On Reliability in a Multicomponent Stress-Strength Model with Power Lindley Distribution

Sobre la fiabilidad en un modelo multicomponente de resistencia al estrés con distribución de power Lindley

Abbas Pak[1,a], Arjun Kumar Gupta[2,b], Nayereh Bagheri Khoolenjani[3,c]

[1]Department of Computer Sciences, Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran

[2]Department of Mathematics and Statistics, Bowling Green State University, New York, United States

[3]Department of Statistics, Isfahan University, Isfahan, Iran

Abstract

In this paper we study the reliability of a multicomponent stress-strength model assuming that the components follow power Lindley model. The maximum likelihood estimate of the reliability parameter and its asymptotic confidence interval are obtained. Applying the parametric Bootstrap technique, interval estimation of the reliability is presented. Also, the Bayes estimate and highest posterior density credible interval of the reliability parameter are derived using suitable priors on the parameters. Because there is no closed form for the Bayes estimate, we use the Markov Chain Monte Carlo method to obtain approximate Bayes estimate of the reliability. To evaluate the performances of different procedures, simulation studies are conducted and an example of real data sets is provided.

Key words: Bayesian inference; Bootstrap confidence interval; Maximum likelihood estimation; Stress-strength model.

Resumen

En este trabajo, estudiamos la fiabilidad de un modelo multicomponente de resistencia al estrés suponiendo que los componentes siguen el modelo Lindley de potencia. Se obtiene la estimación de máxima verosimilitud del

[a]PhD. E-mail: Abbas.pak1982@gmail.com

[b]PhD. E-mail: gupta@bgsu.edu

[c]PhD. E-mail: n.b.khoolenjani@gmail.com

parámetro de confiabilidad y su intervalo de confianza asintótico. Aplicando la técnica Bootstrap paramétrica, se presenta la estimación de intervalo de la confiabilidad. Además, la estimación de Bayes y el intervalo creíble de la densidad posterior más alta del parámetro de confiabilidad se obtienen utilizando los antecedentes adecuados sobre los parámetros. Debido a que no existe una forma cerrada para la estimación de Bayes, utilizamos el método de Markov Chain Monte Carlo para obtener una estimación aproximada de Bayes de la confiabilidad. Para evaluar el rendimiento de diferentes procedimientos, se realizan estudios de simulación y se proporciona un ejemplo de conjuntos de datos reales.

Palabras clave: Inferencia bayesiana; intervalo de confianza Bootstrap; estimación de máxima verosimilitud; modelo de resistencia al estrés.

1. Introduction

Stress-strength models have attracted the attention of statisticians for many years due to their applicability in diverse areas such as medicine, engineering, and quality control, among others. In reliability studies with strength X and stress Y, the parameter $R = P(X > Y)$ measures the reliability of a system (Kotz & Pensky 2003). This model is used in engineering problems for comparing the capability of two workers or comparison of the performances of products from two companies, etc. There is a large amount of literature about the estimation of R using different approaches and distributional assumptions on (X, Y). Estimation of R in the models with correlated stress and strength is conducted by Balakrishnan & Lai (2009). Hanagal (1997) derived maximum likelihood estimate (MLE) of stress-strength parameter R in a bivariate Pareto model. Inference for the stress-strength models in a generalized exponential model is studied by Kundu & Gupta (2005). Pak, Parham & Saraj (2014) have used fuzzy set theory to derive inferences on the parameter R when the observations of the strength and stress are imprecise quantities. Ghitany, Al-Mutairi & Aboukhamseen (2015) studied statistical estimation of R for the power Lindley model . Pak, Khoolenjani & Jafari (2014) developed inference procedures for the stress-strength parameter R in bivariate Rayleigh model. They studied different estimation methods by using the ML and bootstrap techniques. Dey, Raheem & Mukherjee (2017) derived the form of stress-strength reliability parameter for the transmuted Rayleigh distribution. Tarvirdizade & Ahmadpour (2016) considered estimation of the stress-strength reliability for the two-parameter bathtub-shaped lifetime distribution based on upper record values. Mahmoud, El-Sagheer, Soliman & Abd Ellah (2016) discussed Bayesian estimation of R based on record values from the Lomax Distribution. Condino, Domma & Latorre (2016) studied likelihood and Bayesian estimation of $P(Y < X)$ using lower record values from a proportional reversed hazard family. Inference on the Weibull distribution based on record values is considered by Wang & Ye (2015).

Recently, several researchers pay attention to developing inferential procedures for the reliability in multicomponent stress-strength (MSS) models. In the MSS system there are m identical and independent strength components and a common

stress that functions when at least $r(1 \leq r \leq m)$ of the components survive. This MSS model is denoted as r-out-of-m: G system. For example, consider an automobile with a V-8 engine that works if four cylinders are firing. So, it can be represented as 4-out-of-8: G system. Another example may be a suspension bridge with m pairs of vertical cables that survive when at least r number of vertical cables work. Inference on the reliability in MSS models when the stress and strength have Weibull distribution is considered by Kizilaslan & Nadar (2015). Rao (2012) and Rao, Aslam & Kundu (2014) conducted a series of studies to estimate the reliability of MSS models by assuming generalized exponential and Burr XII distributions for the components. They have used classical approaches to compute the reliability estimation in r-out-of-m: G models. Nadar & Kizilaslan (2016) considered Marshal-Olkin bivariate Weibull distribution and provided some inference procedures for MSS models. Dey, Mazucheli & Anis (2017) addressed MSS models consisting of Kumaraswamy distributed random variables. Hassan (2017) provided classical and Bayesian estimation of reliability parameter when the components follow Lindley distribution.

