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Form-Invariance of the Non-Regular Exponential Family of Distributions

Distribuciones de forma invariante de la familia exponencial no regular

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Abstract

The weighted distributions are used when the sampling mechanism records observations according to a nonnegative weight function. Sometimes the form of the weighted distribution is the same as the original distribution except possibly for a change in the parameters that is called the form-invariant weighted distribution. In this paper, by identifying a general class of weight functions, we introduce an extended class of form-invariant weighted distributions belonging to the non-regular exponential family which included two common families of distribution: exponential family and non-regular family as special cases. Some properties of this class of distributions such as the sufficient and minimal sufficient statistics, maximum likelihood estimation and the Fisher information matrix are studied.

***Key words*:** Fisher information matrix; Form-invariance; Non-regular exponential family; Maximum likelihood estimation; Weighted distribution.

Resumen

Las distribuciones ponderadas son usadas cuando el mecanismo de muestreo registra observaciones de acuerdo a una función no negativa. En ocasiones la forma de la función ponderada es igual a la original, excepto, posiblemente, en un cambio de parámetros y se denominan distribuciones ponderadas de forma invariante. En este artículo identificamos una clase general de funciones ponderadas e introducimos una forma extendida de distribuciones ponderadas de forma invariante, la cual incluye dos familias comunes: la familia exponencial y la familia no regular como caso particular. Algunas propiedades de estas distribuciones como las estadísticas suficientes y máximas suficientes, la estimación de máxima verosimilitud y la matriz de información de Fisher son estudiadas.

***Palabras clave*:** Distribución ponderada; estimación de máxima verosimilitud; familia exponencial no regular; invarianza de forma; matriz de información de Fisher.

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1. Introduction

The weighted distributions have been used when the sampling mechanism records observations according to a certain chance. For example, suppose that the random variable X has the probability density function (pdf) $f(x;\theta)$ with parameter $\theta \in \Theta$ and that the probability of the recording observation x of X be proportional to a non-negative weight function $w(x,\theta,\beta)$ where β is a known or unknown parameter, and that assumed $E\big[w(X,\theta,\beta)\big]$ exists. Then the recorded x is an observation of X^w, having a pdf

$$f^w(x;\theta,\beta) = \frac{w(x,\theta,\beta)f(x;\theta)}{E\big[w(X,\theta,\beta)\big]}. \tag{1}$$

The weighted distributions are widely applied in various studies such as line transect sampling, renewal theory, branching process, bio-medicine, statistical ecology and reliability modeling. Some references are Patil & Rao (1978); Rao (1958); Gupta & Kirmani (1990); Oluyede & George (2002) and Gupta & Keating (1986).

An important property of weighted distributions is that the original distribution keeps the same form under weighting except possibly for a change in the parameters. This is known as the form-invariant property of weighted distributions. According to Patil & Ord (1976), the distribution of X is said to be form-invariant under the weight function $w(x,\theta)$ when

$$f^w(x;\theta) = f(x;\eta), \tag{2}$$

where $\eta = \eta(\theta) \in \Theta$. The form-invariance is a useful property for the weighted distributions because the estimate of the population parameters based on a random variable could be used in weighted cases, with considerable modification (Nair & Sunoj (2003)). Patil & Ord (1976) proved that a necessary and sufficient condition for X to be form-invariant under size-biased weight function of order β (i.e. $w(x,\beta) = x^\beta$) is that its distribution belongs to log-exponential family with pdf

$$f(x;\theta) = \exp\{\theta \log x + a(x) - c(\theta)\}.$$

Sankaran & Nair (1993) and Sindu (2002), respectively, derived the conditions which the Pearson and generalized Pearson families of distributions are form-invariant under $w(x) = x$. Alavi & Chinipardaz (2009) developed the Patil & Ord (1976) results and gave the necessary and sufficient conditions for the form-invariance of the distributions belong to exponential family with pdf

$$f(x;\theta) = \exp\{\theta \log \nu(x) + a(x) - c(\theta)\}, \tag{3}$$

under the general weight function $w(x,\beta) = [\nu(x)]^{h(\beta)}$. Esparza (2013) proved that two discrete distribution phase-type and matrix-geometric distributions are form-invariant under the size-biased factorial sampling of order β (i.e. $w(x,\beta) = x^{(\beta)} = x(x-1)\cdots(x-\beta+1)$).

There are the distributions that their pdf form can be written as the function form of (3) and also are form-invariant under the weight function $w(x,\beta) =$

$[\nu(x)]^{h(\beta)}$ but do not belong to the exponential family. For example, consider the exponential distribution with two unknown parameters of α and θ and the pdf

$$f(x;\alpha,\theta)=\exp\{\theta\log\nu^*(x)+a^*(x)-c^*(\theta)\}\,I_{(\alpha,\infty)}(x),$$

where $\nu^*(x)=e^{-x}$, $a^*(x)=0$ and $c^*(\theta)=-(\theta\alpha+\log\theta)$. X is form-invariant under $w(x,\beta)=[\nu^*(x)]^\beta=e^{-\beta x}$ since $X^w\sim f^w(x;\theta)=e^{-(\theta-\beta)(x-\alpha)}\,I(x>\alpha)$, but does not belong to exponential family.

In this paper, we study the form-invariant property of the extended class of distributions, termed as the non-regular exponential family with pdf, given by

$$f(x;\theta)=\exp\Big\{\sum_{i=1}^{p}b_i(\theta)\,d_i(x)+a(x)+c(\theta)\Big\}\,I_{\big(\pi_1(\theta),\pi_2(\theta)\big)}(x), p=1,2, \tag{4}$$

where $\pi_1(\theta)<\pi_2(\theta)$ for all $\theta\in\Theta$. The proposed class of the distributions include two common families of distributions; non-regular family and the exponential family, as particular cases. We identified the class of the weight functions

$$W=\Big\{w(.):\ w(x,\theta,\beta)=\exp\Big\{\sum_{i=1}^{p}\beta_i\,d_i(x)\Big\}\,I_{\big(\pi_1^*(\theta),\pi_2^*(\theta)\big)}(x)\Big\},$$

where $\big(\pi_1^*(\theta),\pi_2^*(\theta)\big)\subseteq\big(\pi_1(\theta),\pi_2(\theta)\big)$ and proved that a necessary and sufficient condition for a distribution to be form-invariant under $w\in W$ is that its pdf is the form (4).

