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Univariate Conditional Distributions of an Open-Loop TAR Stochastic Process

Distribuciones condicionales univariadas de un proceso estocástico TAR sin retroalimentación

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Abstract

Clusters of large values are observed in sample paths of certain open-loop threshold autoregressive (TAR) stochastic processes. In order to characterize the stochastic mechanism that generates this empirical stylized fact, three types of marginal conditional distributions of the underlying stochastic process are analyzed in this paper. One allows us to find the conditional variance function that explains the aforementioned stylized fact. As a by-product, we are able to derive a sufficient condition to have asymptotic weak stationarity in an open-loop TAR stochastic process.

***Key words*:** Conditional heteroscedasticity, Nonlinear stochastic process, Open-loop TAR model, Stationary nonlinear stochastic process.

Resumen

En trayectorias de un proceso estocástico autoregresivo de umbrales (TAR), sin retroalimentación, se observan conglomerados de valores extremos. Con el fin de caracterizar el mecanismo probabilístico que los genera, en este artículo se estudian tres tipos de distribuciones marginales condicionales del proceso subyacente. Uno de ellos permite encontrar la función de varianza condicional que explica ese hecho estilizado del proceso. Como un resultado adicional, se obtiene una condición suficiente para determinar estacionariedad débil asintótica, de un proceso TAR sin retroalimentación.

***Palabras clave*:** heterocedasticidad condicional, modelo TAR sin retroalimentación, proceso estocástico no lineal estacionario.

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1. Introduction

In the context of nonlinear stochastic processes, Tong (1990) proposed the open-loop threshold autoregressive (TAR) process and, among many others, Nieto (2005, 2008) and Nieto, Zhang & Li (2013) developed a Bayesian methodology to analyze particular cases, namely-when the threshold variable is not a covariable. Sample paths in this kind of TAR process can exhibit clusters of (either positive or negative) large values, which is an empirical fact that is also observed, for example, in financial and meteorological/hydrological time series. In terms of financial data, that stylized fact has been very well characterized by Engle's (1982) ARCH model and, then by the GARCH process (Bollerslev 1986) and almost all of its extensions. The key element in these models has been the conditional variance function of the underlying stochastic process. Along these lines, it is important to find a conditional variance function of an open-loop TAR process in order to determine if large-value clusters are explained by some type of conditional probabilistic mechanism.

The issue of characterizing univariate conditional distributions for this type of TAR processes or, even, their univariate marginal distributions, has not in the authors knowledge been studied until to now (see also Tong 2011). Thus, the main goal of this paper is to obtain univariate marginal conditional distributions for the open-loop TAR process in order to explain the presence of large-sample clusters in a sample path. As a by-product, we provide a sufficient condition to have asymptotic weak stationarity of the TAR process which is also considered in this paper. The issues of weak and strict stationarity and ergodicity have been contemplated in other cases related to the general family of TAR processes. For example, Petruccelli & Woolford (1984) and Chen & Tsay (1991) have worked on the so-called SETAR model class. Tong (1978), Wong & Li (2000), and Li, Ling & Tong (2012) have undertaken research in which their threshold moving average (TMA) equation has a stationary solution. Tong's (2011) paper is an excellent and comprehensive review of the literature of these issues.

The paper is organized as follows: In Section 2 we present the basic specification of the considered TAR process. Section 3 includes the main results of our research, and in Section 4 we present three illustrative examples (two are real-data examples). Section 5 concludes.

2. The Proposed Open-Loop TAR Model

Let $\mathbb{Z}$ be the set of integer numbers. Following Nieto (2005) , let $\{X_t : t \in \mathbb{Z}\}$ and $\{Z_t : t \in \mathbb{Z}\}$ be stochastic processes, such that for all $t \in \mathbb{Z}$,

$$X_t \quad = \quad a_0^{(j)} + \sum_{i=1}^{k_j} a_i^{(j)} X_{t-i} + h^{(j)} \varepsilon_t \quad \text{if} \quad r_{j-1} < Z_t \leq r_j, \tag{1}$$

for some $j = 1, \ldots, l$, where l denotes the number of the so-called regimes in the sample space of variable Z_t. These regimes are determined by the extended real numbers (threshold values) $r_0 < r_1 < \cdots < r_{l-1} < r_l$, where $r_0 = -\infty$ and $r_l = \infty$.

Let $R_j = (r_{j-1}, r_j]$, $j = 1, \ldots, l$, with the convention $(r_{l-1}, \infty] = (r_{l-1}, \infty)$; then, the set of regimes $R_1,\ldots,R_l$ is a partition of the real line $\mathbb{R}$.

Here, $a_i^{(j)}$ and $h^{(j)}$, $i = 0, 1, \ldots, k_j$, $j = 1, \ldots, l$, are real numbers and they are called *nonstructural* parameters. The nonnegative number k_j is the autoregressive order in regime j, $j = 1, \ldots, l$, and l,r_1,...,r_{l-1}, k_1,...,k_{l-1} and k_l are called the *structural* parameters of the model.

Additionally, $\{\varepsilon_t\}$ is a Gaussian zero-mean white a noise process with a variance 1 such that $E(X_s\varepsilon_t) = 0$ for $s < t$, Z_s and the set $\{X_t, X_{t-1}, \ldots\}$ are mutually independent for all $s > t$, and $\{Z_t\}$ and $\{\varepsilon_t\}$ are mutually independent. Equation (1) describes a dynamical system without feedback with input $\{Z_t\}$ and output $\{X_t\}$. To describe the stochastic behavior of $\{Z_t\}$, we additionally assume that $\{Z_t\}$ is a homogeneous mth order Markov chain, $m \geq 1$, and, as a sequence of random variables, it converges weakly to the distribution F.

We shall use the symbol TAR$(l; k_1, \ldots, k_l)$ to denote this model and we will say that $\{X_t\}$ is an open-loop TAR process with $\{Z_t\}$ as its threshold process. Notice that if we define, for each $t \in \mathbb{Z}$, the random variable J_t as $J_t = j$, if and only if $Z_t \in R_j$, then $\{J_t\}$ is an indicator process with sample space $\{1, \cdots, l\}$. Hence, the processes $\{X_t\}$ and $\{J_t\}$ conform a particular case of the general TAR process, in Tong's (1990, 2011) sense.

