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The Exponentiated Kumaraswamy-G Class: General Properties and Application

La clase Kumaraswamy-G exponenciada: propiedades generales y aplicación

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Abstract

We propose a new family of distributions called the exponentiated Kumaraswamy-G class with three extra positive parameters, which generalizes the Cordeiro and de Castro's family. Some special distributions in the new class are discussed. We derive some mathematical properties of the proposed class including explicit expressions for the quantile function, ordinary and incomplete moments, generating function, mean deviations, reliability, Rényi entropy and Shannon entropy. The method of maximum likelihood is used to fit the distributions in the proposed class. Simulations are performed in order to assess the asymptotic behavior of the maximum likelihood estimates. We illustrate its potentiality with applications to two real data sets which show that the extended Weibull model in the new class provides a better fit than other generalized Weibull distributions.

***Key words*:** BFGS method; Exponential distribution; Exponentiated Kumaraswamy-G; Kumaraswamy distribution; Maximum likelihood estimation.

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Resumen

Proponemos una nueva clase de distribuciones llamada la clase de Kumaraswamy-G exponenciada con tres parámetros positivos adicionales, que generaliza la familia de Cordeiro y de Castro. Se discuten algunas distribuciones especiales en la nueva clase. Derivamos algunas propiedades matemáticas de la clase propuesta, incluyendo expresiones explícitas para la función cuartil, momentos ordinarios e incompletos, función generadora, desviaciones medias, confiabilidad, entropía de Rényi y entropía de Shannon. El método de máxima verosimilitud se utiliza para ajustar las distribuciones en la clase propuesta. Se realizaron simulaciones para evaluar el comportamiento asintótico de las estimaciones de máxima verosimilitud. Ilustramos su potencialidad con dos aplicaciones a dos conjuntos de datos reales que muestra que el modelo extendido de Weibull en la nueva clase proporciona un mejor ajuste que otras distribuciones generalizadas de Weibull.

***Palabras clave*:** Distribución exponencial; Distribución Kumaraswamy; Estimación de máxima verosimilitud; Kumaraswamy-G Exponenciada; Método BFGS.

1. Introduction

One of the preferred area of research in the filed of distribution is that of generating new distributions starting with a baseline distribution by adding one or more additional parameters. A generalized distribution may be important because it is connected with other special distributions in interesting ways (via transformations, limits, conditioning, etc.). In some cases, a parametric family may be important because it can be used to model a wide variety of random phenomena. In many cases, a special parametric family of distributions will have one or more distinguished standard members, corresponding to specified values of some of the parameters. Usually the standard distributions will be mathematically simpler, and often other members of the family can be constructed from the standard distributions by simple transformations on the underlying standard random variable. An incredible variety of special distributions have been studied over the years, and new ones are constantly being added to the literature. Notable among them are the Azzalini's skewed family (Azzalini 1985), Marshall-Olkin extended (MOE) family (Marshall & Olkin 1997), exponentiated family of distributions (Gupta, Gupta & Gupta 1998), or the composite methods of combining two or more known competing distributions through transformations like beta generated family (Eugene, Lee & Famoye 2002, Jones 2004), gamma-generated familiy (Zografos & Balakrishnan 2009, Ristic & Balakrishnan 2012), Kumaraswamy-G (Kw-G) family (Cordeiro & de Castro 2011), McDonald-G family (Alexander, Cordeiro, Ortega & Sarabia 2012), beta extended-G family (Cordeiro, Silva & Ortega 2012), Kumaraswamy-beta generalized family (Pescim, Cordeiro, Demetrio, Ortega & Nadarajah 2012), exponentiated transformed transformer family (Alzaghal, Felix & Carl 2013), exponentiated generalized family (Cordeiro, Ortega & Cunha 2013), geometric exponential-Poisson family (Nadarajah, Cancho & Ortega 2013), truncated-exponential skew symmetric family (Nadarajah, Jayakumar & Ristic 2013), logistic generated family (Torabi & Montazari 2014), Kumaraswamy Marshall-Olkin-G family (Alizadeh, Tahir, Cordeiro, Zubair & Hamedani 2015), generalized gamma-Weibull distribution (Meshkat, Torabi & Hamedani 2016), generalized odd log-logistic family (Cordeiro, Alizadeh, Tahir, Mansoor, Bourguignon & G. 2017), generalized transmuted-G family (Nofal, Afify, Yousof & Cordeiro 2017) and odd Lindley-G family (Gomes-Silva, Percontini, Brito, Ramos, Silva & Cordeiro 2017). While the additional parameter(s) bring in more flexibility at the same time they also complicate the mathematical form of the resulting distribution, often considerably enough to render it not amenable to further analytical and numerical manipulations. But with the advent of sophisticated powerful mathematical and statistical softwares more complex distributions are getting accepted as useful models for data analysis. Tahir & Nadarajah (2015) provided a detail review of how new families of univariate continuous distributions can be generated through introduction of additional parameter(s).

Cordeiro & de Castro (2011) defined the Kw-G family as follows. If G(x) denotes the cumulative distribution function (cdf) of a random variable, the Kw-G cdf family is

$$H_{a,b}(x) = 1 - \left[1 - G^a(x)\right]^b, \tag{1}$$

where $a > 0$ and $b > 0$ are two additional shape parameters to the G distribution, whose role is to govern skewness and tail weights. The cdf (1) compares extremely favorably in terms of simplicity with the beta cdf. The probability density function (pdf) corresponding to (1) is

$$h_{a,b}(x) = a\,b\,g(x)\,G^{a-1}(x)\left[1 - G^a(x)\right]^{b-1}, \tag{2}$$

where g(x) = dG(x)/dx. Equation (2) does not involve any special function, such as the incomplete beta function, as is the case of the beta-G family pionnered by Eugene et al. (2002). So, the Kw-G family is obtained by adding two shape parameters a and b to the G distribution. The generalization (2) contains distributions with unimodal and bathtub shaped hazard rates. It also contemplates a broad class of models with monotonic hazard rate functions (hrf's).

In this paper, we define a new class of distributions that extends the Kw-G family and derive some of its structural properties. Based on a continuous cdf $H_{a,b}(x)$ given by (1), the class of exponentiated $H_{a,b}$ distributions is defined by

$$F_{a,b,c}(x) = H^c_{a,b}(x) = \left\{1 - \left[1 - G^a(x)\right]^b\right\}^c, \tag{3}$$

where $a > 0$, $b > 0$ and $c > 0$ are three additional shape parameters to the G distribution. The pdf corresponding to (3) is

$$f_{a,b,c}(x) = a\,b\,c\,g(x)\,G^{a-1}(x)\left[1 - G^a(x)\right]^{b-1}\left\{1 - \left[1 - G^a(x)\right]^b\right\}^{c-1} \tag{4}$$

and the associated hrf reduces to

$$\tau_{a,b,c}(x) = \frac{a\,b\,c\,g(x)\,G^{a-1}(x)\left[1 - G^a(x)\right]^{b-1}\left\{1 - \left[1 - G^a(x)\right]^b\right\}^{c-1}}{1 - \left\{1 - \left[1 - G^a(x)\right]^b\right\}^c}.$$

The *exponentiated* Kw-G *class* ("EKw-G" for short) of densities (4) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. We study some structural properties of (4) because it extends several well-known distributions in the literature. In the next sections some mathematical properties of the new class are derived. The density function (4) will be most tractable when the cdf $G(x)$ and pdf $g(x)$ have simple analytic expressions. Note that even if $g(x)$ is a symmetric distribution, the distribution $f(x)$ will not be symmetric. The three extra parameters in (4) can control both tail weights and possibly adding entropy to the center of the EKw-G density function. Henceforth, $X \sim$EKw-G(a,b,c) denotes a random variable having density function (4). Each new EKw-G distribution can be obtained from a specified G model. For $a=b=c=1$, the G distribution is a special model of the EKw-G class with a continuous crossover towards cases with different shapes (e.g., a particular combination of skewness and kurtosis). Some cases of EKw-G have been discussed and explored in recent works. Here, we refer to the papers and baseline distributions: Huang & Oluyede (2014) for the Dagum, Rodrigues & Silva (2015) for the exponential and Rodrigues, Silva & Hamedani (2016) for the inverse Weibull. One major benefit of the family (4) is its ability of fitting skewed data that can not be properly fitted by existing distributions.

We define the exponentiated-G (Exp-G) random variable Z with power parameter $c>0$ from an arbitrary baseline distribution G, say $Z \sim \mathrm{Exp}^c\mathrm{G}$, if Z has cdf and pdf given by $\Pi_c(x)=G(x)^c$ and $\pi_c(x)=cg(x)G(x)^{c-1}$, respectively. This model is also called the Lehmann type I distribution. Note that there is a dual transformation $\mathrm{Exp}^c(1-\mathrm{G})$ referred to as the Lehmann type II distribution corresponding to the cdf $F(x)=1-[1-G(x)]^c$. Thus, equation (3) encompasses both Lehmann type I ($\mathrm{Exp}^c\mathrm{G}$ for $a=b=1$) and Lehmann type II ($\mathrm{Exp}^c(1-\mathrm{G})$ for $a=c=1$) distributions (Lehmann 1953). Clearly, the triple construction $\mathrm{Exp}^c\big\{1\text{-}\mathrm{Exp}^b\big[1\text{-}\mathrm{Exp}^a\mathrm{G}\big]\big\}$ generates the EKw-G class of distributions. Several properties of the EKw-G class can be derived using this triple transformation.