For further insights on the form-invariance, we have extended the study to the two-parameter distributions that belong to the non-regular exponential family and proved the necessary and sufficient conditions for the form-invariance property. Finally, some properties of the distributions belong to the nonregular-exponential family such as the sufficient and minimal sufficient statistics, maximum likelihood estimation and the Fisher information matrix are given.

2. Form-Invariance in One-Parameter Distributions

In this Section, we extend the Patil & Ord (1976) and Alavi & Chinipardaz (2009) results for a larger class of distributions such as nonregular-exponential family.

Theorem 1. *Let X be a random variable that changes over an open interval $I=\big(\pi_1(\theta),\pi_2(\theta)\big)$ with $-\infty\le\pi_1(\theta)<\pi_2(\theta)\le\infty$ where θ is a real scalar parameter. Also let*

$$W^*=\Big\{w(.):\ w(x,\theta,\beta)=\exp\big\{\beta\,d(x)\big\}\,I_{\big(\pi_1^*(\theta),\pi_2^*(\theta)\big)}(x)\Big\},$$

be a class of weight functions which β is constant, $d(x)$ is only function of x with $E[d(X)]<\infty$ and also π_1^* and π_2^* are parametric functions of θ which $\pi_1(\theta)\le\pi_1^*(\theta)<\pi_2^*(\theta)\le\pi_2(\theta)$ for all $\theta\in\Theta$. Suppose the following conditions are satisfied;

(a) $\lim_{\eta\to\theta}\frac{\beta}{\eta-\theta}$ *and* $\lim_{\beta\to 0}\frac{\eta-\theta}{\beta}$ *are finite where* η *is a parametric function of* θ.

(b) $E\Big[\exp\big\{\beta\, d(X)\big\}\, I_{\big(\pi_1^*(\theta),\pi_2^*(\theta)\big)}(X)\Big]$ *and* $\lim_{\eta\to\theta}\frac{\log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)}\exp\{\beta d(x)\}\, f(x;\theta)\, dx}{\eta-\theta}$ *are both finite.*

Then the distribution of X *is form-invariant under* $w\in W^*$ *if and only if its pdf is as*

$$f(x;\theta)=\exp\big\{\theta d(x)+a(x)-c(\theta)\big\}\, I_{\big(\pi_1(\theta),\pi_2(\theta)\big)}(x), \tag{5}$$

which $c(\theta)=\log\int_{\pi_1(\theta)}^{\pi_2(\theta)}\exp\big\{\theta\, d(x)+a(x)\big\}dx<\infty$.

Proof. According to the difinition of form-invarince in (2), it is clear that for each f, η is a function of β with $\eta=\theta$ when $\beta=0$ and

$$\big(\pi_1(\theta),\pi_2(\theta)\big)=\big(\pi_1^*(\theta),\pi_2^*(\theta)\big),\ \forall\theta\in\Theta.$$

Therefore, it can be said that $\eta=\theta$ if and only if $\beta=0$ and $\big(\pi_1(\theta),\pi_2(\theta)\big)=\big(\pi_1^*(\theta),\pi_2^*(\theta)\big)$. Suppose that X has the pdf (5), then

$$f^w(x;\theta)=\exp\big\{\eta d(x)+a(x)-c^w(\eta)\big\}\, I_{\big(\pi_1^*(\eta-\beta),\pi_2^*(\eta-\beta)\big)}(x),$$

where $\eta=\theta+\beta,\ \lim_{\eta\to\theta}\frac{\beta}{\eta-\theta}=\lim_{\beta\to 0}\frac{\eta-\theta}{\beta}=1$ and

$$\begin{aligned} E\Big[\exp\big\{\beta d(x)\big\}\, I_{\big(\pi_1^*(\theta),\pi_2^*(\theta)\big)}(X)\Big] &= \int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)}\exp\big\{(\beta+\theta)d(x)+a(x)-c(\theta)\big\}\, dx\\ &=\exp\big\{-c(\theta)+c^w(\eta)\big\}<\infty. \end{aligned}$$

Also, we have

$$\begin{aligned} c^w(\eta)\Big|_{\eta=\theta} &= \log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)}\exp\big\{\eta\, d(x)+a(x)\big\}dx\Big|_{\eta=\theta}\\ &=\log\int_{\pi_1(\theta)}^{\pi_1(\theta)}\exp\big\{\theta d(x)+a(x)\big\}dx\\ &=c(\theta) \end{aligned}$$

and

$$
\begin{aligned}
\lim_{\eta\to\theta} \frac{\log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} e^{\beta d(x)}\, f(x;\theta)\, dx}{\eta-\theta} &= \lim_{\eta\to\theta} \frac{\log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} \exp\left\{(\beta+\theta)d(x)+a(x)-c(\theta)\right\} dx}{\eta-\theta} \\
&= \lim_{\eta\to\theta} \frac{c^w(\eta)-c(\theta)}{\eta-\theta} \\
&= \lim_{\eta\to\theta} \frac{c^w(\eta)-c^w(\theta)}{\eta-\theta} \\
&= \frac{\partial}{\partial\eta} c^w(\eta)\Big|_{\eta=\theta} \\
&= \lim_{\delta\to 0} \frac{c^w(\eta+\delta)-c^w(\eta)}{\delta}\Big|_{\eta=\theta} \\
&= \lim_{\delta\to 0} \frac{1}{\delta}\Bigg(\log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} \exp\left\{(\eta+\delta)d(x)+a(x)\right\} dx \\
&\qquad - \log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} \exp\left\{(\eta d(x)+a(x)\right\} dx\Bigg)\Big|_{\eta=\theta}. \qquad (6)
\end{aligned}
$$