This TAR model has been applied in the fields of the hydrology and meteorology (Nieto 2005, Nieto 2008, Nieto et al. 2013), the economics (Hoyos 2006), and finance (Moreno 2011). In a multivariate setting, Tsay (1998) among many other authors, has also analyzed open-loop models, where variable Z_t is included as a covariable. In what follows, we refer to the above open-loop TAR process simply as a TAR process (or model).

3. New Characteristics of the TAR Process

In this section, our goal is to obtain the distribution of X_t for all t, conditional on (i) a regime, R_j say, (ii) a regime R_j and the past data $\tilde{x}_{t-1} = (x_{t-1}, \ldots, x_1)$ with $t > \max\{k_j \mid j = 1, \ldots, l\}$, and (iii) $\tilde{x}_{t-1}$. That is to say, we will consider *three* types of conditioning sets. Before finding these univariate conditional distributions, we shall derive a sufficient condition to characterize the univariate marginal distributions of the process $\{X_t\}$. Nieto (2008) presented the first steps in characterizing some of these conditional distributions, as well as the univariate marginal distributions, but the results were shown in an intuitive way without formal analytical proofs.

3.1. Univariate Marginal Distributions and Asymptotic Weak Stationarity

We assume that all the random elements considered in this paper are defined on the same probability space $(\Omega, \mathfrak{F}, P)$. Initially, we note that for all $t \in \mathbb{Z}$, the marginal cumulative distribution function (cdf) of X_t is given by

$$F_t(x) = \sum_{j=1}^{l} p_{t,j} F_{t,j}(x) \tag{2}$$

for any $x \in \mathbb{R}$, where $F_{t,j}(x) = P(X_t \leq x \mid Z_t \in R_j)$ and $p_{t,j} = P(Z_t \in R_j)$, $j = 1, \ldots, l$ (assuming this last probability is positive). Because $\sum_{j=1}^{l} p_{t,j} = 1$, the marginal distribution of X_t is a mixture of conditional distributions, where the conditioning events are the regimes. Moments for X_t can be obtained from this marginal cdf. Indeed, if we denote $\mu_{1,j,t} = \int x dF_{t,j}(x)$ for all $j = 1, \ldots, l$ and all t, then $\mu_t = E(X_t) = \sum_{j=1}^{l} p_{t,j}\mu_{1,j,t}$ (a weighted average of the *regime* means at time t). A similar expression holds for the second moment around zero. Indeed, if we denote $\mu_{2,j,t} = \int x^2 dF_{t,j}(x)$ for all $j = 1, \ldots, l$ and all t, we obtain $\mu_{2,t} = E(X_t^2) = \sum_{j=1}^{l} p_{t,j}\mu_{2,j,t}$. Obviously, $\mathrm{Var}(X_t) = \mu_{2,t} - \mu_t^2$ for all $t \in \mathbb{Z}$.

Now, since $\{Z_t\}$ converges weakly to F, $p_{t,j} \to p_j = F(r_j) - F(r_{j-1})$ as $t \to \infty$ for all $j = 1, \ldots, l$. Of course, if $\{Z_t\}$ has identical univariate marginal distributions, as in the case of a strictly stationary process, then the cdf of Z_t is F for all t and, thus, $p_{t,j} = p_j$ for all t and for all $j = 1, \ldots, l$.

Proposition 1. *Let $\mathbb{C}$ be the complex number set. If for each $j = 1, \ldots, l$, the roots of the polynomial $\phi_j(z) = 1 - \sum_{i=1}^{k_j} a_i^{(j)} z^i$, $z \in \mathbb{C}$, are outside of the unit circle, then*

$$F_{t,j}(x) = \Phi\left(\frac{x - \psi_j(1)a_0^{(j)}}{h^{(j)}\bar{\sigma}_j}\right),\ x \in \mathbb{R}, \tag{3}$$

where $\psi_j(z) = \frac{1}{\phi_j(z)} = \sum_{i=0}^{\infty} \psi_i^{(j)} z^i$ for $\mid z \mid \leq 1$, with $\sum_{i=0}^{\infty} \mid \psi_i^{(j)} \mid < \infty$, $\bar{\sigma}_j^2 = \sum_{i=0}^{\infty} (\psi_i^{(j)})^2$, and $\Phi(\cdot)$ denotes the standard normal cdf.

Proof. See the Appendix. □

This result was quoted by Moreno (2011) but without a formal proof, and in the particular case where the distribution of Z_t is the same for all t. It is important to remark here that, under the assumptions of Proposition 1, the conditional cdf $F_{t,j}$ does not depend on t and that, as is noted in the proof, the Gaussian assumption on the process $\{\varepsilon_t\}$ is the basis for this result. Thus, let $F_j = F_{t,j}$, $\mu_{1,j,t} = \mu_{1j}$, and $\mu_{2,j,t} = \mu_{2j}$ for all t and for all $j = 1, \ldots, l$. Also, we can state that the distribution of X_t conditional on the regime R_j is $N(\psi_j(1)a_0^{(j)}, [h^{(j)}\bar{\sigma}_j]^2)$ for all t and for all $j = 1, \ldots, l$. We note that $\mu_{1j} = \psi_j(1)a_0^{(j)}$, and for future reference we set $\sigma_j^2 = [h^{(j)}\bar{\sigma}_j]^2$. Furthermore, the sequence $\{X_t\}$ converges weakly to a random variable with cdf $F_X = \sum_{j=1}^{l} p_j F_j$. Based on this fact, the mean and the variance function of this process converge, respectively, to a constant value, as $t \to \infty$. As noted above, if the univariate marginal distributions of $\{Z_t\}$ are identical, the cdf of X_t is F_X for all t. Notice that the limit distribution is a *mixture* of conditional normal distributions where the conditioning sets are the regimes, and hence, this distribution is potentially multimodal. Of course, if $\mu_{1j} = \mu_1$ for all $j = 1, \ldots, l$, the distribution is unimodal.

Proposition 2. *Under the conditions in Proposition 1, the autocovariance function (ACVF) of $\{X_t\}$ is given by*

$$Cov(X_t, X_{t-n}) = \sum_{j,k=1}^{l} p_{t,t-n,jk} q_{jk}(n) - \mu_t \mu_{t-h}, \tag{4}$$

for all $t, n \in \mathbb{Z}$, where $p_{t,t-n,jk} = P(Z_t \in R_j, Z_{t-n} \in R_k)$ and

$$q_{jk}(n) = \mu_{1j}\mu_{1k} + h^{(j)}h^{(k)} \sum_{m=0}^{\infty} \psi_m^{(k)} \psi_{n+m}^{(j)}, \tag{5}$$

for $j, k = 1, \ldots, l$.