The EKw-G class shares an attractive physical interpretation whenever a, b and c are positive integers. Consider a device made of c independent components in a parallel system. Further, each component is made of b independent subcomponents identically distributed according to $G(x)^a$ in a series system. The device fails if all c components fail and each component fails if any subcomponent fails. Let $X_{j1},\ldots,X_{jc}$ denote the lifetimes of the subcomponents within the jth component ($j=1,\ldots,b$) with common cdf $G(x)$. Let X_j denote the lifetime of the jth component and let X denote the lifetime of the device. Thus, the cdf of X is given by

$$\begin{aligned}
P(X\le x) &= P(X_1\le x,\ldots,X_c\le x)=P^c(X_1\le x)=[1-P(X_1>x)]^c\\
&=\big\{1-\big[1-P(X_1\le x)\big]\big\}^c=\big\{1-\big[1-P(X_{11}\le x,\ldots,X_{1b}\le x)\big]\big\}^c\\
&=\big\{1-\big[1-P(X_{11}\le x)\big]^b\big\}^c\\
&=\big\{1-\big[1-P(X_{111}\le x,\ldots,X_{11a}\le x)\big]^b\big\}^c\\
&=\big\{1-\big[1-P^a(X_{111}\le x)\big]^b\big\}^c.
\end{aligned}$$

So, the lifetime of the device follows the EKw-G family of distributions.

The rest of the paper is organized as follows. In Section 2, we present four special models of the EKw-G class by extending the Weibull, Gumbel, gamma and Burr XII distributions. Section 3 provides two useful expansions for the EKw-G class. The quantile function and moments of X are derived in Sections 4 and 5, respectively. Generating function and mean deviations are obtained in Sections 6 and 7, respectively. The Rényi and Shannon entropies and the reliability are determined in Sections 8 and 9, respectively. Maximum likelihood estimation is discussed in Section 10. In Section 11, we provide a simulation study. An application to a real data set is performed in Section 12. Some concluding remarks are addressed in Section 13.

2. Special Models

Next, we present four EKw-G distributions.

2.1. Exponentiated Kumaraswamy Weibull (EKwW)

The Weibull cdf with parameters $\beta > 0$ and $\alpha > 0$ is $G(x) = 1 - \mathrm{e}^{-(\beta x)^{\alpha}}$ (for $x > 0$). The cdf of a random variable X having the EKwW distribution, say $X\sim$ EKwW(a, b, c, α, β), can be expressed as

$$F_{\mathrm{EKwW}}(x) = \left\{1 - \left[1 - \left(1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right)^{a}\right]^{b}\right\}^{c},$$

and the associated density function reduces to

$$\begin{aligned} f_{\mathrm{EKwW}}(x) &= a\,b\,c\,\alpha\,\beta^{\alpha}\,x^{\alpha-1}\,\mathrm{e}^{-(\beta x)^{\alpha}}\left[1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right]^{a-1}\left\{1 - \left[1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right]^{a}\right\}^{b-1} \\ &\times \left\{1 - \left[1 - \left(1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right)^{a}\right]^{b}\right\}^{c-1}. \end{aligned} \tag{5}$$

The hrf corresponding to (5) is given by

$$\begin{aligned} \tau_{\mathrm{EKwW}}(x) = {} & a\,b\,c\,\alpha\,\beta^{\alpha}\,x^{\alpha-1}\,\mathrm{e}^{-(\beta x)^{\alpha}}\left[1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right]^{a-1}\left\{1 - \left[1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right]^{a}\right\}^{b-1} \\ & \times \left\{1 - \left[1 - \left(1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right)^{a}\right]^{b}\right\}^{c-1} \\ & \times \left\{1 - \left[1 - \left(1 - \left[1 - \mathrm{e}^{-(\beta x)^{\alpha}}\right]^{a}\right)^{b}\right]^{c}\right\}^{-1}. \end{aligned}$$

For $c = 1$, we obtain as a special model the Kw-Weibull (KwW) distribution. The most important case of (5) is the exponentiated Weibull (EW) (when $b = c = 1$) pioneered by Mudholkar and Srivastawa (Mudholkar & Srivastava 1993). Plots of the density and hrf of the EKwW distribution for some parameter values are displayed in Figures 1 and 2, respectively.

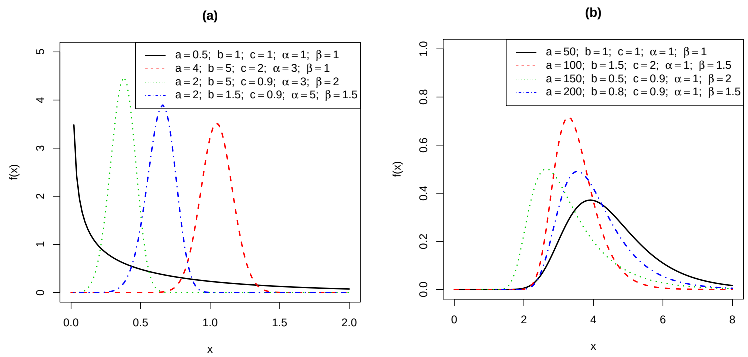

FIGURE 1: Plots of the EKwW density function for some parameter values.

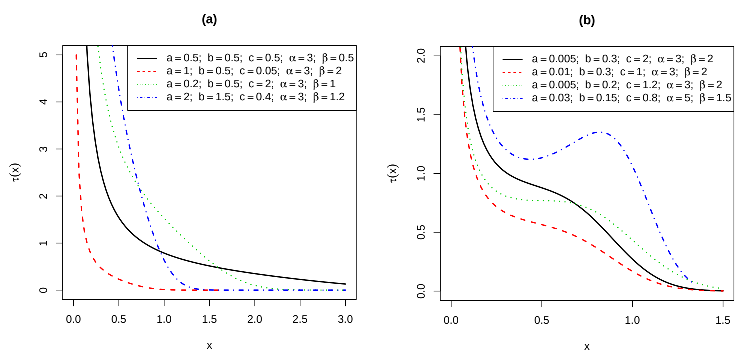

FIGURE 2: Plots of the EKwW hrf for some parameter values.

2.2. Exponentiated Kumaraswamy Gumbel (EKwGu)

The Gumbel cdf (for x real) is $G(x) = \exp\{-\exp[-(\frac{x-\mu}{\sigma})]\}$, where the parameters are μ real and $\sigma > 0$. The EKwGu cdf can be expressed as

$$F_{\text{EKwGu}}(x) = \left\{1 - \left[1 - \exp\left(-a\left\{\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}\right)\right]^b\right\}^c$$

and the associated density function is

$$\begin{aligned} f_{\text{EKwGu}}(x) &= a\,b\,c\,\sigma^{-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right) - a\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\} \\ &\times \left[1 - \exp\left\{-a\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}\right]^{b-1} \\ &\times \left\{1 - \left[1 - \exp\left(-a\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right)\right]^b\right\}^{c-1}. \end{aligned}$$

Plots of the EKwGu density function for some parameter values are displayed in Figure 3.

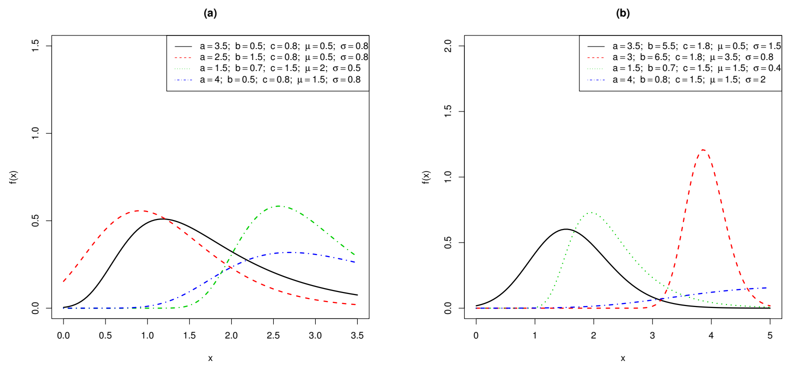

FIGURE 3: Plots of the EKwGu density function for some parameter values.

