac{2}{p}\right) + \gamma\left(\frac{1}{p}\right)\right],$$
$$c'_q(p,q) = c(p,q)\left[\frac{2}{pq} + \gamma\left(q-\frac{2}{p}\right) - \gamma(q)\right].$$

Here, $sgn(\cdot)$ and $\gamma(\cdot)$ denote the sign and the digamma functions, respectively.

Likelihood equations include nonlinear functions, therefore it is not possible to obtain explicit forms of the ML estimators. Thus, they have to be computed by using numerical methods. A lot of software, including optimization toolbox, i.e. S-Plus, R, RMATLAB2010a etc can be used for obtaining the ML estimates of the parameters. In this study, we use the optimization toolbox of RMATLAB2010a which uses "*Nelder-Mead simplex direct search*" algorithm. It should be noted that initial estimates for the parameter estimates are needed to compute the corresponding ML estimates when numerical algorithms are performed. It is clear that choosing wrong initial values causes optimization to end with local maximums. As in Ma & Genton (2004), we try different initial values to obtain global maximum in order to get rid of local maximums. The same discussion is also given by Genc (2013).

For obtaining the standard errors of the ML estimates one should compute the information matrix **I**. It is well known that the elements of **I** are given by

$$\mathbf{I}(i,j) = E\left(\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_i \partial\boldsymbol{\theta}_j}\right), \quad i,j = 1,2,\ldots,5 \tag{25}$$

where $\boldsymbol{\theta} = (\mu,\sigma,\alpha,p,q)'$. Since expectation over ASGT distribution and second order derivatives are not straightforward, numerical methods should be performed to obtain the explicit form of the information matrix **I**. Thus, we use the observed information matrix for calculating the standard errors in the rest of the paper.

4. Applications

In this section, we investigate the modeling performance of the ASGT distribution on two real life examples taken from the literature. The first data set is an example of unimodal data called Roller data. Faithful geyser data is the second one which is bimodal.

4.1. Roller Data

This data set has 1,150 observations which are available at `http://lib.stat.cmu.edu/jasadata/laslett` website, see also Gomez, Elal-Olivero, Salinas & Bolfarine (2011) who analyzed the same data.

In this study, we model roller data by using ASGT distribution. ML estimates of the parameters and Akaike information criterion (AIC) values are given in Table

Table 2: Parameter estimates of roller data based on ASGT and ASN distributions. Standard errors are given in the parentheses.

	μ	σ	α	p	q	AIC
ASN	3.5363	0.6497	0.0025	–	–	2277.7318
	(0.1512)	(0.0002)	(0.3573)	–	–	
ASGT	3.7838	0.6210	0.3877	1.5208	5.8424	2154.3653
	(0.0009)	(0.0016)	(0.0051)	(0.0393)	(10.2366)	

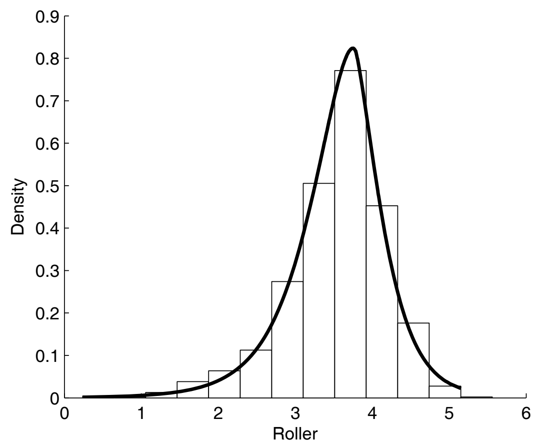


Figure 3: The histogram and the fitted ASGT density for Roller data.

2 along with the results for ASN distribution. The histogram and the fitted density for roller data are illustrated in Figure 3.

AIC values suggest that ASGT distribution is more reliable than ASN distribution for roller data. It is also obvious from Figure 3 that ASGT provides a substantially good fitting.

Results obtained from AIC are also supported by the following likelihood ratio test (LRT) given below. In LRT, we test the null hypothesis

$$H_0 : \text{Distribution of the data is ASN}$$

versus

$$H_1 : \text{Distribution of the data is ASGT.}$$

by using the following likelihood ratio statistic:

$$-2\lambda(\mathbf{x}) = -2\log\frac{L_{ASN}(\hat{\mu},\hat{\sigma},\hat{\alpha})}{L_{ASGT}(\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{p},\hat{q})} \tag{26}$$

where L_{ASN} and L_{ASGT} stand for likelihood functions of ASN and ASGT distributions, respectively. The value of $\lambda(\mathbf{x})$ is calculated as 127.3665 by using the ML estimates given in Table 2. The null hypothesis is rejected at $\alpha = 0.05$ significance level since the critical Chi-square value for two degrees of freedom is 5.99. Therefore, it can be concluded that the distribution of the data is ASGT.