***Proof*.** See the Appendix. □

Moreno (2011) proposed an expression for this ACVF and provided a proof for it. However, in the proof, some conditional probability measures are not well established, there is a mistake in a probability statement, and only the case in which the distribution of Z_t is the same for all t was considered.

Using the Chapman-Kolmogorov equations, we can show that the sequence of two-dimensional random vectors $\{(Z_t, Z_{t-n})\}$ converges weakly to a two-dimensional random vector with cdf F_n as $t \to \infty$. Then, $p_{t,t-n,jk} \to p_{n,jk} = F_n(r_j, r_k) - F_n(r_j, r_{k-1}) - F_n(r_{j-1}, r_k) + F_n(r_{j-1}, r_{k-1})$ as $t \to \infty$ and, consequently, $\text{Cov}(X_t, X_{t-n}) \to \sum_{j,k=1}^{l} p_{n,jk} q_{jk}(n) - \mu^2$ as $t \to \infty$. Of course, if $\{Z_t\}$ has identical univariate marginal distributions, the ACVF of $\{X_t\}$ is given directly by this limit form. We additionally note that $E(X_t^2) < \infty$ for all $t \in \mathbb{Z}$ because of the finiteness of the normal distribution second moment. The results obtained in the above propositions provide us with evidence to state the following definition.

Definition 1. The TAR stochastic process $\{X_t\}$ is said to be an *asymptotic* weak stationary process if it satisfies the conditions given in Proposition 1.

Remarks. (1) If $\{Z_t\}$ has identical marginal univariate distributions, $\{X_t\}$ *is* weakly stationary. (2) We must be careful in trying to use this function for model identification purposes, as in the case of *linear* ARMA processes. The reason is that the usual ACVF only captures linear dependence and the process $\{X_t\}$ is *not* linear. (3) The usefulness of the computation of the ACVF in Proposition 2 is in that it leads to the definition of the asymptotic weak stationarity of the process $\{X_t\}$. (4) In future research, the analytical properties of this ACVF will be investigated, specifically for describing its mathematical properties and, in particular, the rate of convergence towards zero (under ergodicity conditions).

It is worth noticing that the limit form of expression (4) is an extension of the ACVF of a *linear* stochastic process. Indeed, if $l = 1$, the only regime is the real line $\mathbb{R}$ and setting $h^{(1)} = h$, we obtain the limit form of the autocovariance in (4), which is

$$h^2 \sum_{m=0}^{\infty} \psi_m \psi_{n+m}.$$

This expression is analogous to the general form of the ACVF of a linear stochastic process $\{X_t = \sum_{i=0}^{\infty} \psi_i \varepsilon_t\}$, where the real-number sequence $\{\psi_i\}$ is absolutely summable and $\{\varepsilon_t\}$ is a zero-mean white noise process with variance h^2 (see Brockwell & Davis 1991). Obviously, in this particular case, $X_t = \sum_{i=1}^{k} a_i X_{t-i} + h\varepsilon_t$, with $k = k_1$, and the roots of the polynomial $\phi(z) = 1 - a_1 z - \cdots - a_k z^k$ are outside of the unit circle; that is, $\{X_t\}$ *is* an AR(k) linear process. For future reference, we put $\gamma(n) = \lim_{t\to\infty} \mathrm{Cov}(X_t, X_{t-n})$, for all $n \in \mathbb{Z}$.

3.2. Univariate Conditional Distributions

In the previous subsection, the first type of univariate conditional distribution emerged; namely when the conditioning set is the *event* $Z_t \in R_j$ (in the σ-algebra $\mathfrak{F}$) for some $j = 1, \ldots, l$. We have referred to this fact by stating that the conditioning is "on the regime"R_j. As was seen there, under the conditions in Proposition 1, the conditional distribution of X_t given the regime R_j is normal with mean μ_{1j} and variance σ_j^2 for all t and for all $j = 1, \ldots, l$. Also notice that, under this scenario, the conditional variance function of the process $\{X_t\}$ is given by $Var(X_t \mid R_j) = \sigma_j^2$ if at time t the observation $z_t \in R_j$. This is a step function that essentially depends on the variable-Z values.

The second type of conditional distribution for the variable X_t occurs when the conditioning set is $\tilde{x}_{t-1}$ and the regime R_j, for some $j = 1, \ldots, l$. Trivially, this distribution is $N\left(\mu_{t-1,j},\ [h^{(j)}]^2\right)$, for all $j = 1, \ldots, l$ and for all $t > \max\{k_j \mid j = 1, \ldots, l\}$, where $\mu_{t-1,j} = a_0^{(j)} + \sum_{i=1}^{k_j} a_i^{(j)} x_{t-i}$. Here, the conditional variance function for the process $\{X_t\}$ is $Var(X_t \mid \tilde{x}_{t-1}, R_j) = [h^{(j)}]^2)$ for $t > \max\{k_j \mid j = 1, \ldots, l\}$. Clearly, R_j occurs at time t if and only if $z_t \in R_j$. As in the previous case, it is a step function that intrinsically depends on the values of variable Z. Interestingly, under this conditional distribution, we can interpret the parameter $h^{(j)}$ as a *conditional* standard deviation, $j = 1, \ldots, l$.

Now, we only consider $\tilde{x}_{t-1}$ as the conditioning information set. In order to formally establish the main results, we set the following notation: let $\mathbb{N}$ be the set of natural numbers, $\mathbf{X}_t = (X_t, \ldots, X_1)$, $\mathcal{P} = \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$, $\mathfrak{F}_t = \sigma(\mathcal{B}^t \times \mathcal{P})$, the σ-algebra generated by the cartesian product between the t-dimensional Borelian σ-algebra and $\mathcal{P}$, λ_t the Lebesgue measure on the measurable space $(\mathbb{R}^t, \mathcal{B}^t)$, μ the counting measure on the measurable space $(\mathbb{N}, \mathcal{P})$, and $P_{(\mathbf{X}_t, J_t)}$ the probability induced by $(\mathbf{X}_t, J_t)$ on the product measure space $(\mathbb{R}^t \times \mathbb{N}, \mathfrak{F}_t, \lambda_t \times \mu)$.