To our knowledge there are no reports on MSS models based on power Lindley distribution. The interest of this paper is to provide classical and Bayesian inferences on the reliability of r-out-of-m: G models when the strength and stress components are independent random variables distributed as power Lindley model. The ML estimate of the reliability parameter and its asymptotic confidence interval are obtained. Also, by using a parametric Bootstrap approach, two confidence intervals (CI) are derived for the interested parameter. Considering squared error loss function and using gamma priors on the parameters, an expression is provided as the Bayesian estimate of the reliability parameter. Since this expression can not simplified to a nice closed form, we employ a Markov Chain Monte Carlo (MCMC) procedure to obtain random samples from the posterior distributions and in turn use them to derive the Bayes estimate and highest posterior density (HPD) credible interval of the reliability.

A random variable (r.v.) Z follows power Lindley model with the parameters γ and δ if its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$f(z; \gamma, \delta) = \frac{\gamma \delta^2}{\delta + 1}(1 + z^\gamma)z^{\gamma - 1}e^{-\delta z^\gamma}, \quad z > 0, \quad \gamma, \delta > 0. \tag{1}$$

and

$$S(z; \gamma, \delta) = 1 - \left(1 + \frac{\delta}{\delta + 1}z^\gamma\right)e^{-\delta z^\gamma}, \quad z > 0, \quad \gamma, \delta > 0, \tag{2}$$

respectively. From now on power Lindley model with the parameters γ and δ will be denoted as $PL(\gamma, \delta)$.

The layout of this paper is as follows. Section 2 concerns ML estimation of the reliability parameter. In Section 3, two different confidence intervals for the reliability is constructed by using parametric Bootstrap samples. The Bayesian analyses are provided in Section 4. To evaluate the performances of the proposed estimators, simulation studies are conducted in Section 5. Moreover, for illustrative

purposes, analysis of two real data sets is presented. Finally, some comments and conclusions are made in Section 6.

2. Maximum Likelihood Estimation

Suppose that the m strength components of a MSS system are independent r.v.s with the common cdf $G(x)$ and let $F(y)$ be the cdf of the stress r.v. Y. When the strength and stress components of the system follow $PL(\gamma, \delta_1)$ and $PL(\gamma, \delta_2)$, respectively, the reliability of MSS model can be obtained as

$$
\begin{aligned}
R_{r,m} &= \sum_{i=r}^{m} \binom{m}{i} \int_0^\infty (1 - G(y))^i (G(y))^{m-i} dF(y) \\
&= \sum_{i=r}^{m} \binom{m}{i} \int_0^\infty \left(1 + \frac{\delta_1}{\delta_1 + 1} y^\gamma\right)^i e^{-i\delta_1 y^\gamma} \left(1 - \left(1 + \frac{\delta_1}{\delta_1 + 1} y^\gamma\right) e^{-\delta_1 y^\gamma}\right)^{m-i} \\
&\qquad \times \frac{\gamma \delta_2^2}{\delta_2 + 1} (1 + y^\gamma) y^{\gamma-1} e^{-\delta_2 y^\gamma} dy.
\end{aligned}
\tag{3}
$$

After simplification, the expression in (3) is expressed as

$$
\begin{aligned}
R_{r,m} &= \sum_{i=r}^{m} \sum_{j=0}^{i} \sum_{t=0}^{m-i} \sum_{s=0}^{t} \frac{m!}{j! s! (i-j)! (m-i-t)! (t-s)!} (-1)^t \frac{\delta_2^2}{\delta_2 + 1} \left(\frac{\delta_1}{\delta_1 + 1}\right)^{s+j} \\
&\qquad \times \left[\frac{(s+j)!}{((i+t)\delta_1 + \delta_2)^{s+j+1}} + \frac{(s+j+1)!}{((i+t)\delta_1 + \delta_2)^{s+j+2}} \right].
\end{aligned}
\tag{4}
$$

Now assume that $x_1, \ldots, x_n$ and $y_1, \ldots, y_k$ are the ordered random observations from $PL(\gamma, \delta_1)$ and $PL(\gamma, \delta_2)$, respectively. For more details about the experimental design used for generating the stress and strength data, the readers can refer to Bhattacharyya & Johnson (1974). The observed data likelihood function of γ, δ_1 and δ_2 becomes

$$
L(\gamma, \delta_1, \delta_2) = \frac{\gamma^{n+k} \delta_1^{2n} \delta_2^{2k}}{(\delta_1 + 1)^n (\delta_2 + 1)^k} e^{-\delta_1 \sum_{i=1}^{n} x_i^\gamma - \delta_2 \sum_{j=1}^{k} y_j^\gamma} \prod_{i=1}^{n} (1 + x_i^\gamma) x_i^{\gamma-1} \prod_{j=1}^{k} (1 + y_j^\gamma) y_j^{\gamma-1}
\tag{5}
$$

and the corresponding log-likelihood function is

$$
\begin{aligned}
\ell(\gamma, \delta_1, \delta_2) &= \log L(\gamma, \delta_1, \delta_2) \\
&= (n+k) \log \gamma + 2n \log \delta_1 + 2k \log \delta_2 - n \log(\delta_1 + 1) - k \log(\delta_2 + 1) \\
&\quad + \sum_{i=1}^{n} \left\{ \log(1 + x_i^\gamma) + (\gamma - 1) \log x_i - \delta_1 x_i^\gamma \right\} \\
&\quad + \sum_{j=1}^{k} \left\{ \log(1 + y_j^\gamma) + (\gamma - 1) \log y_j - \delta_2 y_j^\gamma \right\}.
\end{aligned}
\tag{6}
$$

The ML estimate of the parameters γ, δ_1 and δ_2, say $\hat{\gamma}$, $\hat{\delta}_1$ and $\hat{\delta}_2$, can be obtained using the following system of equations:

$$
\begin{aligned}
\frac{\partial \ell}{\partial \gamma} &= \frac{n+k}{\gamma} + \sum_{i=1}^{n} \left\{ \frac{x_i^{\gamma} \log x_i}{1 + x_i^{\gamma}} + (1 - \delta_1 x_i^{\gamma}) \log x_i \right\} \\
&\quad + \sum_{j=1}^{k} \left\{ \frac{y_j^{\gamma} \log y_j}{1 + y_j^{\gamma}} + (1 - \delta_2 y_j^{\gamma}) \log y_j \right\} = 0,
\end{aligned}
\tag{7}
$$

$$
\frac{\partial \ell}{\partial \delta_1} = \frac{2n}{\delta_1} - \frac{n}{\delta_1 + 1} - \sum_{i=1}^{n} x_i^{\gamma} = 0
\tag{8}
$$

and

$$
\frac{\partial \ell}{\partial \delta_2} = \frac{2k}{\delta_2} - \frac{k}{\delta_2 + 1} - \sum_{j=1}^{k} y_j^{\gamma} = 0.
\tag{9}
$$