By an application of Dominated Convergence Theorem (see Billingsley 1979) and the Hospital's rule, the equation (6) is as

$$
\begin{aligned}
&= \lim_{\delta\to 0} \frac{\frac{\partial}{\partial\delta}\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} \exp\left\{(\eta+\delta)d(x)+a(x)\right\} dx}{\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} \exp\left\{(\eta+\delta)d(x)+a(x)\right\} dx}\Bigg|_{\eta=\theta} \\
&\overset{DCT}{=} e^{-c^w(\eta)} \int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} d(x)\exp\left\{\eta d(x)+a(x)\right\} dx\Big|_{\eta=\theta} \\
&= e^{-c(\theta)} \int_{\pi_1(\theta)}^{\pi_2(\theta)} d(x)\exp\left\{\theta d(x)+a(x)\right\} dx \\
&= E\Big[d(X)\Big] < \infty,
\end{aligned}
$$

which the notation 'DCT' in the proof is abbrivation of the Domiated Convergence Theoerm. To prove the sufficiency, we note that

$$
\frac{\log f(x;\eta)-\log f(x;\theta)}{\eta-\theta} = \frac{\beta}{\eta-\theta}\, d(x) - \frac{\log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} e^{\beta d(x)}\, f(x;\theta)\, dx}{\eta-\theta}. \qquad (7)
$$

Taking limit of both sides as $\eta\to\theta$ and using the conditions in theorem 1, so that (7) becomes

$$
\frac{\partial}{\partial\theta}\log f(x;\theta) = b_1(\theta)\, d(x) + b_2(\theta), \qquad (8)
$$

where $b_1(\theta) = \lim_{\eta\to\theta}\frac{\beta}{\eta-\theta}$ and

$$
b_2(\theta) = \lim_{\eta\to\theta} \frac{-\log\int_{\pi_1^*(\theta)}^{\pi_2^*(\theta)} e^{\beta d(x)}\, f(x;\theta)\, dx}{\eta-\theta} < \infty.
$$

Now, integrating with respect to θ in (8), we get

$$\log f(x;\theta) = B_1(\theta)\, d(x) + C_1(\theta) + a(x)$$

where $B_1(\theta) = \int b_1(\theta)\, d\theta$, $C_1(\theta) = \int b_2(\theta)\, d\theta$ and $a(x)$ is the constant coefficient of integration. Thus the pdf of X must be the form (5). □

Corollary 1. *When the functions of π_1 and π_2 in* (5) *are independent of θ, the resultant distribution is belong to exponential family. In this case, X is form-invariant under the weight function $w(x;\beta) = \exp\{\beta\, d(x)\}$ if and only if its pdf has the form* (3)*.*

This is studied by Patil & Ord (1976) and Alavi & Chinipardaz (2009) by defining, respectively, $d(x) = \log \nu(x)$ and $d(x) = \log x$.

Corollary 2. *The non-regular family of distributions with the pdf as*

$$f(x;\theta) = c(\theta)h(x)\, I_{(\pi_1(\theta),\pi_2(\theta)\,)}(x)$$

is a special case of (5) *when $d(x) = 0$. In this case, X is form-invariant under the weight function $w(x,\theta) = I_{(\pi_1^*(\theta),\pi_2^*(\theta)\,)}(x)$ with $(\pi_1^*(\theta), \pi_2^*(\theta)\,) \subseteq (\pi_1(\theta), \pi_2(\theta)\,)$ for all $\theta \in \Theta$ if and only if the distribution of X belongs to non-regular family.*

Some examples of the form-invariant one-parameter distributions belong to the non-regular exponential family with the common weight functions are given in Table 1.

3. Form-Invariance in Two-Parameter Distributions

The pdf of the multi-parameter distributions in nonregular exponential family is defined as

$$f(x;\boldsymbol{\theta}) = \exp\Big\{\sum_{i=1}^{p} b_i(\boldsymbol{\theta})d_i(x) + a(x) - k(\boldsymbol{\theta})\Big\}\, I_{\big(U_1(\boldsymbol{\theta}),U_2(\boldsymbol{\theta})\big)}(x), \tag{9}$$

where $\boldsymbol{\theta} \in \boldsymbol{\theta}$ is a vector of parameters. By reparametrizing of (9), as $\alpha_i = b_i(\boldsymbol{\theta})$ and $p = 2$, the pdf of two-parameter distributions in nonregular exponential family is written as

$$f(x;\boldsymbol{\alpha}) = \exp\Big\{\sum_{i=1}^{2} \alpha_i d_i(x) + a(x) - c(\boldsymbol{\alpha})\Big\}\, I_{\big(\pi_1(\boldsymbol{\alpha}),\pi_2(\boldsymbol{\alpha})\big)}(x), \tag{10}$$

which $\boldsymbol{\alpha} = \big(\alpha_1, \alpha_2\big)$ and

$$c(\boldsymbol{\alpha}) = \log \int_{\pi_1(\boldsymbol{\alpha})}^{\pi_2(\boldsymbol{\alpha})} \exp\Big\{\sum_{i=1}^{2} \alpha_i d_i(x) + a(x)\Big\}\, dx < \infty. \tag{11}$$

Alavi & Chinipardaz (2009) gave some examples of the form-invariant two-parameter distributions belong to exponential family but they could not give a general expression. In this Section, we show that the form-invariant property is extensible for all two-parameter distributions belong to non-regular exponential family.

TABLE 1: Form-invariant one-parameter distributions in non-regular exponential family.