4.2. Faithful Geyser Data

Faithful geyser data includes 272 observations which denote the waiting-time between eruptions and the duration-time of these eruptions for Old Faithful geyser in Yellow National Park, Wyoming, USA. This popular data is available at R-system and see also Arellano-Valle et al. (2010) in which it is indicated that data is negatively skewed and bimodal.

In this study, we use ASGT distribution to model this popular data. ML estimates of the parameters and AIC values for both ASGT and ASN distributions are given in Table 3. The histogram and the fitted density for Faithful geyser data are given in Figure 4.

TABLE 3: Parameter estimates of Faithful geyser data based on ASGT and ASN distributions. Standard errors are given in the parentheses.

	μ	σ	α	p	q	AIC
ASN	3.2344	0.6857	-6.0772	–	–	633.9026
	(0.0010)	(0.0003)	(1.6982)	–	–	
ASGT	3.2586	1.5169	-7.7342	16.2196	1.5262	549.3188
	(0.0004)	(0.0011)	(1.5652)	(112.2971)	(3.6750)	

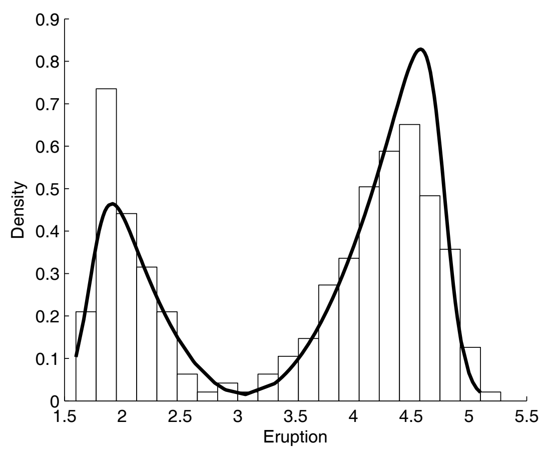


FIGURE 4: The histogram and the fitted densities for Faithful geyser data.

It is clear that AIC values suggest that ASGT distribution is more reliable than ASN distribution for Faithful geyser data. Further, Figure 3 shows that ASGT provides a satisfactory fitting.

LRT test is also used for testing the null hypothesis given in the previous subsection. The calculated value of the LRT test statistic is 88.5838. This result indicates that the null hypothesis is rejected at 0.05 significance level. Thus, ASGT distribution is more suitable for Faithful geyser data.

5. Conclusion

In this study, we propose ASGT distribution as an extension of symmetric GT distribution by using Elal-Olivero's (2010) skewing procedure. ASGT nests some distributions for specified values of the shape parameters. This feature of ASGT distribution makes it very attractive for the practical users since it is flexible for modeling unimodality or bimodality as well as skewness and kurtosis. Moments of new distribution are derived. As a result, the skewness and the kurtosis measures are obtained. Its distribution function is formulated. After discussing ML estimation of the parameters, two real life data taken from literature are modeled by using ASGT distribution. The results show that ASGT distribution provides a good fitting.

Acknowledgements

We thank the referees for their valuable comments and suggestions that improved the paper.

[Received: April 2014 — Accepted: January 2015]

References

Acitas, S., Senoglu, B. & Arslan, O. (2013), Alpha-skew generalized t distribution, International Conference Applied Statistics, Slovenia.

Arellano-Valle, R. B., Cortes, M. A. & Gomez, H. W. (2010), 'An extension of the epsilon-skew-normal distribution', *Communications in Statistics – Theory and Methods* **39**, 912–922.

Arslan, O. (2004), 'Family of multivariate generalized t distributions', *Journal of Multivariate Analysis* **89**, 329–337.

Arslan, O. & Genc, A. I. (2003), 'Robust location and scale estimation based on the univariate generalized t distribution', *Communications in Statistics – Theory and Methods* **32**, 1505–1525.

Butler, R. J., McDonald, J. B., Nelson, R. D. & White, S. (1990), 'Robust and partially adaptive estimation of regression models', *The Review of Economics and Statistics* **72**, 321–327.

Castillo, N. O., Gomez, H. W. & Bolfarine, H. (2011), 'Epsilon Birnbaum-Saunders distribution family: Properties and inference', *Statistical Papers* **52**, 871–883.

Celik, N., Senoglu, B. & Arslan, O. (2015), 'Estimation and testing in one-way ANOVA when the errors are skew-normal', *Revista Colombiana de Estadística* **38**(1), 75–91.