Then, by using the invariance property of the MLEs, the ML estimate of $R_{r,m}$ can be computed as

$$
\hat{R}_{r,m} = \sum_{i=r}^{m} \sum_{j=0}^{i} \sum_{t=0}^{m-i} \sum_{s=0}^{t} \frac{m!}{j!s!(i-j)!(m-i-t)!(t-s)!} (-1)^t \frac{\hat{\delta_2}^2}{\hat{\delta_2} + 1} \left(\frac{\hat{\delta_1}}{\hat{\delta_1} + 1} \right)^{s+j}
$$
$$
\times \left[\frac{(s+j)!}{((i+t)\hat{\delta_1} + \hat{\delta_2})^{s+j+1}} + \frac{(s+j+1)!}{((i+t)\hat{\delta_1} + \hat{\delta_2})^{s+j+2}} \right].
\tag{10}
$$

Once the maximum likelihood estimate of the reliability parameter is obtained, we can use the asymptotic normality of the MLEs to compute the approximate $100(1 - \alpha)\%$ CI of the reliability $R_{r,m}$ as follows:

$$
\hat{R}_{r,m} \pm z_{\frac{\alpha}{2}} \sqrt{\hat{\sigma}_{R_{r,m}}^2}.
\tag{11}
$$

Here, $z_{\frac{\alpha}{2}}$ is an upper percentile of the standard normal variate and the asymptotic variance $\hat{\sigma}_{R_{r,m}}^2$ is obtained as

$$
\hat{\sigma}_{R_{r,m}}^2 = \left\{ \sigma_{\delta_1}^2 \left(\frac{\partial R_{r,m}}{\partial \delta_1} \right)^2 + \sigma_{\delta_2}^2 \left(\frac{\partial R_{r,m}}{\partial \delta_2} \right)^2 \right\} \Big|_{(\hat{\gamma}, \hat{\delta}_1, \hat{\delta}_2)}
$$

where

$$
\sigma_{\delta_1}^2 = \left[E \left(-\frac{\partial^2 \ell}{\partial \delta_1^2} \right) \right]^{-1} = \frac{\delta_1^2 (\delta_1 + 1)^2}{n(\delta_1^2 + 4\delta_1 + 2)},
$$

$$
\sigma_{\delta_2}^2 = \left[E \left(-\frac{\partial^2 \ell}{\partial \delta_2^2} \right) \right]^{-1} = \frac{\delta_2^2 (\delta_2 + 1)^2}{k(\delta_2^2 + 4\delta_2 + 2)},
$$

$$\frac{\partial R_{r,m}}{\partial \delta_1} = \sum_{i=r}^{m} \sum_{j=0}^{i} \sum_{t=0}^{m-i} \sum_{s=0}^{t} \frac{m!}{j!s!(i-j)!(m-i-t)!(t-s)!}(-1)^t \frac{\delta_2^2}{\delta_2+1}\left(\frac{\delta_1}{\delta_1+1}\right)^{s+j}$$

$$\times \left[\frac{s+j}{\delta_1(\delta_1+1)}\left\{\frac{(s+j)!}{((i+t)\delta_1+\delta_2)^{s+j+1}} + \frac{(s+j+1)!}{((i+t)\delta_1+\delta_2)^{s+j+2}}\right\}\right.$$

$$\left. - (i+t)\left\{\frac{(s+j+1)!}{((i+t)\delta_1+\delta_2)^{s+j+2}} + \frac{(s+j+2)!}{((i+t)\delta_1+\delta_2)^{s+j+3}}\right\}\right]$$

and

$$\frac{\partial R_{r,m}}{\partial \delta_2} = \sum_{i=r}^{m} \sum_{j=0}^{i} \sum_{t=0}^{m-i} \sum_{s=0}^{t} \frac{m!}{j!s!(i-j)!(m-i-t)!(t-s)!}(-1)^t \left(\frac{\delta_1}{\delta_1+1}\right)^{s+j}$$

$$\times \left[\frac{\delta_2^2+2\delta_2}{(\delta_2+1)^2}\left\{\frac{(s+j)!}{((i+t)\delta_1+\delta_2)^{s+j+1}} + \frac{(s+j+1)!}{((i+t)\delta_1+\delta_2)^{s+j+2}}\right\}\right.$$

$$\left. - \frac{\delta_2^2}{\delta_2+1}\left\{\frac{(s+j+1)!}{((i+t)\delta_1+\delta_2)^{s+j+2}} + \frac{(s+j+2)!}{((i+t)\delta_1+\delta_2)^{s+j+3}}\right\}\right].$$

Note that the form of confidence interval of R given by Eq. (11) may lead to a confidence interval that does not completely fall inside the unit interval $(0,1)$. In this case, one can apply the logit transformation $g(R) = \log(\frac{R}{1-R})$ (see Ghitany et al. 2015). Then, using the delta method, the asymptotic $100(1-\alpha)\%$ confidence interval for $g(R)$ is derived

$$\log\left(\frac{\hat{R}}{1-\hat{R}}\right) \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\hat{\sigma}_R^2}}{\hat{R}(1-\hat{R})} \equiv (L, U).$$

Finally, the asymptotic $100(1-\alpha)\%$ confidence interval for R is obtained by

$$\left(\frac{e^L}{1+e^L}, \frac{e^U}{1+e^U}\right). \tag{12}$$

3. Bootstrap Confidence Interval

In this section, we describe the extraction of bootstrap CI for the reliability $R_{r,m}$ by applying two types of bootstrap techniques, namely, percentile bootstrap (Boot-p) method on the basis of the idea of Efron (1982) and Student's t bootstrap method (Boot-t) based on the idea of Hall (1988). The performances of these two bootstrap methods will be compared with the asymptotic CI of $R_{r,m}$ in Section 5. For more details about different parametric and non-parametric methods of constructing bootstrap confidence intervals, one can refer to the excellent book of Meeker, Hahn & Escobar (2017) and the references therein.