Distribution	θ	Weight function	η
Normal (μ, σ)	μ	$e^{\beta x}$	$\mu + \beta\sigma^2$
Normal (μ, σ)	μ	$e^{\beta x^2}$	$\frac{\mu}{1-2\beta\sigma^2}$
Uniform$(\theta, \theta+1)$	θ	$I_{(\theta,\theta+\beta+1)}(x)$ $-1 < \beta < 0$	$(\theta, \theta+\beta+1)$
Uniform$(\theta, \theta+1)$	θ	$I_{(\theta+\beta,\theta+1)}(x)$ $0 < \beta < 1$	$(\theta+\beta, \theta+1)$
Uniform$(0, \theta)$	θ	$I_{(0,\beta\theta)}(x)$ $0 < \beta < 1$	$\beta\theta$
Pareto(θ, α)	θ	$I_{(\theta^2+\beta,+\infty)}(x)$ $\beta > 0$	$\theta^2 + \beta$
Pareto(θ, α)	α	x^{β}	$\alpha - \beta$
Exponential(θ)	θ	$e^{-\lvert\beta\rvert x}$	$(\frac{1}{\theta} + \lvert\beta\rvert)^{-1}$
Gamma(α, λ)	α	x^{β}	$\alpha + \beta$
Gamma(α, λ)	λ	$e^{-\beta\, x}$ $\beta > 0$	$(\frac{1}{\lambda} + \beta)^{-1}$
Laplace$(0, \theta)$	θ	$e^{\beta\,\lvert x\rvert}$ $1 - \beta\theta > 0$	$\frac{\theta}{1-\beta\theta}$
Laplace$(0, \theta)$	θ	$\beta^{\frac{-\lvert x\rvert}{\theta}}$ $\beta > e^{-1}$	$\theta\,(\log\beta + 1)^{-1}$
Weibull(τ, θ)	θ	$e^{-\beta(\frac{x}{\theta})^{\tau}}$ $\beta > -1$	$\theta\,(\beta+1)^{\frac{1}{\tau}}$
Beta(α, λ)	α	x^{β}	$\alpha + \beta$
Beta(α, λ)	λ	$(1-x)^{\beta}$	$\lambda + \beta$

Theorem 2. *Let the random variable* X*, with the pdf* $f(x;\boldsymbol{\alpha})$*, changes over an open interval* $I = \big(\pi_1(\boldsymbol{\alpha}), \pi_2(\boldsymbol{\alpha})\big)$*. Also let*

$$W = \left\{ w(.) : \, w(x, \boldsymbol{\alpha}, \beta_1, \beta_2) = \, \exp\left\{\beta_1 d_1(x) + \beta_2 d_2(x)\right\} I_{\left(\pi_1^*(\boldsymbol{\alpha}), \pi_2^*(\boldsymbol{\alpha})\right)}(x) \right\}, \tag{12}$$

be a class of weight functions which $-\infty < \pi_1(\boldsymbol{\alpha}) \le \pi_1^*(\boldsymbol{\alpha}) < \pi_2^*(\boldsymbol{\alpha}) \le \pi_2(\boldsymbol{\alpha}) < \infty$*. If the following conditions are satisfied;*

(a) $E\big[d_i(X)\big]$ *and* $E\Big[\exp\Big\{\sum_{i=1}^{2}\beta_i d_i(X)\Big\} I_{\left(\pi_1^*(\boldsymbol{\alpha}), \pi_2^*(\boldsymbol{\alpha})\right)}(X)\Big]$ *are finite.*

(b) For $i, j = 1$ *and 2,* $\lim_{\eta_i \to \theta_i} \frac{\beta_j}{\eta_i - \theta_i}$ *are finite where* η_i *are functions of* α, β_1 *and* β_2*.*

(c) For $i \neq j = 1$ *and 2,* $\lim_{\eta_i \to \alpha_i} \frac{\log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\left\{\beta_1 d_1(x) + \beta_2 d_2(x)\right\} f(x;\boldsymbol{\alpha})\, dx}{\eta_i - \alpha_i}\Bigg|_{\eta_j = \alpha_j}$ *converge uniformly in* $\big(\pi_1^*(\boldsymbol{\alpha}), \pi_2^*(\boldsymbol{\alpha})\big)$*.*

Then, the distribution of X *is form-invariant under* $w \in W$ *if and only if the pdf is the form of* (10)*.*

Proof. According to (2), for each f, $\boldsymbol{\eta} = \boldsymbol{\alpha}$ if and only if $(\beta_1, \beta_2) = (0, 0)$ and $\big(\pi_1(\boldsymbol{\alpha}), \pi_2(\boldsymbol{\alpha})\big) = \big(\pi_1^*(\boldsymbol{\alpha}), \pi_2^*(\boldsymbol{\alpha})\big)$. Suppose that the pdf of X is of the form (10) then X is form-invariant with $\eta_i = \alpha_i + \beta_i$ $(i = 1, 2)$,

$$\lim_{\eta_i \to \alpha_i} \frac{\beta_j}{\eta_i - \alpha_i} = \begin{cases} \frac{\beta_j}{\beta_i} & i \neq j = 1, 2 \\ 1 & i = j = 1, 2, \end{cases}$$

$$\begin{aligned} E\big[d_i(X)\big] &= \int_{\pi_1(\boldsymbol{\alpha})}^{\pi_2(\boldsymbol{\alpha})} d_i(x)\, f(x; \boldsymbol{\alpha})\, dx \\ &\leq \int_{-\infty}^{\infty} d_i(x) \, \exp\Big\{ \sum_{i=1}^{2} \alpha_i d_i(x) + a(x) - c(\boldsymbol{\alpha}) \Big\}\, dx \\ &= \frac{\partial}{\partial \alpha_i} c(\boldsymbol{\alpha}) < \infty \end{aligned}$$

and

$$\begin{aligned} E\Big[\exp\Big\{\sum_{i=1}^{2} \beta_i d_i(X)\Big\}\, I_{\big(\pi_1^*(\boldsymbol{\alpha}), \pi_2^*(\boldsymbol{\alpha})\big)}(X)\Big] &= \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} d_i(x) \, \exp\Big\{ \sum_{i=1}^{2} \alpha_i d_i(x) + a(x) - c(\boldsymbol{\alpha}) \Big\}\, dx \\ &= \exp\big\{c^w(\boldsymbol{\eta}) - c(\boldsymbol{\alpha})\big\} < \infty, \end{aligned}$$

where $c^w(\boldsymbol{\eta}) = \log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha}} \exp\big\{ \sum_{i=1}^{2} \eta_i d_i(x) + a(x) \}dx$ with $\eta_i = \alpha_i + \beta_i$, $i = 1, 2$. For $i \neq j = 1, 2$, we have

$$\begin{aligned} &\lim_{\eta_i \to \alpha_i} \frac{\log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha}} \exp\big\{\beta_1 d_1(x) + \beta_2 d_2(x)\big\}\, f(x; \boldsymbol{\alpha})\, dx}{\eta_i - \alpha_i}\Big|_{\eta_j = \alpha_j} \\ &\quad = \lim_{\eta_i \to \alpha_i} \frac{\log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\big\{\eta_1 d_1(x) + \eta_2 d_2(x) + a(x) - c(\boldsymbol{\alpha})\big\}\, dx}{\eta_i - \alpha_i}\Big|_{\eta_j = \alpha_j}. \end{aligned} \tag{13}$$