2.3. Exponentiated Kumaraswamy Gamma (EKwGa)

The gamma cdf (for $x > 0$) with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ is $G(x) = \gamma(\alpha, \beta\, x)/\Gamma(\alpha)$, where $\Gamma(\alpha) = \int_0^\infty w^{\alpha-1}\mathrm{e}^{-w}dw$ is the gamma function and $\gamma(\alpha, x) = \int_0^x w^{\alpha-1}\mathrm{e}^{-w}dw$ is the lower incomplete gamma function. The EKwGa cdf can be written as

$$F_{\text{EKwGa}}(x) = \left\{1 - \left[1 - \left(\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right)^a\right]^b\right\}^c,$$

and the associated density function reduces to

$$\begin{aligned} f_{\text{EKwGa}}(x) &= \frac{a\, b\, c\, \beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, \mathrm{e}^{-\beta\, x} \left[\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right]^{a-1} \left\{1 - \left[\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right]^a\right\}^{b-1} \\ &\times \left\{1 - \left[1 - \left(\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right)^a\right]^b\right\}^{c-1}. \end{aligned}$$

The corresponding hrf is

$$\begin{aligned} \tau_{\text{EKwGa}}(x) &= \frac{a\, b\, c\, \beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, \mathrm{e}^{-\beta\, x} \left[\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right]^{a-1} \left\{1 - \left[\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right]^a\right\}^{b-1} \\ &\times \left\{1 - \left[1 - \left(\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right)^a\right]^b\right\}^{c-1} \\ &\times \left\{1 - \left[1 - \left(1 - \left[\frac{\gamma(\alpha, \beta\, x)}{\Gamma(\alpha)}\right]^a\right)^b\right]^c\right\}^{-1}. \end{aligned}$$

Plots of the pdf and hrf of the EKwGa distribution for some parameter values are displayed in Figures 4 and 5, respectively.

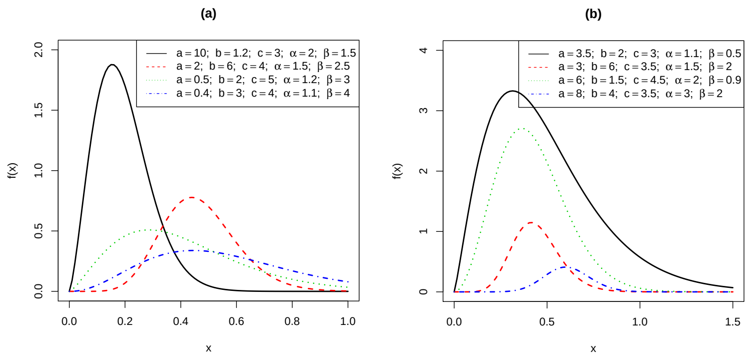

FIGURE 4: Plots of the EKwGa density function for some parameter values.

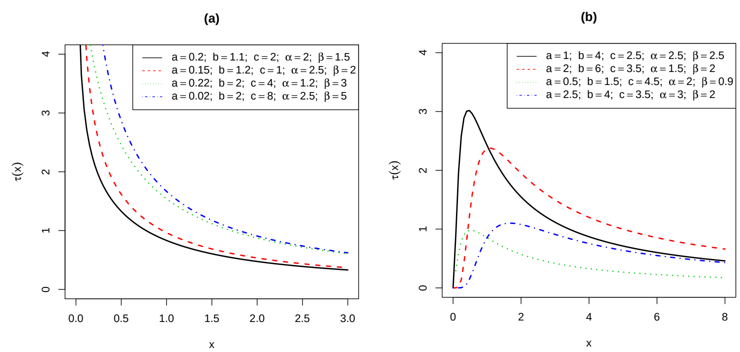

FIGURE 5: Plots of the EKwGa hrf for some parameter values.

2.4. Exponentiated Kumaraswamy Burr XII (EKwBXII)

Zimmer, Keats & Wang (1998) introduced the three parameter Burr XII (BXII) distribution with cdf and pdf (for $x > 0$): $G(x; s, k, p) = 1 - \left[1 + \left(\frac{x}{s}\right)^p\right]^{-k}$ and $g(x; s, k, p) = p\, k\, s^{-p}\, x^{p-1} \left[1 + \left(\frac{x}{s}\right)^p\right]^{-k-1}$, respectively, where $k > 0$ and $p > 0$ are shape parameters and $s > 0$ is a scale parameter. The EKwBXII cdf is

$$F_{\text{EKwBXII}}(x) = \left\{1 - \left[1 - \left(1 - \left[1 + \left(\frac{x}{s}\right)^p\right]^{-k}\right)^a\right]^b\right\}^c,$$

and the associated density function reduces to

$$\begin{aligned} f_{\text{EKwBXII}}(x) = a\, b\, c\, p\, k\, s^{-p}\, x^{p-1} &\left[1 + \left(\frac{x}{s}\right)^p\right]^{-k-1} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^p\right]^{-k}\right\}^{a-1} \\ &\times \left\{1 - \left(1 - \left[1 + \left(\frac{x}{s}\right)^p\right]^{-k}\right)^a\right\}^{b-1} \\ &\times \left\{1 - \left[1 - \left(1 - \left[1 + \left(\frac{x}{s}\right)^p\right]^{-k}\right)^a\right]^b\right\}^{c-1}. \end{aligned}$$

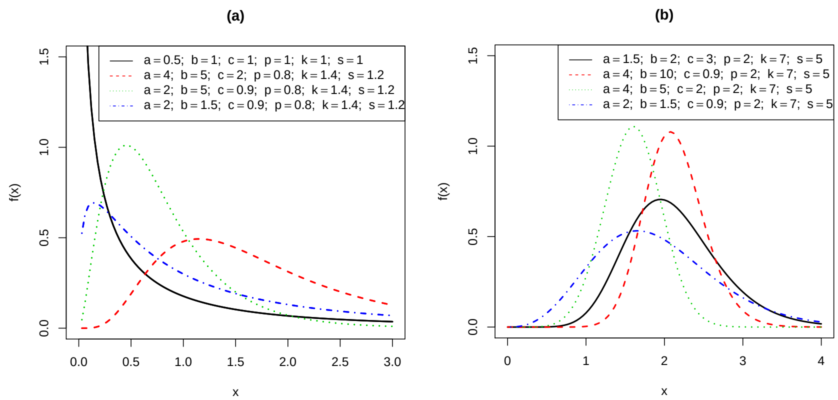

FIGURE 6: Plots of the EKwBXII density function for some parameter values.

The corresponding hrf is given by

$$\begin{aligned}\tau_{\mathrm{EKwBXII}}(x) = {} & a\,b\,c\,p\,k\,s^{-p}\,x^{p-1}\left[1+\left(\frac{x}{s}\right)^{p}\right]^{-k-1}\left\{1-\left[1+\left(\frac{x}{s}\right)^{p}\right]^{-k}\right\}^{a-1}\\ & \times\left\{1-\left(1-\left[1+\left(\frac{x}{s}\right)^{p}\right]^{-k}\right)^{a}\right\}^{b-1}\\ & \times\left\{1-\left[1-\left(1-\left[1+\left(\frac{x}{s}\right)^{p}\right]^{-k}\right)^{a}\right]^{b}\right\}^{c-1}\\ & \times\left\{1-\left[1-\left(1-\left\{1-\left[1+\left(\frac{x}{s}\right)^{p}\right]^{-k}\right\}^{a}\right)^{b}\right]^{c}\right\}^{-1}.\end{aligned}$$

Plots of the density and hrf of the EKwBXII distribution for some parameter values are displayed in Figures 6 and 7, respectively.

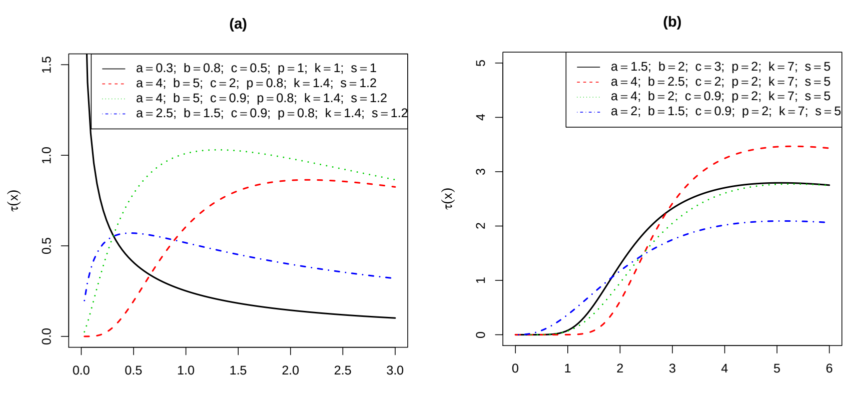

FIGURE 7: Plots of the EKwBXII hrf for some parameter values.

3. Useful Expansions

The pdf (4) can be expressed as a linear combination of Kw-G density functions. Using the generalized binomial expansion, we can rewrite (4) as

$$f(x) = \sum_{j=0}^{\infty} v_j \, h_{a,(j+1)\,b}(x), \tag{6}$$

where

$$v_j = (-1)^j \binom{c-1}{j}, \; j = 0, 1, \ldots$$

The cdf corresponding to (6) can be expressed as

$$F(x) = \sum_{j=0}^{\infty} v_j \, H_{a,(j+1)\,b}(x). \tag{7}$$

Based on equations (6) and (7) some structural properties of the EKw-G class can be obtained from well-established Kw-G properties. These equations can also be expressed as linear combinations of Exp-G distributions. Substituting (2) in equation (6) and using the binomial expansion, we obtain

$$f(x) = \sum_{k=0}^{\infty} w_k \, \pi_{(k+1)\,a}(x), \tag{8}$$

where

$$w_k = \frac{(-1)^k \, b}{k+1} \sum_{j=0}^{\infty} (-1)^j \binom{c-1}{j} \binom{(j+1)\,b-1}{k}.$$

Integrating (8), we have

$$F(x) = \sum_{k=0}^{\infty} w_k \, \Pi_{(k+1)a}(x), \tag{9}$$

where $\Pi_{(k+1)\,a}(x)$ denotes the Exp-G cdf with power parameter $(k+1)\,a$. Equation (8) reveals that the EKw-G density function is a linear combination of Exp-G densities. Thus, some structural properties of the EKw-G class such as the ordinary and incomplete moments and generating function can be obtained from well known Exp-G properties.