First, based on the original samples $x_1, \ldots, x_n$ and $y_1, \ldots, y_k$ from $PL(\gamma, \delta_1)$ and $PL(\gamma, \delta_2)$, we generate bootstrap samples of size B by using the following algorithm.

Step 1: By using the samples $x_1, \ldots, x_n$ and $y_1, \ldots, y_k$, obtain the ML estimates $\hat{\gamma}$, $\hat{\delta}_1$ and $\hat{\delta}_2$ from the system of equations in (7)-(9).

Step 2: Using the new estimates of parameters computed in Step 1, generate bootstrap samples $x_1^*, \ldots, x_n^*$ and $y_1^*, \ldots, y_k^*$ and compute the MLEs $\hat{\gamma}^*$, $\hat{\delta}_1^*$ and $\hat{\delta}_2^*$. Then, by using relation (10), calculate the bootstrap estimate of reliability $R_{r,m}$, say $\hat{R}_{r,m}^*$.

Step 3: Repeat Steps 2 for B times, and obtain $\hat{R}_{r,m}^{l*}$ for $l = 1, \ldots, B$.

Now, define $\bar{R}_{BP}(t) = Q^{-1}(t)$ where $Q(t) = P(\hat{R}_{r,m}^* \leq t)$ is the cdf of $\hat{R}_{r,m}^*$. The $100(1 - \alpha)\%$ Boot-p CI of the reliability $R_{r,m}$ is obtained as

$$\left(\bar{R}_{BP}(\frac{\alpha}{2}), \bar{R}_{BP}(1 - \frac{\alpha}{2}) \right). \tag{13}$$

For constructing Boot-t CI of $R_{r,m}$, first compute the statistic $A_l^* = \frac{\hat{R}_{r,m}^{l*} - \hat{R}_{r,m}}{\sqrt{\widehat{Var(\hat{R}_{r,m})}}}$ for $l = 1, \ldots, B$ in which $\widehat{Var(\hat{R}_{r,m})}$ is derived from expression (10). Then, define

$$\log \left(\frac{\hat{R}_{r,m}}{1 - \hat{R}_{r,m}} \right) \pm W^{-1}(t) \frac{\sqrt{\widehat{Var(\hat{R}_{r,m})}}}{\hat{R}_{r,m}(1 - \hat{R}_{r,m})} \equiv (\bar{L}, \bar{U}) \tag{14}$$

where $W(t) = P(A_l^* \leq t)$. The $100(1 - \alpha)\%$ Boot-t CI of the reliability $R_{r,m}$ is given by

$$\left(\frac{e^{\bar{L}}}{1 + e^{\bar{L}}}, \frac{e^{\bar{U}}}{1 + e^{\bar{U}}} \right). \tag{15}$$

Note that in the calculation of the $(\alpha/2)100\%$ and $(1 - \alpha/2)100\%$ percentiles, the functions $Q(t)$ and $W(t)$ are not found exactly but by the empirical CDF in each case. Moreover, the empirical distributions used in construction of confidence intervals may not provide adequate approximation to the tails. In this case, the empirical distributions based on some generalized pivotal quantities can be used to obtain approximate confidence intervals.

4. Bayesian Analyses

In this section we describe the Bayesian estimation of the reliability $R_{r,m}$ as well as the corresponding HPD credible interval. In order to conduct the Bayesian analysis, some prior distributions on the parameters are required. Here, we assume that γ, δ_1 and δ_2 are independent r.v.s and follow the gamma prior distributions as

$$\begin{cases} \pi_1(\gamma) \propto \gamma^{a_1 - 1} e^{-\gamma b_1} \\ \pi_2(\delta_1) \propto \delta_1^{a_2 - 1} e^{-\delta_1 b_2} \\ \pi_3(\delta_2) \propto \delta_2^{a_3 - 1} e^{-\delta_2 b_3}, \end{cases} \tag{16}$$

respectively, where the hyperparameters a_i, b_i, $i = 1, 2, 3$ are positive (see Pak, Parham & Saraj 2013). By combining (5) with (16), the joint density function of $(\gamma, \delta_1, \delta_2)$ and the *data* becomes

$$\pi(\gamma, \delta_1, \delta_2, data) \propto \frac{\gamma^{n+k+a_1-1}\delta_1^{2n+a_2-1}\delta_2^{2k+a_3-1}}{(\delta_1+1)^n(\delta_2+1)^k}e^{-\gamma b_1}e^{-\delta_1(b_2+\sum\limits_{i=1}^{n}x_i^{\gamma})}e^{-\delta_2(b_3+\sum\limits_{j=1}^{k}y_j^{\gamma})}$$

$$\times \prod_{i=1}^{n}(1+x_i^{\gamma})x_i^{\gamma-1}\prod_{j=1}^{k}(1+y_j^{\gamma})y_j^{\gamma-1}. \qquad (17)$$

Thus, we can write the posterior density function of γ, δ_1 and δ_2 as

$$\pi^*(\gamma, \delta_1, \delta_2 \mid data) = \frac{\pi(\gamma, \delta_1, \delta_2, data)}{\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \pi(\gamma, \delta_1, \delta_2, data)d\gamma d\delta_1 d\delta_2}. \qquad (18)$$

It is well known that, assuming squared error loss function, the Bayes estimate of the reliability $R_{r,m}$ is its posterior mean which is obtained by

$$E(R_{r,m} \mid data) = \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty R_{r,m}\pi^*(\gamma, \delta_1, \delta_2 \mid data)d\gamma d\delta_1 d\delta_2. \qquad (19)$$

Since the posterior density function $\pi^*(\gamma, \delta_1, \delta_2 \mid data)$ has a complex form, it is difficult to derive a nice closed form for the Bayes estimate of $R_{r,m}$. Therefore, we adopt Gibbs sampling method to extract random samples from the conditional densities of the parameters and use them to obtain the Bayes estimate and HPD credible interval of $R_{r,m}$.