Proposition 3. *Suppose that $P_{(\mathbf{X}_t, J_t)}$ has a Radon-Nikodym derivative with respect to the product measure $\lambda_t \times \mu$, then, (i) for all $t > \max\{k_j \mid j = 1, \ldots, l\}$, the conditional distribution of X_t given $\tilde{x}_{t-1}$ is a mixture of conditional normal distributions and (ii) the conditional variance function of the process $\{X_t\}$ is given by*

$$Var(X_t \mid \tilde{x}_{t-1}) = \sum_{j=1}^{l} p_{t,j} \left[h^{(j)}\right]^2 + \sum_{j=1}^{l} p_{t,j} \mu_{t-1,j}^2 - \left(\sum_{j=1}^{l} p_{t,j} \mu_{t-1,j}\right)^2, \quad (6)$$

for $t > \max\{k_j \mid j = 1, \ldots, l\}$.

Proof. See the appendix. $\square$

As in Propositions 1 and 2, Moreno (2011) and Moreno & Nieto (2014) proposed and used in their applications the results shown in Proposition 3. However, no formal analytic proof, such as the one presented here that is based on measure theory, was made to support those important results. Because of this, some not-well defined expressions were used in those papers.

We note that independent of the TAR-model parameter values, we have that the right-hand side of equality (6) is always nonnegative. Because $p_{t,j} \approx p_j$ for a t that is large enough when $\{X_t\}$ is asymptotically weak stationary, the approximate summand $\sum_{j=1}^{l} p_j [h^{(j)}]^2$ in expression (6) can be interpreted as a *communality* term in this type of conditional variance. It is very interesting to note that in the context of Mixture Autoregressive Models (MAR), expression (6) is very similar to Wong & Li's (2000) conditional variance function and that our communality term is analogous to their *base-line* conditional variance. It is worth noting that, additionally on $\tilde{x}_{t-1}$, the function given by expression (6) depends on all the regimes via the parameters $p_{t,j}$ and $[h^{(j)}]^2$, $t > \max\{k_j \mid j = 1, \ldots, l\}$ and $j = 1, \ldots, l$. Hence, this function involves more information about the conditional probabilistic mechanisms of the TAR process than the previous two.

In the following section, we shall call the above three conditional distributions Type I, Type II, and Type III, respectively.

4. Some Examples

In order to illustrate the three types of univariate conditional distributions of a TAR process and how the Type-III conditional distribution can explain the presence of large-value clusters in its sample paths, we present the following examples.

4.1. A Simulated Model

We consider the TAR(2;1,1) model given by

$$X_t = \begin{cases} -0.5 - 0.6X_{t-1} + \varepsilon_t & \text{if } Z_t \leq 0, \\ 0.9 - 0.7X_{t-1} + 10.0\varepsilon_t & \text{if } Z_t > 0, \end{cases}$$

where $Z_t = 0.5Z_{t-1} + a_t$ and $\{a_t\}$ is a zero-mean Gaussian white noise process with variance 1, which is independent of $\{\varepsilon_t\}$. Notice that $\{Z_t\}$ is a first-order Markov chain and a strict stationary stochastic process. The length of the simulated time series is 200. Figure 1 (top) shows the graph of the simulated time series $\{x_t\}$ and clusters of large values can be seen.

Since the roots of the polynomials $\phi_1(z) = 1 + 0.6z$ and $\phi_2(z) = 1 + 0.7z$, $z \in \mathbb{C}$, are outside of the unit circle, the process $\{X_t\}$ is weakly stationary. Now, $\mu_{11} = -0.31$ (rounding to two decimal digits) and $\mu_{12} = 0.53$. Clearly, $p_1 =$

$p_2 = 0.5$, thus $\mu = E(X_t) = 0.11$. Because $\psi_1(z) = \sum_{i=0}^{\infty}(-0.6)^i z^i$ and $\psi_2(z) = \sum_{i=0}^{\infty}(-0.7)^i z^i$, $\bar{\sigma}_1^2 = 1.56$ and $\bar{\sigma}_2^2 = 1.96$. In this way, the Type-I conditional variances are $\sigma_1^2 = 1.56$ and $\sigma_2^2 = 196.08$, and the marginal variance is $\mathrm{Var}(X_t) = 99$ for all t.

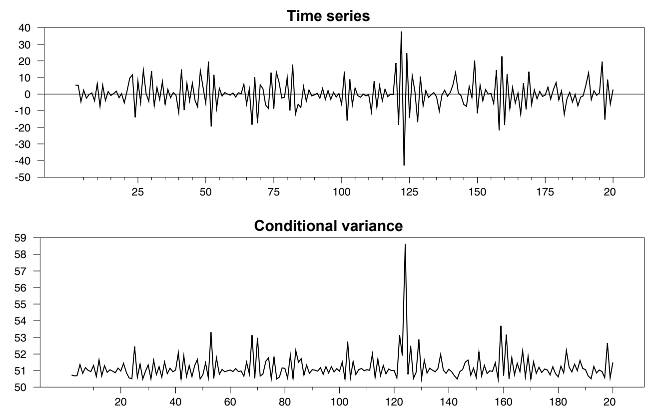

Figure 1: Time series and Type-III conditional variance function for Example 1.

As was signaled in the above remarks (2), (3), and (4) we have to be careful with the interpretation and use of the ACVF of the process $\{X_t\}$. Here, and only for theoretical illustration purposes, we describe the way in which this function can be obtained. In order to do this, we need to compute the quantities $p_{n,jk}$ and $q_{jk}(n)$ of expression (5), for $j, k = 1, 2$. Initially, we observe that $q_{12}(n) = -0.16+17.24(-0.7)^n$, $q_{21}(n) = -0.16+17.24(-0.6)^n$, $q_{11}(n) = 0.10+1.56(-0.6)^n$, and $q_{22}(n) = 0.28 + 196.08(-0.7)^n$. To calculate $p_{n,jk} = P(Z_t \in R_j, Z_{t-n} \in R_k)$, we remark that the process $\{Z_t\}$ is Gaussian; hence, all of its bivariate distributions are multinormal and $p_{n,jk}$ is a double integral of the joint pdf on the bidimensional set $R_j \times R_k$. In this paper we do not compute these values. Thus, $\gamma(n) = \sum_{j,k=1}^{2} p_{n,jk} q_{jk}(n) - 0.01$ for all $n \in \mathbb{Z}$.