Several Exp-G properties have been studied by many authors in recent years, see Mudholkar and Srivastava (Mudholkar & Srivastava 1993) and Mudholkar *et al.* (Mudholkar, Srivastava & Kollia 1996) for exponentiated Weibull, Gupta *et al.* (Gupta et al. 1998) for exponentiated Pareto, Gupta and Kundu (Gupta & Kundu 1999) for exponentiated exponential, Nadarajah (Nadarajah 2005) for exponentiated Gumbel, Nadarajah and Gupta (Nadarajah & Gupta 2007) for exponentiated gamma and Lemonte *et al.* (Lemonte, Barreto-Souza & Cordeiro 2013) for exponentiated Kumaraswamy distributions. See, also, Nadarajah and Kotz (Nadarajah & Kotz 2006), among others. Equations (6)-(9) are the main results of this section.

4. Quantile Function

The EKw-G quantile function, say $Q(u) = F^{-1}(u)$, is straightforward to be computed by inverting (3) provided a closed-form expression for the baseline quantile function $Q_G(u) = G^{-1}(u)$ is available. From equation (3), we can write

$$Q(u) = Q_G\Big(\Big[1 - \Big(1 - u^{1/c}\Big)^{1/b}\Big]^{1/a}\Big). \tag{10}$$

For example, the EKwBXII quantile function comes by inverting the cdf in Section 2.4 as

$$Q(u) = s\left\{\Big(1 - \Big[1 - \Big(1 - u^{1/c}\Big)^{1/b}\Big]^{1/a}\Big)^{\frac{-1}{k}} - 1\right\}^{\frac{1}{p}}. \tag{11}$$

The shortcomings of the classical kurtosis measure are well-known. There are many heavy-tailed distributions for which this quantile is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of classical skewness and kurtosis for several generalized distributions. The Bowley skewness (Kenney & Keeping 1962) is based on quartiles

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)},$$

whereas the Moors kurtosis (Moors 1988) is based on octiles

$$M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

where $Q(\cdot)$ denotes the quantile function given by (10). Plots of the B and M functions for the EKwBXII distribution computed from (11) are displayed (for some parameter values) in Figures 8 and 9, respectively. These plots indicate a high dependence of these measures on the generator parameters of the new class.

Here, we derive a power series for the quantile function (10). If the baseline quantile function $Q_G(u) = G^{-1}(u)$ does not have a closed-form expression, it can usually be expressed in terms of a power series

$$Q_G(u) = \sum_{i=0}^{\infty} a_i\, u^i, \tag{12}$$

where the coefficients a_i are suitably chosen real numbers which depend on the parameters of the G distribution. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions, but it can be expanded as in equation (12). As a simple example, for the normal $N(0,1)$ distribution, $a_i = 0$ for $i = 0, 2, 4, \ldots$ and $a_1 = 1$, $a_3 = 1/6$, $a_5 = 7/120$ and $a_7 = 127/7560, \ldots$

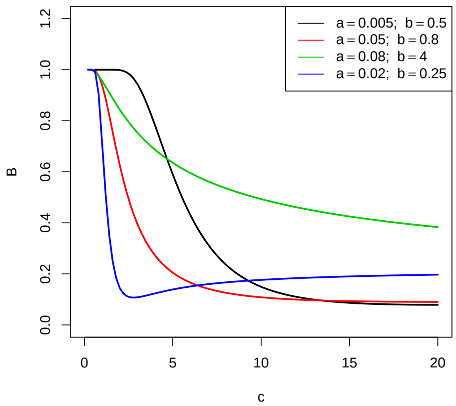

FIGURE 8: The B function for $s = 5.0$, $k = 7.0$ and $p = 2.0$.

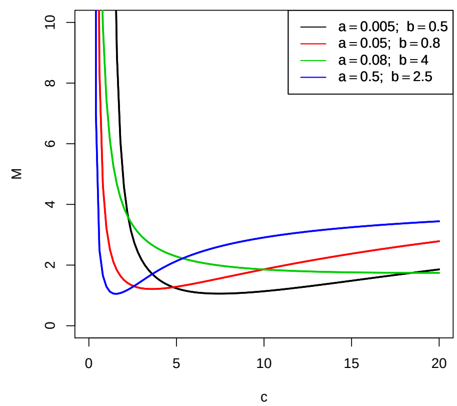

FIGURE 9: The M function for $s = 5.0$, $k = 7.0$ and $p = 2.0$.

Using the binomial expansion, we obtain

$$\left[1-(1-u^{1/c})^{1/b}\right]^{i/a} = \sum_{j,k=0}^{\infty} (-1)^{j+k} \binom{i/a}{j} \binom{j/b}{k} u^{k/c},$$

and then using (12), the EKw-G quantile function can be expressed as

$$Q(u) = \sum_{i,j,k=0}^{\infty} (-1)^{j+k} a_i \binom{i/a}{j} \binom{j/b}{k} u^{k/c} = \sum_{k=0}^{\infty} g_k u^{k/c}, \tag{13}$$

where

$$g_k = \sum_{i,j=0}^{\infty} (-1)^{j+k} a_i \binom{i/a}{j} \binom{j/b}{k}.$$

For $0 < u < 1$, we have an expansion for u^ρ which holds for $\rho > 0$ real non-integer

$$u^\rho = \sum_{l=0}^{\infty} s_l(\rho)\, u^l, \tag{14}$$

where

$$s_l(\rho) = \sum_{m=l}^{\infty} (-1)^{m+l} \binom{\rho}{m} \binom{m}{l}.$$

Setting $\rho = k/c$ in (14) and substituting in (13), we can write

$$Q(u) = \sum_{k=0}^{\infty} g_k \sum_{l=0}^{\infty} s_l(k/c)\, u^l = \sum_{l=0}^{\infty} q_l\, u^l, \tag{15}$$

where $q_l = \sum_{k=0}^{\infty} g_k\, s_l(k/c)$.

Equation (15) is the main result of this section since it allows to obtain various mathematical quantities for the EKw-G class as proved in the next sections.

The formula derived throughout the paper can be easily handled in most symbolic computation software platforms such as Maple and Mathematica since they have currently the ability to deal with analytic expressions of formidable size and complexity. The infinity limit in the sums can be substituted by a large positive integer such as 20 or 30 for most practical purposes.

5. Moments

Hereafter, we shall assume that G is the cdf of a random variable Y and that F is the cdf of a random variable X having density function (4). The moments of X can be obtained from the (r,s) th probability weighted moments (PWMs) of Y defined by

$$\tau_{r,s} = E[Y^r\, G^s(Y)] = \int_{-\infty}^{\infty} y^r\, G^s(y)\, g(y)\, dy.$$

An alternative formula for $\tau_{r,s}$ can be based on the baseline quantile function $Q_G(x) = G^{-1}(x)$. Setting $G(x) = u$, we obtain

$$\tau_{r,s} = \int_0^1 Q_G^r(u)\, u^s\, du. \tag{16}$$

We use throughout the paper a result of Gradshteyn and Ryzhik (Gradshteyn & Ryzhik 2007) for a power series raised to a positive integer k (for $k \geq 1$)

$$\left(\sum_{l=0}^{\infty} b_l\, u^l\right)^k = \sum_{l=0}^{\infty} c_{k,l}\, u^l, \tag{17}$$

where the coefficients $c_{k,l}$ (for $l = 1, 2, \ldots$) are easily obtained from the recurrence equation (with $c_{k,0} = b_0^k$)

$$c_{k,l} = (l\, b_0)^{-1} \sum_{m=1}^{l} [m(k+1) - l]\, b_m\, c_{k,l-m}. \tag{18}$$

Clearly, $c_{k,l}$ can be determined from $c_{k,0}, \ldots, c_{k,l-1}$ and then from the quantities $b_0, \ldots, b_l$.

We can write using (12), (17) and (18)

$$Q_G^r(u) = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^r = \sum_{i=0}^{\infty} e_{r,i}\, u^i, \tag{19}$$

where $e_{r,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(r+1) - i]\, a_m\, e_{r,i-m}$ and $e_{r,0} = a_0^r$ and then an alternative expression for $\tau_{r,s}$ follows as

$$\tau_{r,s} = \sum_{i=0}^{\infty} \frac{e_{r,i}}{i+s+1}. \tag{20}$$

From equation (8), we can write

$$E(X^r) = \sum_{k=0}^{\infty} p_k \, \tau_{r,(k+1)a-1}, \tag{21}$$

where

$$p_k = (-1)^k \, a\, b\, c \sum_{j=0}^{\infty} (-1)^j \binom{c-1}{j} \binom{(j+1)\,b-1}{k}.$$

Thus, the EKw-G moments can be expressed as an infinite weighted sum of the baseline PWMs. The ordinary moments of several EKw-G distributions can be determined directly from equations (16) and (21). Expressions for moments of several exponentiated distributions are given by Nadarajah and Kotz (Nadarajah & Kotz 2006), which can be useful to produce $E(X^r)$. Table 1 gives some values from equation (21) for the EKw-exponential model (parameter λ) with different parameter values.