From (17), the conditional posterior densities of γ, δ_1 and δ_2 can be extracted as

$$\pi_1^*(\gamma \mid \delta_1, \delta_2, data) \propto Gamma(n+k+a_1, b_1)e^{-\delta_1\sum\limits_{i=1}^{n}x_i^{\gamma}}e^{-\delta_2\sum\limits_{j=1}^{n}y_j^{\gamma}}$$

$$\times \prod_{i=1}^{n}(1+x_i^{\gamma})x_i^{\gamma-1}\prod_{j=1}^{k}(1+y_j^{\gamma})y_j^{\gamma-1}, \qquad (20)$$

$$\pi_2^*(\delta_1 \mid \gamma, data) \propto Gamma(2n+a_2, b_2+\sum_{i=1}^{n}x_i^{\gamma})\frac{1}{(\delta_1+1)^n}, \qquad (21)$$

and

$$\pi_3^*(\delta_2 \mid \gamma, data) \propto Gamma(2k+a_3, b_3+\sum_{j=1}^{k}y_j^{\gamma})\frac{1}{(\delta_2+1)^k}, \qquad (22)$$

respectively. Note that the conditional densities in (20)-(22) are not in the form of known distributions and therefore it is not possible to sample from these distributions by standard methods. If the posterior density function be unimodal

and roughly symmetric, then it is often convenient to approximate it by a normal distribution (see Gelman, Carlin, Stern & Rubin 2003). To show that the posterior densities of γ, δ_1 and δ_2 are roughly symmetric and log-concave (so unimodal), plots of the conditional densities along with the asymmetry and kurtosis coefficients of the posterior distributions are given in Figure 1 for the different sets of parameter values. It is observed that the plot of the posterior densities are similar to normal distribution. Therefore, in the following algorithm we employ Metropolis-Hastings (M-H) technique with normal proposal distribution to generate samples from these distributions.

1) Let initial values of the parameters to be $(\gamma^0, \delta_1^0, \delta_2^0)$ and set $l = 1$.

2) Considering the proposal distribution $q(\delta_1) \equiv I(\delta_1 > 0)N(\delta_1^{l-1}, 1)$ for the M-H method, generate δ_1^l from $\pi_2^*(\delta_1 \mid \gamma^{l-1}, data)$.

3) Generate δ_2^l, from $\pi_3^*(\delta_2 \mid \gamma^{l-1}, data)$ using M-H method with the proposal distribution $q(\delta_2) \equiv I(\delta_2 > 0)N(\delta_2^{l-1}, 1)$.

4) Generate γ^l, from $\pi_1^*(\gamma \mid \delta_1^l, \delta_2^l, data)$ using M-H method with $q(\gamma) \equiv I(\gamma > 0)N(\gamma^{l-1}, 1)$.

5) Compute $R_{r,m}^l$ from (4) and Set $l = l + 1$.

6) Repeat Steps 2-5 for M times, and obtain γ^l, δ_1^l, δ_2^l and $R_{r,m}^l$ for $l = 1, \ldots, M$.

The steps of M-H technique used in the above algorithm can be described as follows:

- Set $\sigma = \mu^{l-1}$.

- Generate τ using the proposal distribution $q(\mu) \equiv I(\mu > 0)N(\mu^{l-1}, 1)$.

- Let $p(\sigma, \tau) = \min\{1, \pi_i^*(\tau)q(\sigma)/\pi_i^*(\sigma)q(\tau)\}$.

- Accept τ with probability $p(\sigma, \tau)$, or accept σ with probability $1 - p(\sigma, \tau)$.

Then, by using the generated random samples from the above Gibbs procedure, the approximate Bayes estimate of the reliability parameter $R_{r,m}$ becomes $R_{r,m}^M = \sum_{l=1}^{M} R_{r,m}^l / M$.

Also, let $R_{r,m}^{(1)} < \cdots < R_{r,m}^{(M)}$ be the ordered values of $R_{r,m}^l$ for $l = 1, \ldots, M$. The HPD credible interval of $R_{r,m}$ will be derived by selecting the interval with the shortest length through the following $100(1 - \alpha)\%$ credible intervals of $R_{r,m}$:

$$(R_{r,m}^{(1)}, R_{r,m}^{((1-\alpha)M)}), \ldots, (R_{r,m}^{(\alpha M)}, R_{r,m}^{(M)}).$$
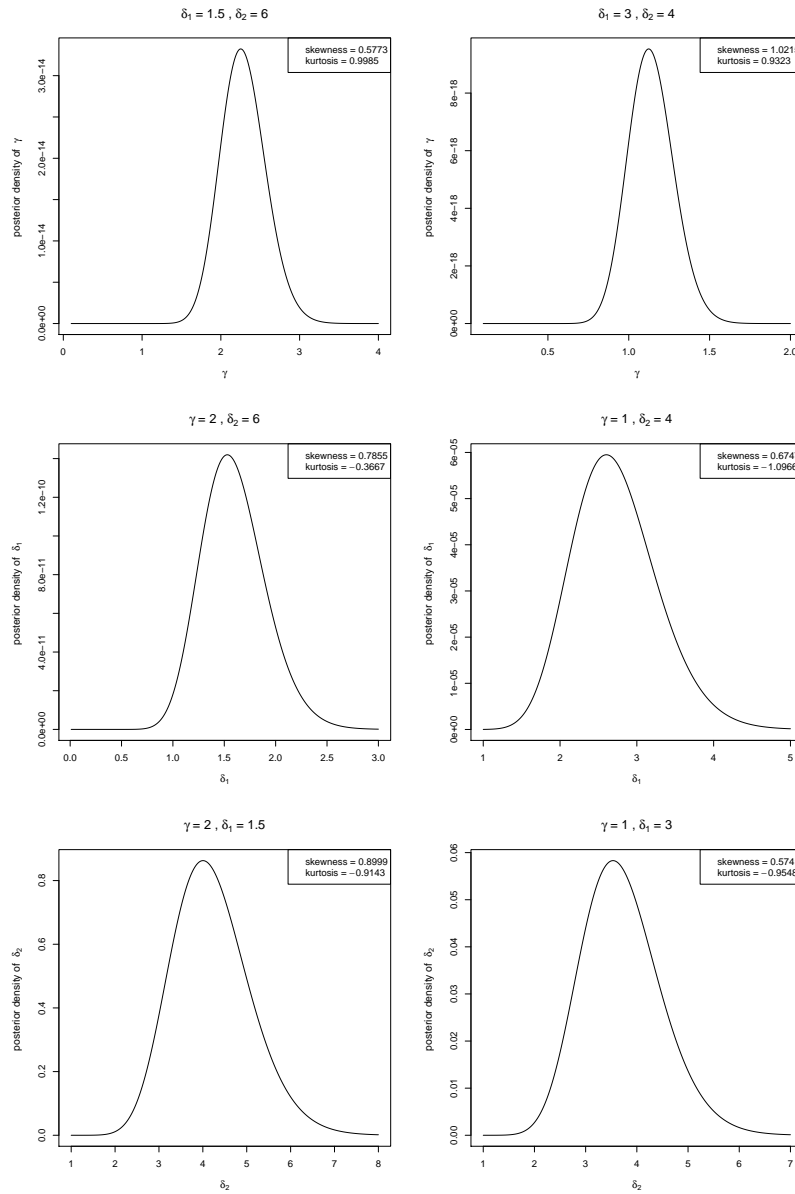
FIGURE 1: Posterior densities of γ, $\delta 1$ and $\delta 2$ for various sets of parameter values.

It must be noted that, for implementing the above MCMC procedure, some of the points must be considered. Firstly, specification of starting values is generally required for MCMC simulations. Sample draws at the beginning of the MCMC sequence cannot be expected to represent draws from the desired posterior distribution. Thus, it is common practice to drop some number of the initial sample draws, so that the remaining draws more accurately represent a sample from the limiting distribution (Meeker et al. 2017). The discarded sample draws are referred to as *burn-in* draws. Secondly, if there is a strong autocorrelation between the generated samples, it may be advisable to use *thinning* by retaining every 1 in 10 or 1 in 50 of the sample draws (depending on the strength of the autocorrelation). Thinning will increase the number of draws that need to be generated but reduce the number of draws that need to be stored.

TABLE 1: MLE and different CIs of the reliability $R_{r,m}$ when $(\gamma, \delta_1, \delta_2) = (2, 1, 1)$.

(r, m)	(n, k)	MLE		Asymptotic CI		Boot-p CI		Boot-t CI	
		AV	MSE	AL	CP	AL	CP	AL	CP
(1,3)	(15,15)	0.73107	0.00954	0.36841	0.9233	0.39128	0.9436	0.41007	0.9681
	(20,20)	0.73214	0.00743	0.32274	0.9327	0.333637	0.9285	0.38145	0.9725
	(30,30)	0.73887	0.00568	0.26678	0.9476	0.27410	0.9067	0.30632	0.9643
	(40,40)	0.74362	0.00376	0.23214	0.9516	0.23317	0.9653	0.25714	0.9715
	(50,50)	0.74425	0.00307	0.21218	0.9631	0.21309	0.9417	0.22985	0.9652
(2,4)	(15,15)	0.59035	0.01309	0.41502	0.9327	0.43947	0.9255	0.44780	0.9613
	(20,20)	0.59247	0.00968	0.36826	0.9419	0.38507	0.9289	0.3903	0.9641
	(30,30)	0.59705	0.00718	0.30826	0.9467	0.31927	0.9061	0.32043	0.9403
	(40,40)	0.59727	0.00524	0.27154	0.9533	0.27546	0.9513	0.27850	0.9592
	(50,50)	0.59814	0.00411	0.24496	0.9579	0.24568	0.9438	0.24825	0.9517

TABLE 2: MLE and different CIs of the reliability $R_{r,m}$ when $(\gamma, \delta_1, \delta_2) = (2, 1.5, 2)$.

(r, m)	(n, k)	MLE		Asymptotic CI		Boot-p CI		Boot-t CI	
		AV	MSE	AL	CP	AL	CP	AL	CP
(1,3)	(15,15)	0.82995	0.00814	0.30497	0.9194	0.31014	0.9349	0.338097	0.9438
	(20,20)	0.83141	0.00552	0.26814	0.9258	0.27136	0.9271	0.34175	0.9619
	(30,30)	0.83275	0.00318	0.21640	0.9471	0.21593	0.9067	0.26074	0.9735
	(40,40)	0.83546	0.00240	0.18708	0.9513	0.18786	0.9526	0.22027	0.9654
	(50,50)	0.83773	0.00168	0.15652	0.9607	0.16744	0.9290	0.17308	0.9548
(2,4)	(15,15)	0.70742	0.01118	0.38452	0.9128	0.40376	0.9243	0.43164	0.9581
	(20,20)	0.70985	0.00854	0.34287	0.9265	0.35714	0.9281	0.37572	0.9537
	(30,30)	0.71028	0.00513	0.28384	0.9341	0.28926	0.9076	0.30358	0.9578
	(40,40)	0.71126	0.00438	0.24846	0.9406	0.25269	0.9507	0.26394	0.9622
	(50,50)	0.71270	0.00339	0.17158	0.9451	0.18947	0.9236	0.19245	0.9608

TABLE 3: Bayes estimates of the reliability $R_{r,m}$ when $(\gamma, \delta_1, \delta_2) = (2, 1, 1)$.