From the above results, the pdf of X_t, for all t, is given by $f(x) = 0.5[f_1(x) + f_2(x)]$, $x \in \mathbb{R}$, where the pdf f_j corresponds to a normal distribution with mean μ_{1j} and variance σ_j^2, $j = 1, 2$. The distribution of X_t, conditional on past data up to $t-1$ and a regime, is normal with mean $-0.5 - 0.6x_{t-1}$ and variance 1 in the first regime and mean $0.9 - 0.7x_{t-1}$ and variance 100 in the second. Now, the Type-III conditional variance function is given by (rounding to two decimal digits)

$$\mathrm{Var}(X_t \mid \tilde{x}_{t-1}) = 50.50 - 0.58x_{t-1} + 0.75x_{t-1}^2, \ t \geq 2.$$

Here, the communality value is 50.50 and, as we can see in Figure 1, the base line for the Type-III conditional variance function is around that value. Furthermore, we can see there that this function has local extreme values in the time periods where large-value clusters of the simulated time series have occurred.

4.2. Analysis of BOVESPA Index Returns

In order to illustrate with financial data how well our proposed TAR model might explain clusters of large values in a time series, we consider the daily Dow Jones index (DJ) as the threshold variable and the daily BOVESPA index (Sao Paulo's stock exchange) as the output variable. More exactly, we set $X_t = \ln(BI_t) - \ln(BI_{t-1})$ and $Z_t = \ln(DJ_t) - \ln(DJ_{t-1})$, where BI denotes the BOVESPA index. The sample period is December 8, 2000-June 2, 2010 (2473 daily data), and in Figure 2 we plot the BOVESPA and Dow Jones returns. Morettin (2008) has undertaken extensive statistical analysis of the BOVESPA time series.

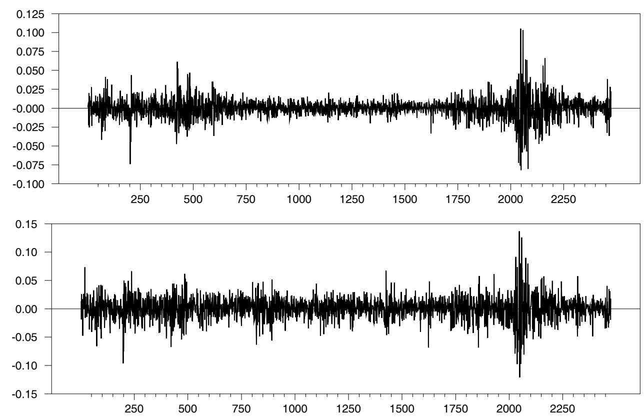

Figure 2: Dow Jones (top) and BOVESPA (bottom) returns.

Using Nieto's (2005) fitting approach, Moreno (2011) found the following TAR(3;2,0,4) model for BOVESPA returns:

$$X_t = \begin{cases} -0.0127 + 0.1113X_{t-1} - 0.0685X_{t-2} + 0.0198\varepsilon_t \ \text{ if } Z_t \leq -0.0054, \\ 6.81 \times 10^{-4} + 0.0137\varepsilon_t \ \text{ if } \ -0.0054 < Z_t \leq 0.0057, \\ 0.0135 - 0.0837X_{t-1} - 0.0684X_{t-2} - 0.1687X_{t-3} \\ -0.0633X_{t-4} + 0.0191\varepsilon_t \ \text{ if } \ 0.0057 < Z_t. \end{cases}$$

Here, the thresholds are the percentiles 25 and 75 of the Dow Jones returns (which is a strict stationary process); hence, $p_1 = p_3 = 0.25$ and $p_2 = 0.5$. Since the conditions in Proposition 1 are fulfilled, $\{X_t\}$ is a weakly stationary process, as expected. To obtain its marginal first two moments, we analyze the Type-I conditional distributions. For the first regime, the expected return is $\mu_{11} = -1.32\%$ with s.d. of 2%; for the second, the expected return is $\mu_{12} = 0.07\%$ with s.d. of 1.41%; and for the third, BOVESPA has an expected return of $\mu_{13} = 0.97\%$ with s.d. of 2%. Then, the marginal expected return is -0.05% (the empirical is 0.058%) and its marginal standard deviation is 1.7% (empirical is 2%).

Using the Type-II conditional distributions, we found that the conditional expected return at time t is 0.068% (constant) for the second regime, for the first it depends on the past two data, and for the third, on the past four observations.

The conditional variances in this case are 0.0198^2, 0.0137^2, and 0.0191^2, indicating that there is more Type-II conditional variability in the first and third regime (almost the same) than in the second regime.

Conditioning by only using the information set $\tilde{x}_{t-1}$, we found that the Type-III conditional variance function is given by

$$\begin{aligned}\operatorname{Var}(X_t \mid \tilde{x}_{t-1}) = 0.0003 + 0.25\mu_{t-1,1}^2 + 0.50\mu_{t-1,2}^2 + 0.25\mu_{t-1,3}^2 \\ - (0.25\mu_{t-1,1} + 0.50\mu_{t-1,2} + 0.25\mu_{t-1,3})^2,\end{aligned}$$

for each $t > 4$, where $\mu_{t-1,1} = -0.0127 + 0.1113x_{t-1} - 0.0685x_{t-2}$, $\mu_{t-1,2} = 0.0007$ (rounding to four decimal digits), and $\mu_{t-1,3} = 0.0135 - 0.0837x_{t-1} - 0.0684x_{t-2} - 0.1687x_{t-3} - 0.0633x_{t-4}$.

Note. Moreno (2011) fitted the following GARCH model to BOVESPA returns:

$$\begin{aligned}X_t &= 0.0012 - 0.0475X_{t-3} + a_t,\\ a_t &= \epsilon_t\sigma_t,\\ \sigma_t^2 &= 0.00001 + 0.0719a_{t-1}^2 + 0.8977\sigma_{t-1}^2,\end{aligned}$$

where $\{\epsilon_t\}$ is a zero-mean Gaussian white noise process with variance 1. We computed the GARCH-model based conditional variance function and plotted it jointly with the Type-III conditional variance function in Figure 3. *Interestingly*, the two functions locally signal the large-value clusters in the BOVESPA returns by means of their extreme values, although the GARCH-model function is smoother than that of the TAR model. Additionally, we can observe that they have almost the same base line. It is important to remark here that, *by no means*, are we trying to compare the TAR and GARCH models. Our main motivation was only to signal that the two conditional variance functions (obtained from two conceptually different models) have, approximately, the same behavior.