Table 1: The moments of EKw-exponential for ($a = 2$, $b = 1$ and $c = 1$) and some values of r and λ.

λ	1	2	3	4
r				
1	1.5	0.75	0.5	0.37
2	3.5	0.87	0.389	0.218
3	11.25	1.41	0.41	0.17
4	46.5	2.91	0.56	0.18

For lifetime models, it is usually of interest to compute the hth incomplete moment of X defined by $m_h(z) = \int_0^z x^h \, f(x)dx$. The quantity $m_h(z)$ can be calculated from (8) as

$$m_h(z) = \sum_{k=0}^{\infty} p_k \int_0^z x^h \, g(x) \, G^{(k+1)\,a-1}(x) \, dx.$$

Setting $u = G(x)$, we obtain

$$m_h(z) = \sum_{k=0}^{\infty} p_k \, \vartheta_{h,k}(z), \tag{22}$$

where

$$\vartheta_{h,k}(z) = \int_0^{G(z)} Q_G^h(u) \, u^{(k+1)\,a-1} \, du.$$

The quantity $\vartheta_{h,k}(z)$ is available for some baseline distributions and can also be computed numerically for most of them.

6. Generating Function

The moment generating function (mgf) $M(t) = E(\mathrm{e}^{tX})$ of X comes from (8) as an infinite weighted sum

$$M(t) = \sum_{k=0}^{\infty} w_k \, M_{(k+1)\,a}(t),$$

where $M_{(k+1)\,a}(t)$ is the mgf of $Y_{(k+1)\,a} \sim$ Exp-G($(k+1)a$). Hence, $M(t)$ can be determined from the mgf of $Y_{(k+1)\,a}$ given by

$$M_{(k+1)\,a}(t) = (k+1)\,a \int_0^{\infty} \mathrm{e}^{tx} g(x) G(x)^{(k+1)a-1} dx.$$

Setting $G(x) = u$, we can write $M_k(t)$ in terms of the baseline quantile function $Q_G(x)$

$$M_{(k+1)\,a}(t) = (k+1)\,a \int_0^{1} \exp[t\, Q_G(u)]\, u^{(k+1)a-1} du.$$

Now, we provide four representations for $M(t)$. The first one comes from (8) as

$$M(t) = \sum_{k=0}^{\infty} w_k \, M_{(k+1)\,a}(t).$$

A second representation for $M(t)$ is obtained from

$$f(x) = g(x) \sum_{k=0}^{\infty} t_k \, G^{(k+1)\,a-1}(x), \tag{23}$$

where $t_k = (-1)^k \, a \, b \sum_{j=0}^{\infty} (-1)^j \binom{c}{j+1} \binom{(j+1)b-1}{k}$. Thus,

$$M(t) = E\big(\mathrm{e}^{\mathrm{tX}}\big) = \sum_{\mathrm{k}=0}^{\infty} \mathrm{t_k} \, \xi_{\mathrm{k}}(\mathrm{t}; \mathrm{a}),$$

where $\xi_k(t; a) = \int_{-\infty}^{\infty} \mathrm{e}^{tx} \, g(x) \, G^{(k+1)\,a-1}(x) \, dx$.

Setting $x = Q_G(u) = G^{-1}(u)$ in (23), a third representation follows as

$$M(t) = \sum_{k=0}^{\infty} t_k \, \rho_{(k+1)\,a-1}(t),$$

where

$$\rho_a(t) = \int_0^{1} \exp[t\, Q_G(u)]\, u^a \, du.$$

A fourth representation can be determined from (19) by expanding the exponential function in the last equation

$$\rho_a(t) = \sum_{n,i=0}^{\infty} \frac{e_{n,i}\, t^n}{(i+a+1)\, n!}.$$

The best representation to derive a closed-form expression for $M(t)$ depends essentially on the forms of the pdf, cdf and quantile function of G.

7. Mean Deviations

The mean deviations about the mean $\delta_1(X) = E(|X - \mu_1'|)$ and about the median $\delta_2(X) = E(|X - M|)$ of X can be expressed as

$$\delta_1(X) = 2\mu_1' F(\mu_1') - 2m_1(\mu_1') \qquad \text{and} \qquad \delta_2(X) = \mu_1' - 2m_1(M),$$

respectively, where $\mu_1' = E(X)$ comes from (21) with $r = 1$, $F(\mu_1')$ is easily calculated from the cdf (3), $m_1(z) = \int_0^z x\, f(x)dx$ is the first incomplete moment of X computed from (22) and M is the median calculated from (10) as

$$M = Q_G\Big\{\Big[1 - \Big(1 - 2^{\frac{-1}{c}}\Big)^{\frac{1}{b}}\Big]^{\frac{1}{a}}\Big\}. \tag{24}$$

Applications of $m_1(z)$ include Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = m_1(q)/(\pi\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $q = Q(\pi)$ comes from (10).

8. Entropies

The entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. Two popular entropy measures are due to Shannon and Rényi (Shannon 1951, Rényi 1961). A large value of the entropy indicates the greater uncertainty in the data. The Rényi entropy is defined by (for $\gamma > 0$ and $\gamma \neq 1$)

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left(\int_0^\infty f^\gamma(x)dx\right).$$

Based on the pdf (4), the Rényi entropy of the EKw-G distribution is given by

$$\begin{aligned}\mathcal{J}_R(\gamma) &= \frac{1}{1-\gamma}\log\left(\int_0^\infty c^\gamma\, h_{a,b}^\gamma(x)\, H_{a,b}^{(c-1)\gamma}(x)dx\right)\\ &= \frac{1}{1-\gamma}\log(c^\gamma\, I_k),\end{aligned}$$

where

$$I_k = \int_0^\infty h_{a,b}^{\gamma}(x)\, H_{a,b}^{(c-1)\gamma}(x) dx.$$

We can determine the Rényi entropy using the integral

$$\begin{aligned}\int_0^\infty f^{\gamma}(x)dx = \quad & (a\, b\, c)^{\gamma} \int_0^\infty g^{\gamma}(x)\, G^{(a-1)\gamma}(x) \left[1 - G^{a}(x)\right]^{(b-1)\gamma} \\ \times \quad & \left\{1 - \left[1 - G^{a}(x)\right]^{b}\right\}^{(c-1)\gamma} dx\end{aligned}$$

and then expanding the binomial and changing variables

$$\int_0^\infty f^{\gamma}(x)dx = (a\, b\, c)^{\gamma} \sum_{j,k=0}^{\infty} (-1)^{j+k} \binom{(c-1)\gamma}{j} \binom{(j+\gamma)b-\gamma}{k} K(\gamma, k). \tag{25}$$

Here, $K(\gamma, k)$ denotes the integral

$$K(\gamma, k) = \int_0^1 g^{\gamma-1}[Q_G(u)]\, u^{(k+\gamma)\, a-\gamma}\, du,$$

which can be calculated for each G model. If $\gamma > 1$ and $a > 1$, the EKw-exponential, where $G(x) = 1 - \mathrm{e}^{-\lambda\, x}$ (with parameter λ), EKw-standard logistic, where $G(x) = (1 - \mathrm{e}^{-\nu\, x})^{-1}$, and EKw-Pareto, where $G(x) = 1 - x^{-\nu}$ (with parameter ν), distributions, are given by

$$K(\gamma, k) = \lambda^{\gamma-1}\, B((\gamma + k)\, a - \gamma + 1, \gamma), \quad K(\gamma, k) = \nu^{\gamma-1}\, B((\gamma + k)\, a, \gamma),$$

and

$$K(\gamma, k) = \nu^{\gamma-1} B((\gamma + k)\, a - \gamma + 1, (1 + \nu^{-1})(\gamma - 1) + 1),$$

respectively. In Table 2 we present a small illustration in which we calculate the Rényi entropy for EKw-exponential with some values of γ and λ.

Table 2: The Rényi entropy of EKw-exponential for ($a = 1$, $b = 2$ and $c = 1$) and some values of γ and λ.

λ	1.2	1.4	1.6	1.8
γ				
2	6.0199	2.6246	1.5271	0.9981
3	7.1356	2.7970	1.4196	0.7702
4	7.6624	2.6751	1.1157	0.3951
5	7.8665	2.3917	0.7043	−0.061

The Shannon entropy is given by

$$E\left\{-\log[f(X)]\right\} = -\log(c) - E\left\{\log[h_{a,b}(X)]\right\} - (c-1)\, E\left\{\log\left[H_{a,b}(X)\right]\right\}.$$

A general expression for $E[H_{a,b}(X)]$ follows from $f(x) = c\, h_{a,b}(x)\, H_{a,b}^{c-1}(x)$ by setting $H_{a,b}(x) = u$. We have

$$E[H_{a,b}(X)] = \int_{-\infty}^{\infty} c\, h_{a,b}(x) H_{a,b}^{c}(x) dx = \frac{c}{c+1}.$$

The quantity $\delta_X = E\left\{\log[h_{a,b}(X)]\right\}$ can be determined for special forms of $h_{a,b}(x)$. Thus, we obtain

$$E\left\{-\log[f(X)]\right\} = -\log\left(c\right) - \delta_X - \frac{c\left(c-1\right)}{c+1}. \tag{26}$$

Equations (25) and (26) are the main results of this section.