(r,m)	(n,k)	prior I				prior II			
		AV	MSE	AL	CP	AV	MSE	AL	CP
(1,3)	(15,15)	0.73410	0.01078	0.36231	0.9085	0.83638	0.00774	0.35257	0.9463
	(20,20)	0.73695	0.00748	0.31959	0.9254	0.73879	0.00627	0.31246	0.9480
	(30,30)	0.73722	0.00520	0.26625	0.9371	0.74019	0.00461	0.26143	0.9511
	(40,40)	0.74158	0.00365	0.23102	0.9436	0.74375	0.00346	0.22818	0.9535
	(50,50)	0.74469	0.00301	0.20758	0.9477	0.74492	0.00280	0.20543	0.9581
(2,4)	(15,15)	0.59371	0.01257	0.42390	0.9249	0.59475	0.01005	0.41332	0.9448
	(20,20)	0.59511	0.00991	0.37417	0.9372	0.59653	0.00821	0.36657	0.9493
	(30,30)	0.59613	0.00679	0.31137	0.9413	0.59771	0.00603	0.30624	0.9517
	(40,40)	0.59685	0.00502	0.27166	0.9458	0.59812	0.00463	0.26837	0.9566
	(50,50)	0.59791	0.00408	0.24431	0.9479	0.59896	0.00380	0.24178	0.9607

TABLE 4: Bayes estimates of the reliability $R_{r,m}$ when $(\gamma, \delta_1, \delta_2) = (2, 1.5, 2)$.

(r,m)	(n,k)	prior I				prior II			
		AV	MSE	AL	CP	AV	MSE	AL	CP
(1,3)	(15,15)	0.82687	0.00736	0.29646	0.9143	0.83119	0.00574	0.27567	0.9462
	(20,20)	0.83315	0.00545	0.25719	0.9287	0.83393	0.00460	0.23450	0.9514
	(30,30)	0.83402	0.00351	0.21530	0.9426	0.83455	0.00327	0.21186	0.9573
	(40,40)	0.83524	0.00241	0.18845	0.9480	0.83543	0.00228	0.18378	0.9621
	(50,50)	0.83692	0.00193	0.16797	0.9513	0.83870	0.00175	0.16643	0.9637
(2,4)	(15,15)	0.70349	0.01243	0.39629	0.9221	0.70840	0.00873	0.37985	0.9386
	(20,20)	0.70668	0.00892	0.34763	0.9267	0.71053	0.00716	0.33089	0.9407
	(30,30)	0.70922	0.00561	0.29108	0.9315	0.71085	0.00544	0.28413	0.9432
	(40,40)	0.71104	0.00432	0.25493	0.9357	0.71160	0.00384	0.25127	0.9487
	(50,50)	0.71247	0.00328	0.22761	0.9373	0.71291	0.00302	0.22540	0.9511

5. Numerical Comparisons

5.1. Simulation Study

In the preceding sections different point and interval estimation techniques are used for estimating the reliability of MSS system under classical and Bayesian perspectives. In this section, we performed Monte Carlo simulations to investigate the behaviour of the proposed methods for various sample sizes. The performance of the competitive estimates has been compared in terms of their average values (AV) and mean squared errors (MSE). In addition, the confidence and credible intervals are compared on the basis of their average width and coverage percentages. The calculations are conducted using R 2.14.0 (R Development Core Team 2011).

We have considered two sets of parameter values as $(\gamma, \delta_1, \delta_2) = (2, 1, 1)$, (2,1.5,2) and different choices of sample sizes as $(n, k) = (15, 15), (20, 20), (30, 30),$ $(40, 40), (50, 50)$. With these choices of the parameter values, the true value of reliability $R_{r,m}$ for $(r, m) = (1, 3)$ becomes, respectively, 0.75 and 0.839455 and for $(r, m) = (2, 4)$ become 0.6 and 0.713061. First, in each case, different random

samples are generated from PL model and the ML estimates of the unknown parameters γ, δ_1 and δ_2 are obtained from the system of equations in (7)-(9). Then, the ML estimates of the reliability $R_{r,m}$ are computed from relation (10). The AVs and MSEs of the MLEs obtained from 10000 replications are presented in Tables 1-2. We have also derived 95% approximate CIs of the reliability parameter by using the asymptotic variance of $\hat{R}_{r,m}$ provided in Section 2. In addition, employing the Boot-p and Boot-t techniques, we have obtained the 95% CIs of $R_{r,m}$ based on 1000 bootstrap replications. For different sample sizes, the coverage probabilities (CP) and average length (AL) of these classical CIs based on 10000 replications are provided in Tables 1-2.

To evaluate the Bayes estimates, we take two different sets of hyper-parameters values as

Prior I: $a_1 = a_2 = b_1 = b_2 = 0$.

Prior II: $a_1 = a_2 = b_1 = b_2 = 2$.

Note that the prior I corresponds to the common noninformative prior which is non-proper also. Press (Press 2001) suggested using small positive choices for the hyperparameters that result in proper priors. So, in this case, we have considered $a_1 = a_2 = b_1 = b_2 = 0.0001$. For the above cases, the approximate Bayes estimates of $R_{r,m}$ are obtained by applying Gibbs sampling technique. To this end, Markov chains of size 7500 are generated and the first 2500 of the observations are removed to eliminate the effect of the starting distribution. Then, in order to reduce the dependence among the generated samples, we take every 5th sampled value which result in final chains of size 1000. To monitor the convergence of MCMC simulations, the scale reduction factor estimate is used. The estimate is given by $\sqrt{Var(\phi)/W}$, where ϕ is the estimand of interest, $Var(\phi) = (n-1)W/n + B/n$ with the iteration number n for each chain, the between-sequence variance B, and the within-sequence variance W; (see Gelman et al. 2003). In our case, the scale factor values of the MCMC estimators are found to be below 1.1, which is an acceptable value for their convergence. The means of the simulated samples are recorded as Bayes estimates of $R_{r,m}$. Tables 3-4 present the AVs and MSEs of the Bayes estimates obtained from 10000 replications. Further, for the generated samples, we have derived 95% credible intervals and counted the ones that cover the true value of the reliability $R_{r,m}$. The number of such intervals divided by 10000 is reported as estimated coverage probabilities. For different sample sizes, the coverage probabilities (CP) and average length of credible intervals are also provided in Tables 3-4.