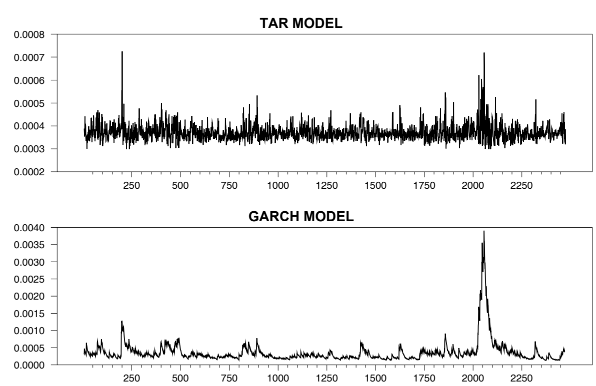

Figure 3: Conditional variance functions for BOVESPA returns.

4.3. River Flow and Rainfall Time Series

Nieto et al. (2013) fitted a TAR model to the daily rainfall (in mm.), as the input or threshold variable, and the daily flow of the Bedon river (in m^3/sec), as the output or target variable, in a geographical region on southern Colombia. Specifically, the data were collected at a meteorological station with coordinates 2.23 north (latitude) and 76.23 west (longitude) and at a hydrological station with coordinates 2.19 north and 76.15 west. These stations are located in the San Rafael's lagoon neighborhood. The sample period is January 1, 1992-November 30, 2000 (3256 data), and the data was assembled by IDEAM, the official Colombian agency for hydrological and meteorological studies. In Figure 4, one can see the two time series, where the relationship between the two variables is clear as it is the presence of large-value clusters in the river flow time series.

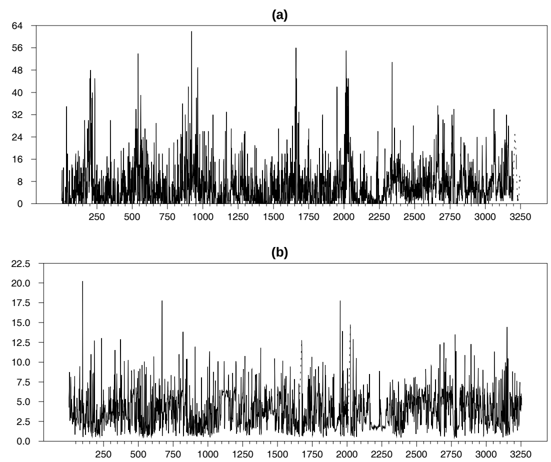

Figure 4: Rainfall (top) and river flow (bottom) time series.

Let P_t and $\tilde{X}_t$, respectively, be the rainfall and river flow at day t. Because of the universal convention for measuring these two variables, we need to make $Z_t = P_{t-1}$. As was suggested by Nieto (2005), the river flow variable needs two transformations, namely- square root of the data and an adjustment for conditional heteroscedasticity via an ARCH(1) model. We denote the adjusted river flow variable as X_t. The two time series had missing data but they were estimated using Nieto's (2005) procedure. From now on, we work with the complete (or interpolated) time series. The fitted model was a TAR(4;3,2,2,2) that is given by

$$X_t = \begin{cases} 1.35 + 0.74X_{t-1} - 0.27X_{t-2} + 0.12X_{t-3} + 1.30\varepsilon_t & \text{if } Z_t \leq 6.0 \ , \\ 1.96 + 0.75X_{t-1} - 0.30X_{t-2} + 1.66\varepsilon_t & \text{if } 6.0 < Z_t \leq 10.3 \ , \\ 2.11 + 0.80X_{t-1} - 0.28X_{t-2} + 2.15\varepsilon_t & \text{if } 10.3 < Z_t \leq 17.18 \ , \\ 2.96 + 0.62X_{t-1} - 0.35X_{t-2} + 3.18\varepsilon_t & \text{if } 17.18 < Z_t. \end{cases}$$

The stochastic dynamic of Z_t was described by means of a first-order Markov chain with identical univariate marginal distribution, which is given by $f_n(z) =$

$ph_n(z)+(1-p)g(z)$, where $p = \mathrm{P}(Z=0)$, $g(\cdot)$ denotes a truncated normal density at $z=0$ with mean 3.24 and variance 7.76^2, and

$$h_n(z) = \begin{cases} 0\ , & -\infty < z < -1/n \\ n\pi/2[\cos(nz\pi+\pi/2)]\ , & -1/n \leq z \leq 0 \\ 0\ , & z > 0, \end{cases}$$

with n a positive integer number (see Nieto's (2005) paper for its interpretation). Nieto et al. (2013) found that $p_1 = 0.60$, $p_2 = 0.20$, $p_3 = 0.12$, and $p_4 = 0.08$ for all t.

Using expression (6), we computed the conditional variance function of the river flow process, which is plotted in Figure 5. As can be see there, and from a local point of view, the time periods at which extreme values of both the conditional variance and the time series occur are similar. The communality value is 2.93.

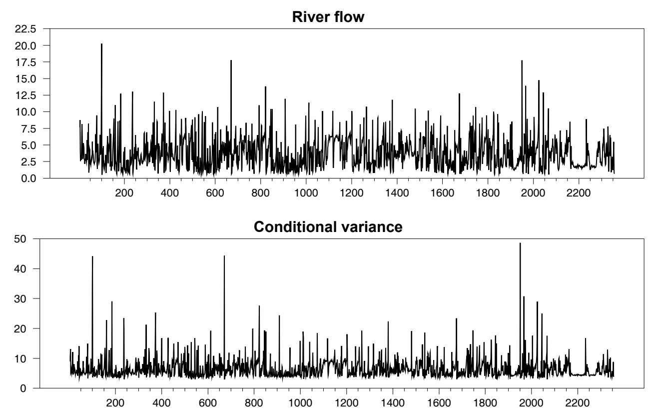

Figure 5: Conditional variance function for the river flow.

5. Conclusions

In this paper, we have characterized three types of univariate conditional distributions for a particular case of the open-loop TAR stochastic processes. We have called them Type I, Type II, and Type III conditional distributions. In the first type, the conditioning information set is a regime; in the second, the conditioning set is constituted by past information of the output variable and a regime; and in the third, the distribution is only conditional on the past information of the output variable. As a by-product, we have found that there is a sufficient condition to have asymptotic weak stationarity in the process and, under that condition and the fact that the model white noise is Gaussian, the Type-I conditional distribution is normal. The Type-III conditional distribution is a mixture of normal distributions. Using some examples, we have illustrated the fact that the Type-III conditional variance function can explain the presence of large-value clusters in time series that obey the TAR model of this paper. An interesting future investigation could be the ACVF characterization of these kinds of TAR processes.