9. Reliability

The component fails at the instant that the random stress X_2 applied to it exceeds the random strength X_1, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = P(X_2 < X_1)$ is a measure of component reliability. It has many applications especially in the area of engineering. We derive the reliability R when X_1 and X_2 have independent EKw-G(a_1, b_1, c_1) and EKw-G(a_2, b_2, c_2) distributions with the same parameter vector $\boldsymbol{\eta}$ for G. The reliability is defined by

$$R = \int_0^\infty f_1(x) F_2(x) dx.$$

The pdf of X_1 and cdf of X_2 are obtained from equations (8) and (9) as

$$f_1(x) = g(x) \sum_{k=0}^\infty p_k(a_1, b_1, c_1)\, G(x)^{(k+1)a_1 - 1}$$

and

$$F_2(x) = \sum_{q=0}^\infty w_q(b_2, c_2)\, G(x)^{(q+1)a_2},$$

where

$$p_k(a_1, b_1, c_1) = (k+1)\, a_1\, w_k(b_1, c_1)$$

and

$$w_q(b_2, c_2) = \frac{(-1)^q\, b_2\, c_2}{q+1} \sum_{j=0}^\infty (-1)^j \binom{c_2 - 1}{j} \binom{(j+1)\, b_2 - 1}{q}.$$

Hence,

$$R = \sum_{k,q=0}^\infty p_k(a_1, b_1, c_1)\, w_q(b_2, c_2) \int_0^\infty g(x)\, G(x)^{(k+1)\, a_1 + (q+1)\, a_2 - 1}\, dx.$$

Setting $u = G(x)$, we obtain

$$R = \sum_{k,q=0}^\infty p_k(a_1, b_1, c_1)\, w_q(b_2, c_2) \int_0^1 u^{(k+1)\, a_1 + (q+1)\, a_2 - 1}\, du.$$

Finally, the reliability of the X reduces to

$$R=\sum_{k,q=0}^{\infty}\frac{p_k(a_1,b_1,c_1)\,w_q(b_2,c_2)}{(k+1)\,a_1+(q+1)\,a_2}. \tag{27}$$

Table 3 gives some values of R for different parameter values.

Table 3: The Reliability in (27) (with $a_1=a_2=2$ and $b_1=b_2=3$) for some values of c_1 and c_2

c_2 \ c_1	1	2	3	4
1	0.5000	0.1667	0.0833	0.0500
2	0.6667	0.2500	0.1333	0.0833
3	0.7500	0.3000	0.1667	0.1071
4	0.8000	0.3333	0.1905	0.1250

10. Estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the EKw-G distribution from complete samples only. Let $x_1,\dots,x_n$ be a observed sample of size n from the EKw-G$(a,b,c,\boldsymbol{\eta})$ distribution, where $\boldsymbol{\eta}$ is a $p\times 1$ vector of unknown parameters in the baseline distribution $G(x;\boldsymbol{\eta})$. The log-likelihood function for the vector of parameters $\boldsymbol{\theta}=(a,b,c,\boldsymbol{\eta})^T$ can be expressed as

$$\begin{aligned}\ell(\boldsymbol{\theta}) \;=\;& n\log(a)+n\log(b)+n\log(c)+\sum_{j=1}^{n}\log[g\left(x_j;\boldsymbol{\eta}\right)]\\ &+(a-1)\sum_{j=1}^{n}\log[G\left(x_j;\boldsymbol{\eta}\right)]\\ &+(b-1)\sum_{j=1}^{n}\log\left[1-G^a\left(x_j;\boldsymbol{\eta}\right)\right]\\ &+(c-1)\sum_{j=1}^{n}\log\left\{1-\left[1-G^a\left(x_j;\boldsymbol{\eta}\right)\right]^b\right\}.\end{aligned} \tag{28}$$

The components of the score vector $U(\boldsymbol{\theta})$ are

$$\begin{aligned}U_a(\boldsymbol{\theta})=\frac{n}{a}+\sum_{j=1}^{n}\log[G\big(x_j;\boldsymbol{\eta}\big)]-(b-1)\sum_{j=1}^{n}\frac{G^a\big(x_j;\boldsymbol{\eta}\big)\log[G\big(x_j;\boldsymbol{\eta}\big)]}{1-G^a\big(x_j;\boldsymbol{\eta}\big)}\\ +\,b\,(c-1)\sum_{j=1}^{n}\frac{G^a\big(x_j;\boldsymbol{\eta}\big)\big[1-G^a\big(x_j;\boldsymbol{\eta}\big)\big]^{b-1}\log[G\big(x_j;\boldsymbol{\eta}\big)]}{1-\big[1-G^a\big(x_j;\boldsymbol{\eta}\big)\big]^b},\end{aligned}$$

$$
\begin{aligned}
U_b(\boldsymbol{\theta}) = \frac{n}{b} &+ \sum_{j=1}^{n} \log\left\{1 - G^a\big(x_j;\boldsymbol{\eta}\big)\right\} \\
&- (c-1)\sum_{j=1}^{n} \frac{\left[1 - G^a\big(x_j;\boldsymbol{\eta}\big)\right]^b \log\left[1 - G^a\big(x_j;\boldsymbol{\eta}\big)\right]}{1 - \left[1 - G^a\big(x_j;\boldsymbol{\eta}\big)\right]^b},
\end{aligned}
$$

$$
U_c(\boldsymbol{\theta}) = \frac{n}{c} + \sum_{j=1}^{n} \log\left\{1 - \left[1 - G^a\big(x_j;\boldsymbol{\eta}\big)\right]^b\right\}
$$

and

$$
\begin{aligned}
U_{\eta_k}(\boldsymbol{\theta}) = &\sum_{j=1}^{n} \frac{1}{g\left(x_j;\boldsymbol{\eta}\right)} [\dot{g}(x_j;\boldsymbol{\eta})]_{\eta_k} + (a-1)\sum_{j=1}^{n} \frac{1}{G\left(x_j;\boldsymbol{\eta}\right)} [\dot{G}(x_j;\boldsymbol{\eta})]_{\eta_k} \\
&- a(b-1)\sum_{j=1}^{n} \frac{G^{a-1}\left(x_j;\boldsymbol{\eta}\right)}{1 - G^a\left(x_j;\boldsymbol{\eta}\right)} [\dot{G}(x_j;\boldsymbol{\eta})]_{\eta_k} \\
&+ ab(c-1)\sum_{j=1}^{n} \frac{G^{a-1}\left(x_j;\boldsymbol{\eta}\right)\left[1 - G^a\left(x_j;\boldsymbol{\eta}\right)\right]^{b-1}}{1 - \left[1 - G^a\left(x_j;\boldsymbol{\eta}\right)\right]^b} [\dot{G}(x_j;\boldsymbol{\eta})]_{\eta_k},
\end{aligned}
$$

where $[\dot{g}(x_j;\boldsymbol{\eta})]_{\eta_k} = \frac{\partial g(x_j;\boldsymbol{\eta})}{\partial \eta_k}$ and $[\dot{G}(x_j;\boldsymbol{\eta})]_{\eta_k} = \frac{\partial G(x_j;\boldsymbol{\eta})}{\partial \eta_k}$ for $k = 1,\ldots,p$. For interval estimation of the model parameters, we require the total observed information matrix $J_n(\boldsymbol{\theta})$, whose elements can be obtained from the authors upon request. Let $\widehat{\boldsymbol{\theta}}$ be the MLE of $\boldsymbol{\theta}$. Under standard regularity conditions (Cox & Hinkley 1974), we can approximate the distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ by the multivariate normal $N_{(p+3)}(0, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta}) = \lim_{n\to\infty} n^{-1} J_n(\boldsymbol{\theta})$ is the unit information matrix. Based on the approximate multivariate normal $N_{(p+3)}(\boldsymbol{\theta}, J_n(\widehat{\boldsymbol{\theta}})^{-1})$ distribution of $\widehat{\boldsymbol{\theta}}$, where $J_n(\widehat{\boldsymbol{\theta}})$ is the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, we can construct approximate confidence regions for the model parameters.

11. Simulations

We perform some simulations with the objective to note the behavior of the MLEs obtained by the BFGS method. It is used for maximizing the log-likelihood function of a probabilistic model. In some complex distributions the task of optimization can be quite complicated.

We consider ten thousand replicas of Monte Carlo (MC) under different sample sizes ($n = 20, 60, 100, 200, 500$ and 1000). For each sample size, we compute the average MLEs obtained by the BFGS method and correct these estimates by using non-parametric bootstrap. We also compute the bootstrap errors and the biases of the MLEs for the model parameters by considerig the exponential distribution with parameter $\lambda > 0$ as the baseline in Table 4. Similarly, we perform other simulations with the same scenarios by considering the Weibull distribution with parameters $\alpha = 1.1$ and $\beta = 1.5$ (fixed) as the baseline (Table 5).