From Tables 1-4 it is observed that the estimates obtained based on larger sample sizes have smaller MSEs, as we expected. The estimates of the the reliability $R_{r,m}$ computed using the Bayesian procedure with the non-informative priors and the MLEs are almost identical. However, using the informative gamma prior distributions, results in reasonable improvements in the performances of Bayes estimates. It can be further observed that asymptotic results of the MLEs have satisfactory performances unless the sample is small. In most of the cases, the coverage percentages of the approximate CIs are near to the predetermined nominal level.

Comparing different classical CIs, we observe that the lengthes of asymptotic CIs are somewhat shorter than that of bootstrap CIs. Both coverage percentages and average lengthes of Boot-t CIs are greater than that of Boot-p intervals. It is to be noted that the lengthes of all confidence and credible intervals decrease as the observed sample sizes increase. Moreover, the credible intervals of $R_{r,m}$ attained smaller lengthes compared to the approximate and bootstrap CIs. Overall, the performances of the credible intervals of $R_{r,m}$ based on informative priors are superior than all the other confidence intervals as shown by their ALs and CPs.

5.2. Data Analysis

To display the application of the different methods to real data, let us consider the two data sets reported in Bader & Priest (1982) on the failure stresses of single carbon fibers of lengths 20 mm and 50 mm, as follows:

Data set 1: (20 mm, ($n = 69$)) 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Data set 2: (50 mm, ($k = 65$)) 1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.840, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051,2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.270, 2.272, 2.280, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.390, 2.410, 2.430, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.620, 2.633, 2.670, 2.682, 2.699, 2.705, 2.735, 2.785, 3.020, 3.042, 3.116, 3.174.

Ghitany et al. (2015) showed that PL model fits data sets 1 and 2 very well. Here, assuming three different choices of (r,m) for the MSS system, we compute the estimates of reliability parameter $R_{r,m}$ by using classical and Bayesian procedures developed in this paper. First, from the the above data sets, the ML estimates of the parameters are obtained as $\hat{\gamma} = 4.029990$, $\hat{\delta}_1 = 0.042273$ and $\hat{\delta}_2 = 0.061771$. Then, the MLEs of the reliability $R_{r,m}$ are computed from the expression in (10). Further, by using the MLEs of the parameters, we have obtained boot-p and Boot-t CIs of the reliability parameter based on 1000 bootstrap replications in both cases.

To analyze the data from the Bayesian perspective, two different sets of values for the hyper-parameters are considered as

$$(a_1 = a_2 = b_1 = b_2) = (0.0001, 0.0001, 0.0001, 0.0001), (2, 2, 2, 2). \qquad (23)$$

Random samples of 20,000 realizations are generated from the posterior densities in (20)-(22) and the first 10000 realizations are deleted to diminish the trace of initial samples. Then, one observation in every 10 iterations is saved to break the autocorrelation between generated samples. Tables 5-6 reports different estimates of the reliability $R_{r,m}$ as well as the 95% confidence and credible intervals. It is

observed that the Bayes estimates of $R_{r,m}$ based on noninformative priors and the ML estimates are about the same, however, the width of credible intervals are somewhat shorter than that of approximate CIs.

TABLE 5: Classical estimates of $R_{r,m}$.

(r,m)	MLE	Asymptotic CI	Boot-p CI	Boot-t CI
(1,3)	0.873880	(0.796951,0.924424)	(0.805811,0.931284)	(0.788861,0.927796)
(2,4)	0.764123	(0.659042,0.844462)	(0.669598,0.855831)	(0.657230,0.845517)
(3,5)	0.676097	(0.561555,0.772821)	(0.571876,0.786770)	(0.564630,0.790619)

TABLE 6: Bayes estimates and HPD intervals of $R_{r,m}$.

(r,m)	$a_1 = a_2 = b_1 = b_2 = 0.0001$		$a_1 = a_2 = b_1 = b_2 = 2$	
	$R_{r,m}^M$	CRI	$R_{r,m}^M$	CRI
(1,3)	0.871363	(0.837419,0.945532)	0.856925	(0.829527,0.911584)
(2,4)	0.753290	(0.714031,0.892473)	0.751942	(0.691382,0.853991)
(3,5)	0.671425	(0.582975,0.786451)	0.643154	(0.620763,0.819327)

6. Conclusions

In the literature, there are well-developed estimation techniques for the reliability parameter of MSS models when the components follow different well-known lifetime distributions, however, as we observed the PL model has not been considered. In this paper, considering independent strength and stress random variables distributed as power Lindley model, we have developed inferential procedures for the MSS systems. The MLE and asymptotic CI for the reliability parameter are computed. Also, by using the parametric bootstrap approach, interval estimation of the reliability is provided. Further, employing a Gibbs sampling procedure, the Bayes estimate and HPD credible interval of the involved parameter have been derived. In order to assess the accuracy of the various approaches, Monte Carlo simulations are conducted. By increasing the sample sizes, expected improvements are observed in the performances of all estimators. Moreover, for larger sample sizes, the CIs constructed from the MLEs work well and their CPs are close to the nominal level. The coverage percentages of Boot-p CIs increase when the sample sizes increase, but are less than 95%. Overall, using informative priors in computing the Bayes estimates of $R_{r,m}$, we observed better performances of the point and interval estimates of the reliability parameter.

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