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Appendix

Proof of Proposition 1. Because for each $j = 1, \dots, l$ the roots of the polynomial $\phi_j(z) = 1 - \sum_{i=1}^{k_j} a_i^{(j)} z^i$, $z \in \mathbb{C}$ are outside the unit circle, there is an inverse operator of $\phi_j(B)$. Let $\psi_j(B) = \phi_j(B)^{-1} = \sum_{i=0}^{\infty} \psi_i^{(j)} B^i$, with $\psi_0^{(j)} = 1$. Then, for all $x \in \mathbb{R}$,

$$F_{t,j}(x) = P(X_t \leq x \mid Z_t \in R_j) = P(a_0^{(j)}\psi_j(1) + h^{(j)}\psi_j(B)\varepsilon_t \leq x \mid Z_t \in R_j).$$

Now, the random variable $\psi_j(B)\varepsilon_t$ is well defined for all t and it has a normal distribution with mean 0 and variance $\sum_{i=0}^{\infty}[\psi_i^{(j)}]^2 = \bar{\sigma}_j^2$, given $Z_t \in R_j$ or, more precisely, $J_t = j$. Consequently,

$$\begin{aligned} F_{t,j}(x) &= P(\psi_j(B)\varepsilon_t \leq \frac{x - a_0^{(j)}\psi_j(1)}{h^{(j)}} \mid Z_t \in R_j) \\ &= P(\frac{\psi_j(B)\varepsilon_t}{\bar{\sigma}_j} \leq \frac{x - a_0^{(j)}\psi_j(1)}{h^{(j)}\bar{\sigma}_j} \mid Z_t \in R_j). \end{aligned}$$

This ends the proof.

Proof of Proposition 2. Initially, we note that from expression (2) and Proposition 1, the mean function of $\{X_t\}$ is $E(X_t) = \sum_{j=1}^{l} p_{t,j}\ \mu_{1j}$ for all t, where

$\mu_{1j} = a_0^{(j)}\psi_j(1)$. Now, in order to obtain the autocovariance function of the stochastic process $\{X_t\}$, we first obtain the bivariate cdf of variables X_t and X_{t-h} for any integer numbers h and t, which we denote as $F_{t,t-h}$.

Let $(x_t, x_{t-h}) \in \mathbb{R}^2$, then

$$\begin{aligned} F_{t,t-h}(x_t, x_{t-h}) &= P(X_t \leq x_t, X_{t-h} \leq x_{t-h}) \\ &= P_{X_t,X_{t-h}}\left\{\Big((-\infty, x_t] \times (-\infty, x_{t-h}]\Big) \bigcap \Big(\bigcup_{j,k=1}^{l} (R_j \times R_k)\Big)\right\}, \end{aligned}$$

where $P_{X_t,X_{t-h}}$ is the probability measure induced by the random vector (X_t, X_{t-h}). Now, it is easy to show that $\{R_j \times R_k : j, k = 1, \ldots, l\}$ constitutes a partition of $\mathbb{R}^2$ because the regimes R_j, $j = 1, \ldots, l$, constitute a partition of $\mathbb{R}$. In this sense, we can say that $\{R_j \times R_k : j, k = 1, \ldots, l\}$ is the set of bidimensional regimes in $\mathbb{R}^2$. Hence,

$$F_{t,t-h}(x_t, x_{t-h}) = \sum_{j,k=1}^{l} P_{X_t,X_{t-h}}\Big\{\Big((-\infty, x_t] \times (-\infty, x_{t-h}]\Big) \bigcap \Big(R_j \times R_k\Big)\Big\}.$$

Equivalently,

$$\begin{aligned} F_{t,t-h}(x_t, x_{t-h}) = \sum_{j,k=1}^{l} & P(X_t \leq x_t, X_{t-h} \leq x_{t-h} \mid Z_t \in R_j, Z_{t-h} \in R_k) \\ & \times P(Z_t \in R_j, Z_{t-h} \in R_k). \end{aligned}$$

Let $P(Z_t \in R_j, Z_{t-h} \in R_k) = p_{t,t-h,jk}$ for all $j, k = 1, \ldots, l$ and $t \in \mathbb{Z}$ and

$$F_{t,t-h;jk}(x_t, x_{t-h}) = P(X_t \leq x_t, X_{t-h} \leq x_{t-h} \mid Z_t \in R_j, Z_{t-h} \in R_k),$$

then, we obtain that

$$F_{t,t-h}(x_t, x_{t-h}) = \sum_{j,k=1}^{l} p_{t,t-h,jk} F_{t,t-h;jk}(x_t, x_{t-h}),$$

where $\sum_{j,k=1}^{l} p_{t,t-h,jk} = 1$. This means that the joint cdf of X_t and X_{t-h} is a mixture of conditional bivariate cdf's. Thus,

$$E(X_t X_{t-h}) = \sum_{j,k=1}^{l} p_{t,t-h,jk} E(X_t X_{t-h} \mid Z_t \in R_j, Z_{t-h} \in R_k).$$

Following the proof of Proposition 1, we obtain $X_t = \mu_{1j} + h^{(j)} \sum_{s=0}^{\infty} \psi_s^{(j)} \varepsilon_{t-s}$ if $Z_t \in R_j$ ($\psi_0^{(j)} = 1$), and $X_{t-h} = \mu_{1k} + h^{(k)} \sum_{m=0}^{\infty} \psi_m^{(k)} \varepsilon_{t-h-m}$ if $Z_{t-h} \in R_k$. Then,

$$E(X_t X_{t-h} \mid Z_t \in R_j, Z_{t-h} \in R_k) = \mu_{1j}\mu_{1k} + h^{(j)} h^{(k)} \sum_{m=0}^{\infty} \psi_m^{(k)} \psi_{h+m}^{(j)},$$

where, without loss of generality, we assumed that $h > 0$. Consequently,

$$E(X_t X_{t-h}) = \sum_{j,k=1}^{l} p_{t,t-h,jk} \left(\mu_{1j}\mu_{1k} + h^{(j)}h^{(k)} \sum_{m=0}^{\infty} \psi_m^{(k)} \psi_{h+m}^{(j)} \right),$$

and

$$\begin{aligned}\operatorname{Cov}(X_t, X_{t-h}) = \sum_{j,k=1}^{l} p_{t,t-h,jk} & \left(\mu_{1j}\mu_{1k} + h^{(j)}h^{(k)} \sum_{m=0}^{\infty} \psi_m^{(k)} \psi_{h+m}^{(j)} \right) \\ & - \left(\sum_{j=1}^{l} p_{t,j}\mu_{1j} \right) \left(\sum_{j=1}^{l} p_{t-h,j}\mu_{1j} \right).\end{aligned}$$

This completes the proof.