The corrected MLEs are based on 500 replicas of bootstrap and the true parameter values considered are $a = b = c = \lambda = 1.1$. For each MC iteration, we consider only samples in which there have been convergence of the BFGS method. It is very important to eliminate the non-convergence of the simulations, for not mislead the results obtained and with the penalty of having simulations in some problems involving the new class of distributions. Computationally intensive of akwards log-likelihood for som families of distributions are fairly complicated, having many regions approximately flat. In our case, to carry out the simulations we use a computer with 32 GB of RAM memory, operating system Arch Linux with processor Intel Core i7-4710QM octa core with each core working at a frequency of 3.5 GHz.

We implement the code using the `Julia` version 0.6.3 programming language which is a pseudo-compiled general-purpose language and has several functions that facilitate its use in scientific computing. `Julia` is a language that provides several advantages when considering the implementation in statistical simulations that are usually computationally intensive. Among these advantages can be highlighted its computational efficiency by using a Low Level Virtual Machine (LLVM) based compiler with run-time compilation (JIT). JIT computing solutions exist in language like `R` (see `compiler` package), however it is something that has been recently developed unlike `Julia` which is a compiled language by definition. Several benckmarks can be obtained on the web showing the superiority with respect to the computational efficiency of the language `Julia` comparing it with other languages.

In addition to being a compiled language by definition, `Julia` is a programming language that is being designed for parallel computing and does not impose any specific parallelism to the programmer. Instead, it provides several important building blocks for distributed computing, making it flexible enough to support multiple styles of parallelism and allowing users to add other styles. In order to use all the computational resources of the available hardware, each Monte Carlo iteration was broken into threads using the macro `@threads`. In this way, it was possible to perform eight iterations simultaneously at every step. In order to be able to execute the code below it is necessary to install the libraries `Distributions` and `Optim` to have access to some functions of probability distributions and global optimization, respectively. In Appendix we provide the `Julia` script setting the exponential distribution for G. By the simulations it is possible to observe improvements in the MLEs obtained when using bootstrap correction. However, the corrections are more significant and perhaps more justifiable in small sample sizes (20 or 60).

12. Applications

In this section, we fit the EKwW distribution to two real data sets and for illustrative purposes also present a comparative study with the fits of some nested and non-nested models. These applications prove empirically the flexibility of the new distribution in modeling positive data. All the computations are performed using the `R` software (*R Development Core Team, R: A Language and*

Table 4: Estimates of maximum likelihood corrected, errors and biases obtained by bootstrap in different sizes of samples for EKw-exponential distribution.

n	$\boldsymbol{\theta}$	$\hat{\boldsymbol{\theta}}$	$\widehat{Bias}(\hat{\boldsymbol{\theta}})$	Error	$\hat{\boldsymbol{\theta}}_c$	Time (Hours)
$\boldsymbol{n=20}$	$a=1.1$	1.8342	0.4123	1.8453	1.4426	3.6543
	$b=1.1$	2.6436	0.4444	1.9453	2.2253	
	$c=1.1$	2.5625	0.3564	1.7653	2.3742	
	$\lambda=1.1$	1.4390	0.4523	1.9486	1.3524	
$\boldsymbol{n=60}$	$a=1.1$	1.7544	0.4526	1.7544	1.3540	12.2435
	$b=1.1$	1.4566	0.6543	1.7643	1.2535	
	$c=1.1$	1.8885	0.6533	1.3455	1.2454	
	$\lambda=1.1$	1.5540	0.9653	1.5464	1.6535	
$\boldsymbol{n=100}$	$a=1.1$	1.6534	0.2123	1.1235	1.4311	13.6543
	$b=1.1$	1.3635	0.2347	1.7654	1.2633	
	$c=1.1$	1.6549	0.3543	1.4326	1.1533	
	$\lambda=1.1$	1.5445	0.1114	1.4326	1.4634	
$\boldsymbol{n=200}$	$a=1.1$	1.1453	0.0911	0.9688	1.0409	17.5061
	$b=1.1$	1.2134	0.3542	1.2145	1.1355	
	$c=1.1$	1.3214	0.3443	1.3413	1.5421	
	$\lambda=1.1$	1.1211	0.1111	0.7544	1.3234	
$\boldsymbol{n=500}$	$a=1.1$	1.0033	0.0234	0.5432	0.6433	38.1216
	$b=1.1$	1.1995	0.1543	0.6545	1.1546	
	$c=1.1$	1.4345	0.0542	0.5637	1.1100	
	$\lambda=1.1$	1.2352	0.05664	0.4556	1.1100	
$\boldsymbol{n=1,000}$	$a=1.1$	1.1001	0.0134	0.2435	1.0999	51.5454
	$b=1.1$	1.2542	0.2452	0.5463	1.1013	
	$c=1.1$	1.2453	0.1351	0.3545	1.0987	
	$\lambda=1.1$	1.1165	0.1344	0.4453	1.1012	

Environment for Statistical Computing 2012). First, we consider the data on the waiting times between 65 consecutive eruptions of the Kiama Blowhole (Silva, Andrade, Maciel, Campos & Cordeiro 2013). These data can be obtained at http://www.statsci.org/data/oz/kiama.html:
83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35,47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Second, we also consider also the data on 101 observations corresponding to the failure times of Kevlar 49/epoxy strands with pressure at 90%. The failure times in hours were originally given in Barlow, Toland & Freeman (1984), Andrews & Herzberg (2012) and analyzed by Cooray & Ananda (2008): 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

TABLE 5: Estimates of maximum likelihood corrected, errors and biases obtained by bootstrap in different sizes of samples for EKwW distribution (with $\beta = 1.5$ fixed).

n	$\boldsymbol{\theta}$	$\hat{\boldsymbol{\theta}}$	$\widehat{Bias}(\hat{\boldsymbol{\theta}})$	Error	$\hat{\boldsymbol{\theta}}_c$	Time (Hours)
$\boldsymbol{n = 20}$	$a = 1.1$	1.3443	0.6543	1.6543	1.2344	4.4353
	$b = 1.1$	1.6543	0.3235	0.7533	1.3450	
	$c = 1.1$	1.6530	0.6343	0.6534	1.3564	
	$\alpha = 1.1$	1.5483	0.7342	1.4328	1.3205	
$\boldsymbol{n = 60}$	$a = 1.1$	1.4453	1.4345	1.3452	1.1134	9.7654
	$b = 1.1$	1.3123	0.5233	1.1455	1.1323	
	$c = 1.1$	1.4345	0.5654	1.5543	1.1340	
	$\alpha = 1.1$	1.3984	0.1546	1.2565	1.1024	
$\boldsymbol{n = 100}$	$a = 1.1$	1.1041	0.1545	0.6753	1.1013	14.5434
	$b = 1.1$	1.1010	0.1535	1.3345	1.1007	
	$c = 1.1$	1.0789	0.0356	1.1445	1.1000	
	$\alpha = 1.1$	1.1345	0.0745	0.6744	1.1031	
$\boldsymbol{n = 200}$	$a = 1.1$	1.1543	0.1963	0.8764	1.1001	18.5445
	$b = 1.1$	1.1044	0.4353	1.1835	1.1003	
	$c = 1.1$	1.1011	0.1452	1.3233	1.1000	
	$\alpha = 1.1$	1.0234	0.0434	0.8654	1.1054	
$\boldsymbol{n = 500}$	$a = 1.1$	1.1035	0.0543	0.5435	1.1112	37.5266
	$b = 1.1$	1.1004	0.0535	0.4324	1.0988	
	$c = 1.1$	1.1420	0.0544	0.2345	1.1654	
	$\alpha = 1.1$	1.4354	0.0465	0.6532	1.1012	
$\boldsymbol{n = 1000}$	$a = 1.1$	1.1008	0.0045	0.0033	1.1001	52.7833
	$b = 1.1$	1.1124	0.0656	0.3454	1.1054	
	$c = 1.1$	1.1023	0.0433	0.4533	1.0345	
	$\alpha = 1.1$	1.1034	0.0345	0.0653	1.1000	