For example for $i = 1$, the equation (13) is

$$
\begin{aligned}
&= \lim_{\eta_1 \to \alpha_1} \frac{\log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\left\{\eta_1 d_1(x) + \alpha_2 d_2(x) + a(x) - c(\boldsymbol{\alpha})\right\} dx}{\eta_1 - \alpha_1} \\
&= \lim_{\eta_1 \to \alpha_1} \frac{-c(\boldsymbol{\alpha}) + \log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\left\{\eta_1 d_1(x) + \alpha_2 d_2(x) + a(x)\right\}}{\eta_1 - \alpha_1} \\
&= \lim_{\eta_1 \to \alpha_1} \frac{-c(\boldsymbol{\alpha}) + c^w(\eta_1, \alpha_2)}{\eta_1 - \alpha_1} \\
&= \lim_{\eta_1 \to \alpha_1} \frac{-c^w(\alpha_1, \alpha_2) + c^w(\eta_1, \alpha_2)}{\eta_1 - \alpha_1} \\
&= \frac{\partial}{\partial \eta_1} c^w(\eta_1, \alpha_2)\bigg|_{\eta_1 = \alpha_1} \\
&= \lim_{\delta \to 0} \frac{c^w(\eta_1 + \delta, \alpha_2) - c^w(\eta_1, \alpha_2)}{\delta}\bigg|_{\eta_1 = \alpha_1} \\
&= \lim_{\delta \to 0} \frac{1}{\delta}\Bigg(\log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\left\{(\eta_1 + \delta) d_1(x) + \alpha_2 d_2(x) + a(x)\right\} dx \\
&\qquad - \log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\left\{\eta_1 d_1(x) + \alpha_2 d_2(x) + a(x)\right\} dx \Bigg)\Bigg|_{\eta_1 = \alpha_1} \\
&\overset{Hospital's\, rule}{=} \lim_{\delta \to 0} \frac{\frac{\partial}{\partial \delta} \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\left\{(\eta_1 + \delta) d_1(x) + \alpha_2 d_2(x) + a(x)\right\} dx}{\int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\left\{(\eta_1 + \delta) d_1(x) + \alpha_2 d_2(x) + a(x)\right\} dx}\Bigg|_{\eta_1 = \alpha_1} \\
&\overset{DCT}{=} \frac{\int_{\pi_1(\boldsymbol{\alpha})}^{\pi_2(\boldsymbol{\alpha})} d_1(x) \exp\left\{\alpha_1 d_1(x) + \alpha_2 d_2(x) + a(x)\right\} dx}{\int_{\pi_1(\boldsymbol{\alpha})}^{\pi_2(\boldsymbol{\alpha})} \exp\left\{\alpha_1 d_1(x) + \alpha_2 d_2(x) + a(x)\right\} dx} \\
&= E\big[d_1(X)\big] < \infty.
\end{aligned}
$$

To prove the sufficiency, we note that

$$
\begin{aligned}
\frac{\log f(x; \eta_1, \alpha_2) - \log f(x; \alpha_1, \alpha_2)}{\eta_1 - \alpha_1} &= \sum_{i=1}^{2} \frac{\beta_i}{\eta_1 - \alpha_1} d_i(x) \\
&\quad - \frac{\log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\{\beta_1 d_1(x) + \beta_2 d_2(x)\} f(x; \boldsymbol{\alpha})\, dx}{\eta_1 - \alpha_1}. \quad (14)
\end{aligned}
$$

Now, when $\eta_1 \to \alpha_1$ in (14), by using the conditions (a) and (b) in theorem, we have

$$
\frac{\partial}{\partial \alpha_1} \log f(x, \boldsymbol{\alpha}) = a_1(\boldsymbol{\alpha}) d_1(x) + a_2(\boldsymbol{\alpha}) d_2(x) + t(\boldsymbol{\alpha}),
$$

where

$$
a_j(\boldsymbol{\alpha}) = \lim_{\eta_1 \to \theta_1} \frac{\beta_j}{\eta_1 - \theta_1},\ j = 1, 2,
$$

and

$$t(\boldsymbol{\alpha}) = \lim_{\eta_1 \to \alpha_1} \frac{-\log \int_{\pi_1^*(\boldsymbol{\alpha})}^{\pi_2^*(\boldsymbol{\alpha})} \exp\{\beta_1 d_1(x) + \beta_2 d_2(x)\} f(x;\boldsymbol{\alpha})\, dx}{\eta_1 - \alpha_1}.$$

By integrating to α_1, we get

$$\log f(x;\boldsymbol{\alpha}) = B_1(\boldsymbol{\alpha}) d_1(x) + B_2(\boldsymbol{\alpha}) d_2(x) + C_1(\boldsymbol{\alpha}) + A_1(x,\alpha_2). \tag{15}$$

where $B_i(\boldsymbol{\alpha}) = \int a_i(\boldsymbol{\alpha})\, d\alpha_1$, $C_1(\boldsymbol{\alpha}) = \int t(\boldsymbol{\alpha})\, d\alpha_1$ and $A_1(x,\alpha_2)$ is the constant of integration. Thus,

$$f(x;\boldsymbol{\alpha}) = \exp\left\{B_1(\boldsymbol{\alpha}) d_1(x) + B_2(\boldsymbol{\alpha}) d_2(x) + C_1(\boldsymbol{\alpha}) + A_1(x,\alpha_2)\right\}. \tag{16}$$

Similarly, when $\eta_1 = \alpha_1$, $\eta_2 \to \alpha_2$ in (14), it is yield that

$$f(x;\boldsymbol{\alpha}) = \exp\left\{P_1(\boldsymbol{\alpha}) d_1(x) + P_2(\boldsymbol{\alpha}) d_2(x) + C_2(\boldsymbol{\alpha}) + A_2(x,\alpha_1)\right\}. \tag{17}$$

From (16) and (17),

$$A_1(x,\alpha_2) = A_2(x,\alpha_1). \tag{18}$$

Since (18) holds for all α_1 and α_2, both sides must be of the form $a(x)$ and independent of $\boldsymbol{\alpha}$. Thus the density must be the form as (10). □

Some examples of the form-invariant two-parameter distributions belong to the non-regular exponential family with the common weight functions are given in Table 2.