Choy, S. T. B. & Chan, J. S. K. (2008), 'Scale mixtures distributions in statistical modelling', *Australian & New Zealand Journal of Statistics* **50**(2), 135–146.

Diaz-Garcia, J. A. & Leiva-Sanchez, V. (2005), 'A new family of life distributions based on the contoured elliptically distributions', *Journal of Statistical Planning and Inference* **128**(2), 445–457.

Elal-Olivero, D. (2010), 'Alpha-skew normal distribution', *Proyecciones Journal of Mathematics* **29**, 224–240.

Fung, T. & Seneta, E. (2010), 'Extending the multivariate generalised t and generalised VG distributions', *Proyecciones Journal of Mathematics* **101**, 154–164.

Genc, A. I. (2013), 'The generalized t Birnbaum-Saunders Family', *Statistics* **47**, 613–625.

Genton, G. G. (2004), *Skew-Elliptical Distributions and their Applications: A Journey beyond Normality*, Chapman & Hall/CRC, Boca Raton, Florida.

Gomez, H. W., Elal-Olivero, D., Salinas, H. S. & Bolfarine, H. (2011), 'Bimodal extension based on skew-normal distribution with application to pollen data', *Environmetrics* **22**, 50–62.

Gomez, H. W., Olivares-Pacheco, J. F. & Bolfarine, H. (2009), 'An extension of the generalized Birnbaum-Saunders distribution', *Statistics & Probability Letters* **79**, 331–338.

Johnson, N. L., Kotz, S. & Balakrishnan, N. (2004), *Continuous Univariate Distributions*, Vol. 2, 2 edn, John Wiley, New York.

Kantar, Y. M., Usta, I. & Acitas, S. (2011), 'A Monte Carlo simulation study on partially adaptive estimators of linear regression models', *Journal of Applied Statistics* **38**, 1681–1699.

Kasap, P., Senoglu, B., Arslan, O. & Acitas, S. (2011), 'Estimating the location and scale parameters of the generalized t distribution', *E-journal of New World Science Academy Physical Sciences* **6**, 103–111.

Lye, J. N. & Martin, V. L. (1993), 'Robust estimation, nonnormalities and generalized exponential distributions', *Journal of the American Statistical Association* **88**, 261–267.

Ma, Y. & Genton, M. G. (2004), 'Flexible class of skew-symmetric distributions', *Scandinavian Journal of Statistics* **31**, 459–468.

Martínez-Flórez, G., Vergara-Cardozo, S. & González, L. M. (2013), 'The family of log-skew-normal alpha-power distributions using precipitation data', *Revista Colombiana de Estadística* **36**(1), 43–57.

McDonald, J. B. & Nelson, R. D. (1989), 'Alternative beta estimation for the market model using partially adaptive techniques', *Communications in Statistics – Theory and Methods* **18**(11), 4039–4058.

McDonald, J. B. & Nelson, R. D. (1993), 'Beta estimation in the market model: Skewness and leptokurtosis', *Communications in Statistics – Theory and Methods* **22**(10), 2843–2862.

McDonald, J. B. & Newey, W. K. (1988), 'Partially adaptive estimation of regression models via the generalized t distribution', *Econometric Theory* **4**, 428–457.

Nadarajah, S. (2008), 'On the generalized t (GT) distribution', *Statistics* **42**, 467–473.

Pereira, J. R., Marques, L. A. & da Costa, J. M. (2012), 'An empirical comparison of EM initialization methods and model choice criteria for mixtures of skew-normal distributions', *Revista Colombiana de Estadística* **35**(3), 457–478.

Theodossiou, P. (1998), 'Financial data and the skewed generalized t distribution', *Management Science Part 1* **12**, 1650–1661.

Tiku, M. L., Islam, M. Q. & Selcuk, S. A. (2001), 'Nonnormal regression. II. Symmetric distributions', *Communications in Statistics – Theory and Methods* **30**(6), 1021–1045.

Venegas, O., Rodríguez, F., Gomez, H. W., Olivares-Pacheco, J. F. & Bolfarine, H. (2012), 'Robust modeling using the generalized epsilon-skew-t distribution', *Journal of Applied Statistics* **39**(12), 2685–2698.

Vilca-Labra, F. & Leiva-Sánchez, V. (2006), 'A new fatigue life model based on the family of skew-elliptical distributions', *Communications in Statistics – Theory and Methods* **35**(2), 229–244.

Wang, D. & Romagnoli, J. A. (2005), 'Generalized t distribution and its applications to process data reconciliation and process monitoring', *Transactions of the Institute of Measurement and Control* **27**(5), 367–390.