Proof of Proposition 3. Let $t > \max\{k_j \mid j = 1, \ldots, l\}$ be given, then we consider $\tilde{x}_{t-1}$ as a realization of the random vector $\mathbf{X}_{t-1}$. In order to use Billingsley's (1995) concept of a conditional probability *measure*, we set the following additional notation and theoretical framework. Let $(\mathbb{R}, \mathcal{B})$ be the unidimensional Borel space, λ the corresponding Lebesgue measure, and P_{J_t} the probability induced by J_t on the measure space $(\mathbb{N}, \mathcal{P}, \mu)$. Finally, let $P_{X_t|\tilde{x}_{t-1}}$ be the conditional probability measure associated with X_t, given $\tilde{x}_{t-1}$, on the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, $P_{X_t,J_t|\tilde{x}_{t-1}}$ be the conditional probability measure corresponding to the random vector (X_t, J_t) given $\tilde{x}_{t-1}$ on the product measure space $(\mathbb{R} \times \mathbb{N}, \mathfrak{F}_1, \lambda \times \mu)$, $P_{X_t|j,\tilde{x}_{t-1}}$ the conditional probability of X_t given $j \in \{1, \ldots, l\}$ and $\tilde{x}_{t-1}$ on the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, and $P_{J_t|\tilde{x}_{t-1}}$ the conditional probability of J_t given $\tilde{x}_{t-1}$ on the measure space $(\mathbb{N}, \mathcal{P}, \mu)$.

Let g be the Radon-Nikodym derivative of $P_{(\mathbf{X}_t, J_t)}$ with respect to $\lambda_t \times \mu$, then g is a probability density function (pdf) of $(\mathbf{X}_t, J_t)$ with respect to $\lambda_t \times \mu$. Using Fubini's theorem,

$$f(\mathbf{x}) = \int_{\mathbb{N}} g(\mathbf{x}, j) d\mu(j), \ \mathbf{x} \in \mathbb{R}^t,$$

is a nonnegative Borel function on the measurable space $(\mathbb{R}^t, \mathcal{B}^t)$ and integrable w.r.t. λ_t. Now, let $A \in \mathcal{B}^t$ and $P_{\mathbf{X}_t}$ be the induced probability of $\mathbf{X}_t$ on $(\mathbb{R}^t, \mathcal{B}^t)$; then,

$$\begin{aligned}\int_A f d\lambda_t &= \int_{A \times \mathbb{N}} g d(\lambda_t \times \mu) \\ &= P_{(\mathbf{X}_t, J_t)}(A \times \mathbb{N}) \\ &= P_{\mathbf{X}_t}(A),\end{aligned}$$

because of $(\mathbf{X}_t, J_t)^{-1}(A \times \mathbb{N}) = \mathbf{X}_t{}^{-1}(A)$, with the exponent -1 denoting inverse image. Hence, f is a pdf of $\mathbf{X}_t$ w.r.t. λ_t.

Using Shao's (2003, pp. 44) book and letting h be the pdf of $\mathbf{X}_{t-1}$ w.r.t. λ_{t-1}, we obtain that $X_t \mid \tilde{x}_{t-1}$ has a pdf with respect to the Lebesgue measure on the measurable space $(\mathbb{R}, \mathcal{B})$. This is given by

$$
\begin{aligned}
f(x \mid \tilde{x}_{t-1}) &= \frac{f[(x, \tilde{x}_{t-1})]}{h(\tilde{x}_{t-1})} \\
&= \frac{\sum_{j=1}^{l} g[(x, \tilde{x}_{t-1}), j]}{h(\tilde{x}_{t-1})}
\end{aligned}
$$

if $h(\tilde{x}_{t-1}) > 0$.

Again, using Shao's (2003, pp. 44) result, we obtain that (X_t, J_t) conditional on $\tilde{x}_{t-1}$ has a pdf w.r.t. the product measure $\lambda \times \mu$, $g(x, j \mid \tilde{x}_{t-1})$ say. Then, because of Shao's (2003) Theorem 1.7, $g[(x, \tilde{x}_{t-1}), j] = g(x, j \mid \tilde{x}_{t-1})h(\tilde{x}_{t-1})$. In the same manner, we obtain that $X_t \mid j, \tilde{x}_{t-1}$ and $J_t \mid \tilde{x}_{t-1}$ have a pdf w.r.t. λ and μ, respectively. Let them be $g(x \mid j, \tilde{x}_{t-1})$ and $k(j \mid \tilde{x}_{t-1})$, respectively. Then, $g(x, j \mid \tilde{x}_{t-1}) = g(x \mid j, \tilde{x}_{t-1})k(j \mid \tilde{x}_{t-1})$, where $k(j \mid \tilde{x}_{t-1}) = P_{J_t|\tilde{x}_{t-1}}(\{j\})$. Since Z_t (consequently J_t) is independent of $\mathbf{X}_{t-1}$, $P_{J_t|\tilde{x}_{t-1}}(\{j\}) = P_{J_t}(\{j\}) = p_{t,j}$. In this way,

$$
f(x \mid \tilde{x}_{t-1}) = \sum_{j=1}^{l} p_{t,j} g(x \mid j, \tilde{x}_{t-1}),
$$

where, for each $j = 1, \ldots, l$, $g(x \mid j, \tilde{x}_{t-1})$ is the pdf of the $N(\mu_{t-1,j}, [h^{(j)}]^2)$. Now, $\mathrm{Var}(X_t \mid \tilde{x}_{t-1}) = \mathrm{E}(X_t^2 \mid \tilde{x}_{t-1}) - (\mathrm{E}(X_t \mid \tilde{x}_{t-1}))^2$. This completes the proof.