TABLE 2: Form-invariant two-parameter distributions in non-regular exponential family.

Distribution	$\boldsymbol{\theta} = (\theta_1, \theta_2)$	Weight function	η
Normal(μ, σ^2)	(μ, σ^2)	$\exp\{\beta_1 x + \beta_2 x^2\}$ $\beta_2 < \frac{1}{2\sigma^2}$	$(\frac{\mu+\beta_1\sigma^2}{1-2\beta_2\sigma^2}, \frac{\sigma^2}{1-2\beta_2\sigma^2})$
Gamma(α, λ)	(α, λ)	$x^{\beta_1} \exp\{\beta_2 x\}$	$(\alpha + \beta_1, \lambda + \beta_2)$
Exponential(α, λ)	(α, λ)	$e^{-\beta_2 x}\, I_{(\alpha+\lvert\beta_1\rvert,\infty)}(x)$	$(\alpha + \lvert\beta_1\rvert, (\beta_2 + \frac{1}{\lambda})^{-1})$
Beta(α, λ)	(α, λ)	$x^{\beta_1}\, (1-x)^{\beta^2}$	$(\alpha + \beta_1, \lambda + \beta_2)$
Uniform $(\theta - \xi, \theta + \xi)$	(θ, ξ)	$I_{(\theta-\xi, \theta+\beta_1+\beta_2\xi)}(x)$	$(\theta - \xi, \theta + \beta_1 + \beta_2\xi)$ $-(1+\beta_2)\xi \le \beta_1 \le (1-\beta_2)\xi$
Pareto(θ, α)	(θ, α)	$x^{\beta_2}\, I_{(\theta+\beta_1, +\infty)}(x)$ $\beta_1 > 0$	$(\theta + \beta_1, \alpha - \beta_2)$

The shapes of some common form-invariant distributions in the nonregular exponential family with their weighted versions are given in Figure 1.

4. Some Properties

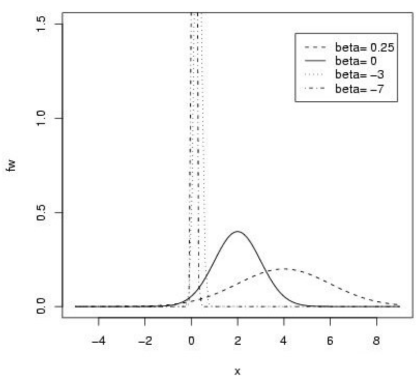
In this Section, some properties of the form-invariant two-parameter distributions with pdf (10) are derived.

(a) The sufficient and minimal sufficient statistics under the random sampling and also the form-invariant weighted sampling of size n have the same form, respectively $T(X)$ and $T(X^w)$, as

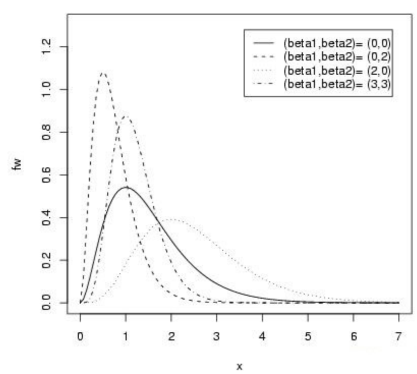
$$T(X) = \Big(\sum_{i=1}^{n} d_1(X_i), \sum_{i=1}^{n} d_2(X_i), X_{(1)}, X_{(n)}\Big)$$

and

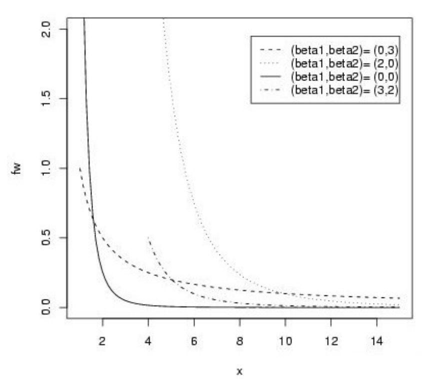
$$T(X^w) = \Big(\sum_{i=1}^{n} d_1(X_i^w), \sum_{i=1}^{n} d_2(X_i^w), X_{(1)}^w, X_{(n)}^w\Big).$$



(a) Weighted normal distribution with $w(x,\beta) = e^{\beta x^2}$



(b) Weighted gamma distribution with $w(x,\beta_1,\beta_2) = x^{\beta_1}\, e^{\beta_2 x}$



(c) Weighted pareto distribution $w(x,\beta_1,\beta_2) = x^{\beta_2}\, I_{(\theta+\beta_1,\infty)(x)}$

FIGURE 1: Some form-invariant distributions in nonregular exponential family.

Example 1. Let X have the exponential distribution with pdf as

$$f(x;\alpha,\theta) = \theta \exp\big\{-\theta(x-\alpha)\big\}\, I_{(\alpha,+\infty)}(x), \tag{19}$$

then $(\sum_{i=1}^{n} X_i, X_{(1)})$ and $(\sum_{i=1}^{n} X_i^w, X_{(1)}^w)$ are sufficient, minimal sufficient and also complete statistics for (α,θ) in (19) under the random sampling and the weighted sampling with the weight function $e^{\beta x}$, respectively.

(b) For the random sample $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ and its form-invariant weighted versions $\boldsymbol{x}^w=(x_1^w,x_2^w,\ldots,x_n^w)$ under the weight functions in (12), the likelihood function, $L(\boldsymbol{\alpha})$ and $L^w(\boldsymbol{\alpha},\boldsymbol{\beta})$, respectively, are calculated as

$$L(\boldsymbol{\alpha}) = \exp\Big\{ \sum_{j=1}^{n}\sum_{i=1}^{2} b_i(\boldsymbol{\theta}) d_i(x_j) + \sum_{j=1}^{n} a(x_j) - nc(\boldsymbol{\alpha})\Big\}\, I_{(\pi_1(\boldsymbol{\alpha}),+\infty)}\,(x_{(1)})\, I_{(-\infty,\pi_2(\boldsymbol{\alpha}))}\,(x_{(n)})$$

and

$$L^w(\boldsymbol{\alpha},\boldsymbol{\beta}) = \exp\Big\{ \sum_{j=1}^{n}\sum_{i=1}^{2} (\alpha_i+\beta_i) d_i(x_j^w) + \sum_{j=1}^{n} a(x_j^w) - nc^w(\boldsymbol{\alpha})\Big\} I_{(\pi_1(\boldsymbol{\alpha}),\infty)}(x_{(1)}^w)\, I_{(-\infty,\pi_2(\boldsymbol{\alpha}))}(x_{(n)}^w),$$

where $\boldsymbol{\beta}=(\beta_1,\beta_2)$ and $c^w(\boldsymbol{\alpha}) = c(\boldsymbol{\alpha}) + \log E\big[w(X,\boldsymbol{\alpha},\boldsymbol{\beta})\big]$. The distinction that $L(\boldsymbol{\alpha})$ and $L^w(\boldsymbol{\alpha},\boldsymbol{\beta})$ are the increasing or decreasing functions with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is not easy since this is depend on that the functions $c(.)$, $c^w(.)$, $\pi_1(.)$ and $\pi_2(.)$ are or not increasing or decreasing.

Example 2. Suppose that $\boldsymbol{x}$ is a random sample from a population with the pdf as

$$f(x;\alpha) = c(\alpha) h(x)\, I_{(0,\alpha)}(x),$$

where $c(.)$ and $h(.)$ are positive functions. Also, let $\boldsymbol{x}^w$ is a weighted version of $\boldsymbol{x}$ under the weight function $w(x,\theta,\beta) = I_{(0,\beta\alpha)}(x)$ where $\beta\in(0,1)$ is known. The likelihood functions in $\boldsymbol{x}$ and $\boldsymbol{x}^w$ is calculated, respectively, as

$$L(\alpha) = [c(\alpha)]^n \prod_{i=1}^{n} h(x_i)\, I_{(0,\alpha)}(x_{(n)})$$

and

$$L^w(\alpha,\beta) = [c(\beta\alpha)]^n \prod_{i=1}^{n} h(x_i^w)\, I_{(0,\beta\alpha)}(x_{(n)}^w).$$

Since $\frac{\partial}{\partial\alpha} c(\alpha) = \frac{-h(\alpha)}{\left(\int_0^\alpha h(x)\,dx\right)^2} < 0$, thus the functions $c(\alpha)$, $c(\beta\alpha)$, $L(\alpha)$ and $L^w(\alpha)$ are decreasing functions with respect to α. Therefore, the maximum likelihood estimator for α under the random sampling and the weighted samples are, respectively, $X_{(n)}$ and $\frac{X_{(n)}^w}{\beta}$.

(c) Suppose that $\{P_{\boldsymbol{\alpha}}\}$ is non-regular exponential family with the pdf of the form (10). The partial derivatives of any order can be obtained by differentiating inside the integral sign.

$$E_{\boldsymbol{\alpha}}[d_i(X)] = \frac{\partial}{\partial \tau_i} c(\boldsymbol{\alpha}) \qquad and \qquad Cov_{\boldsymbol{\alpha}}(d_1(X), d_2(X)) = \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} c(\boldsymbol{\alpha}).$$

Now, let the functions π_1 and π_2 is satisfied in following conditions

(i) $f\big(\pi_1(\boldsymbol{\alpha}); \boldsymbol{\alpha}\big) = f\big(\pi_2(\boldsymbol{\alpha}); \boldsymbol{\alpha}\big) = 0.$

(ii) $\frac{\partial}{\partial x} f(x; \boldsymbol{\alpha})\big|_{x=\pi_1(\boldsymbol{\alpha})} = \frac{\partial}{\partial x} f(x; \boldsymbol{\alpha})\big|_{x=\pi_2(\boldsymbol{\alpha})} = 0$

then, according to the regularity condition of Rao (1965), the Fisher information matrix on $\boldsymbol{\alpha}$ in f, I, and also in its weighted version (f^w), I^w, is calculated as

$$I(\boldsymbol{\alpha}) = \begin{pmatrix} E\left[\frac{\partial^2}{\partial \alpha_1^2} \log f(x; \boldsymbol{\alpha})\right] & E\left[\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \log f(x; \boldsymbol{\alpha})\right] \\ E\left[\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \log f(x; (\alpha))\right] & E\left[\frac{\partial^2}{\partial \alpha_2^2} \log f(x; \boldsymbol{\alpha})\right] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial^2}{\partial \alpha_1^2} c(\boldsymbol{\alpha}) & \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} c(\boldsymbol{\alpha}) \\ \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} c(\boldsymbol{\alpha}) & \frac{\partial^2}{\partial \alpha_2^2} c(\boldsymbol{\alpha}) \end{pmatrix}$$

and

$$I^w(\boldsymbol{\alpha}) = \begin{pmatrix} \frac{\partial^2}{\partial \alpha_1^2} c^w(\boldsymbol{\alpha}) & \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} c^w(\boldsymbol{\alpha}) \\ \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} c^w(\boldsymbol{\alpha}) & \frac{\partial^2}{\partial \alpha_2^2} c^w(\boldsymbol{\alpha}) \end{pmatrix}.$$

According to Patil & Taillie (1987), an intrinsic comparison between the weighted observations and original observations is possible when the difference of the matrices, $I^w(\boldsymbol{\alpha}) - I(\boldsymbol{\alpha})$, is either positive definite or negative definite. Positive (negative) definiteness of the difference matrix means that every scalar-valued function of $\boldsymbol{\alpha}$ can be estimated with smaller (larger) asymptotic standard error under the weighted observations. Therefore, the weighted observations can be or not favorable. When $I^w(\boldsymbol{\alpha}) - I(\boldsymbol{\alpha})$ is indefinite, comparison of the weighted and the original observations can be made in terms of the comparison of scalar-valued measure of the generalized variance, i.e., the determinant of the Fisher information matrix. Based on this measure, the observations (X or X^w) that have the smallest generalized variance will be more favored.

Example 3. Let $X \sim N(\mu, \sigma^2)$. Under the weight function $w(x) = \exp\{\beta_1 x + \beta_2 x^2\}$,

$$X^w \sim N\Big(\frac{\mu + \beta_1 \sigma^2}{1 - 2\beta_2 \sigma^2}, \frac{\sigma^2}{1 - 2\beta_2 \sigma^2}\Big),$$

and therefore is form-invariant. The Fisher information matrix on $\boldsymbol{\alpha} = (\mu, \sigma)$ in X and X^w is, respectively, as

$$I(\boldsymbol{\alpha}) = \frac{1}{\sigma^2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$I^w(\boldsymbol{\alpha}) = \frac{1}{1-2\beta_2\sigma^2}\begin{pmatrix} \frac{1}{\sigma^2} & \frac{2(\beta_1+2\beta_2\mu)}{\sigma(1-2\beta_2\sigma^2)} \\ \frac{2(\beta_1+2\beta_2\mu)}{\sigma(1-2\beta_2\sigma^2)} & \frac{2}{1-2\beta_2\sigma^2}\left(\frac{1}{\sigma^2} + \frac{2(\beta_1+2\beta_2\mu)^2}{1-2\beta_2\sigma^2}\right) \end{pmatrix}.$$

Therefore

$$I^w(\boldsymbol{\alpha}) - I(\boldsymbol{\alpha}) = \frac{2}{1-2\beta_2\sigma^2}\begin{pmatrix} \beta_2 & \frac{2\left(\beta_1+2\beta_2\mu\right)}{\sigma(1-2\beta_2\sigma^2)} \\ \frac{2\left(\beta_1+2\beta_2\mu\right)}{\sigma(1-2\beta_2\sigma^2)} & \frac{2(\beta_1+2\beta_2\mu)^2}{(1-2\beta_2\sigma^2)^2} + \frac{4\beta_2(1-\beta_2\sigma^2)}{1-2\beta_2\sigma^2} \end{pmatrix},$$

which

$$\det\Big[I^w(\boldsymbol{\alpha}) - I(\boldsymbol{\alpha})\Big] = -\frac{(\beta_1+2\mu\beta_2)^2}{\sigma^2\,(1-2\beta_2\sigma^2)} + \frac{4\beta_2^2(1-\beta_2\sigma^2)}{1-2\beta_2\sigma^2},$$

where $det[]$ stands for the determinant. As $|\beta_1 + 2\beta_2\mu| < \frac{1}{\sigma}$, the difference matrix $I^w(\boldsymbol{\alpha}) - I(\boldsymbol{\alpha})$ is a positive definite matrix and therefore, the form-invariant normal distribution is more informative than the normal distribution. This result, also, is fixed for opposite direction of inequality based on the measure of the generalized variances since

$$\det\big[I(\boldsymbol{\alpha})\big] = \frac{2}{\sigma^4}$$

and

$$\det\big[I^w(\boldsymbol{\alpha})\big] = \frac{\det[I(\boldsymbol{\alpha})]}{(1-2\beta_2\sigma^2)^3}, \quad 1-2\beta_2\sigma^2 > 0.$$

Example 4. Consider the generalized gamma distribution with the pdf

$$f(x;\theta,\sigma) = \frac{1}{\sigma^\rho\,\Gamma(\rho)}\left(\frac{x-\theta}{\sigma}\right)^{\rho-1}\exp\Big\{-\frac{x-\theta}{\sigma}\Big\},\ x>\theta,\ \rho>2, \sigma>0,$$

where ρ is known. The Fisher information matrix on $\boldsymbol{\alpha} = (\theta, \sigma)$ in f and f^w under $w(x) = e^{\beta x}$ are, respectively

$$\begin{pmatrix} \frac{1}{\sigma^2(\rho-2)} & \frac{1}{\sigma^2} \\ \frac{1}{\sigma^2} & \frac{\rho}{\sigma^2} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{(1-\beta\sigma)^2}{\sigma^2\,(\rho-2)} & \frac{1}{\sigma^2} \\ \frac{1}{\sigma^2} & \frac{\rho}{\sigma^2(1-\beta\sigma)^2} \end{pmatrix}.$$

Therefore

$$I^w(\boldsymbol{\alpha}) - I(\boldsymbol{\alpha}) = \begin{pmatrix} \frac{\beta\sigma(\beta\sigma-2)}{(\rho-2)} & 0 \\ \\ 0 & \frac{\rho\beta\sigma(2-\beta\sigma)}{(1-\beta\sigma)^2} \end{pmatrix}.$$

The difference matrix is a indefinite matrix. Since $det\big[I^w(\boldsymbol{\alpha})\big] = det\big[I(\boldsymbol{\alpha})\big] = \frac{\rho}{\sigma^4(\rho-2)}$, thus the weighted generalized gamma distribution and the generalized gamma distribution are uniformly equally informative for $\boldsymbol{\alpha}$ in terms of the generalized variance.

5. Conclusion

In the paper, we studied the problem of form-invariance in the non-regular exponential family when the original distribution is subjected to a weighted distribution. The class includes many common distributions, such as Patil & Ord (1976) and Billingsley (1979). It was shown that the maximum likelihood estimator could be obtained differently than original ones. The Fisher information matrix for weighted distribution is compared with original distribution to show which one is more (less) informative.

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