Table 6 lists the MLEs (and the corresponding standard errors in parentheses) of the unknown parameters of the EKwW, *Kumaraswamy Weibull* (KwW), *exponentiated Weibull* (EW), *gamma Weibull* (GW) (Zografos & Balakrishnan 2009) and *Weibull* (W) models for the eruptions times. Table 7 lists the MLEs (and the corresponding standard errors in parentheses) of the unknown parameters of the EKwW, KwW, *beta Weibull* (BW), EW, *generalized power Weibull* (GPW) (Bagdonavicius & Nikulin 2002), *exponentiated Nadarajah-Haghighi* (ENH) (Lemonte 2013), *Weibull* (W), *flexible Weibull* (FW) (Bebbington, Lai & Zitikis 2007) and *Nadarajah-Haghighi* (NH) (Nadarajah & Haghighi 2011) models for the failure times. It is important to emphasize that the BW and EW distributions are popular model for the analysis of lifetime data. The pdf's of the EW, GPW, ENH, FW and NH distributions are given by

$$f_{\mathrm{EW}}(x) = a\,\alpha\,\beta^{\alpha}\,x^{\alpha-1}\,\mathrm{e}^{-(\beta\,\mathrm{x})^{\alpha}}\,[1-\mathrm{e}^{-(\beta\,\mathrm{x})^{\alpha}}]^{\mathrm{a}-1},$$

$$f_{\mathrm{GPW}}(x) = a\,\alpha\,\beta\,x^{\beta-1}\,(1+\alpha\,x^{\beta})^{a-1}\,\exp\{1-(1+\alpha\,x^{\beta})^{a}\},$$

$$f_{\mathrm{ENH}}(x) = a\,\alpha\,\beta\,(1+\beta\,x)^{\alpha-1}\,\mathrm{e}^{1-(1+\beta\,x)^{\alpha}}\,\{1-\mathrm{e}^{1-(1+\beta\,x)^{\alpha}}\}^{a-1},$$

$$f_{\mathrm{FW}}(x) = \left(\alpha + \frac{\beta}{x^2}\right) \exp\left(\alpha\, x + \frac{\beta}{x}\right) \exp\left\{-\exp\left(\alpha\, x + \frac{\beta}{x}\right)\right\}$$

and

$$f_{\mathrm{NH}}(x) = \alpha\,\beta\,(1+\beta\, x)^{\alpha-1} \exp\{1-(1+\beta\, x)^{\alpha}\},$$

respectively.

TABLE 6: MLEs (standard errors in parentheses).

Distributions	**Estimates**				
EKwW(a, b, c, α, β)	29.042506	0.1173240	0.9751186	0.7531988	0.6848540
	(0.117038)	(0.019383)	(0.158727)	(0.003279)	(0.003326)
KwW(a, b, α, β)	19.864245	0.1347696	0.8060101	0.5112256	
	(5.713535)	(0.017590)	(0.002583)	(0.008164)	
EW(a, α, β)	18.290334	0.4087716	0.6603427		
	(37.19796)	(0.235609)	(2.125421)		
GW(a, α, β)	13.684984	0.3228614	103.91514		
	(9.681577)	(0.115217)	(528.3076)		
W(α, β)	1.2726761	0.0231553			
	(0.120163)	(0.002405)			

TABLE 7: MLEs (standard errors in parentheses).

Distributions	**Estimates**				
EKwW(a, b, c, α, β)	0.514602	0.204198	1.103498	1.015556	4.310142
	(0.2237)	(0.0479)	(0.2664)	(0.0026)	(0.9996)
KwW(a, b, α, β)	1.703998	0.217699	1.012092	4.362698	
	(0.4540)	(0.0414)	(0.0031)	(0.9099)	
BW(a, b, α, β)	0.761867	2.462283	1.093998	0.332994	
	(0.3179)	(9.7398)	(0.2981)	(1.2666)	
EW(a, α, β)	0.792934	1.060442	0.821001		
	(0.2870)	(0.2398)	(0.2651)		
GPH(a, α, β)	1.2659	0.7182	0.8696		
	(0.4483)	(0.3485)	(0.1039)		
ENH(a, α, β)	1.0732	0.7762	0.8426		
	(0.2760)	(0.3582)	(0.1238)		
W(α, β)	0.925888	1.010156			
	(0.0725)	(0.1140)			
FW(α, β)	0.3287	0.0838			
	(0.0246)	(0.0133)			
NH(α, β)	0.8898	1.1810			
	(0.1853)	(0.4270)			

Next, we apply formal goodness-of-fit tests in order to verify which distribution fits better to these data. We consider the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics described in Chen & Balakrishnan (1995). In general, the smaller the values of these statistics, the better the fit to the data. Let $F(x;\boldsymbol{\theta})$ be the cdf, where the form of F is known but $\boldsymbol{\theta}$ (a k-dimensional parameter vector, say) is unknown. To obtain the statistics W^* and A^*, we can proceed as follows: (i) Compute $v_i = F(x_i; \widehat{\boldsymbol{\theta}})$, where the x_i's are in ascending order; (ii) Compute $y_i = \Phi^{-1}(v_i)$, where $\Phi(\cdot)$ is the standard normal cdf and $\Phi^{-1}(\cdot)$ its inverse; (iii) Compute $u_i = \Phi\{(y_i - \bar{y})/s_y\}$, where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$; (iv) Calculate

$$W^2 = \sum_{i=1}^{n}\left\{u_i - \frac{(2i-1)}{2n}\right\}^2 + \frac{1}{12n}$$

and

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\{(2i-1)\log(u_i) + (2n+1-2i)\log(1-u_i)\};$$

(v) Modify W^2 into $W^* = W^2(1 + 0.5/n)$ and A^2 into $A^* = A^2(1 + 0.75/n + 2.25/n^2)$.

The statistics W^* and A^* for all the models are given in Tables 8 and 9 for the current data. The proposed EKwW model fits these data better than the other models based on the values of W^* and A^*. This model may be an interesting alternative to other models available in the literature for modeling positive real data.

TABLE 8: Formal statistics

Distributions	$\mathbf{A}^*$	$\mathbf{W}^*$
EKwW	**0.7594**	**0.1037**
GW	0.7927	0.1088
EW	0.8413	0.1148
KwW	0.9098	0.1304
W	1.0081	0.1471

TABLE 9: Formal statistics

Distributions	$\mathbf{W}^*$	$\mathbf{A}^*$
EKwW	**0.1204**	**0.7657**
KwW	0.2630	1.4264
BW	0.1628	0.9480
EW	0.1652	0.9586
GPH	0.1730	0.9930
ENH	0.1670	0.9667
W	0.1986	1.1111
FW	1.1130	5.9971
NH	0.2053	1.1434

More information is provided by a visual comparison of the histograms of the data sets and the main fitted densities (Figures 10 and 11). These plots indicate that the new distribution provides a good fit to both data and it is a very competitive model to the classical EW and BW distributions.

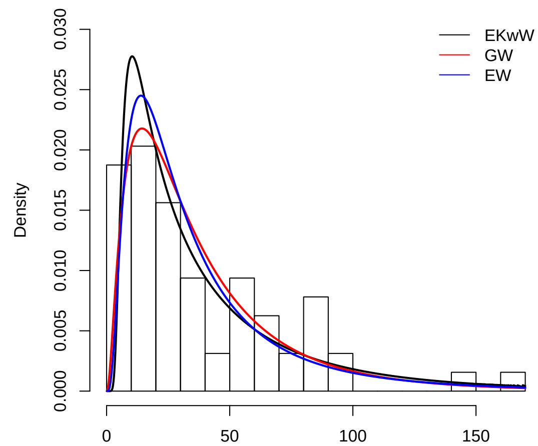

Figure 10: Fitted densities for the eruptions times.

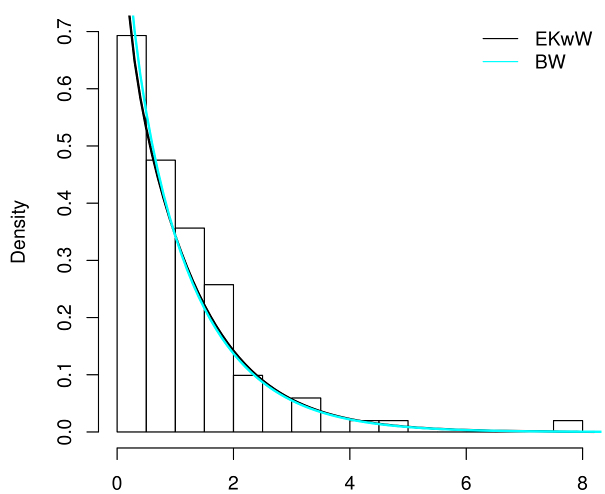

Figure 11: Fitted densities for the failure times.

13. Conclusions

A new family of continuous distributions called the *exponentiated Kumaraswamy-G* ("EKw-G") class is introduced and studied. The proposed class contains three parameters more than those in the baseline distribution. Several new models can be generated based on this family by considering special cased for G. We demonstrate that the EKw-G density function can be expressed as a linear combination of exponentiated-G (Exp-G) density functions. This result allows us to obtain general explicit expressions for some measures of the EKw-G class such as the ordinary and incomplete moments, generating function and mean deviations.

Explicit expressions for two types of entropy and reliability are given. We provide a power series for the quantile function which holds in generality. We discuss maximum likelihood estimation. A simulation study is performed by means of Monte Carlo experiments with the objective to observe the behavior of the maximum likelihood estimates obtained by the BFGS method. The inference on the model parameters is based on Cramér-von Mises and Anderson-Darling statistics. Two applications of a special model of the proposed class to real data demonstrates its potentiality. We hope this generalization may attract several applications in statistics, biology, engineering and other areas.

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Appendix - Simulation codes

```
using Distributions
using Optim

function gexp(x,par)
lambda = par[1]
lambda * exp(-lambda * x)
end

function Gexp(x,par)
```

```
lambda = par[1]
1- exp(-lambda * x)
end

function QGexp(x,par)
lambda = par[1]
quantile.(Exponential(1/lambda),x)
end

function sample_ekwg(QG, n, par0, id, par1...)
a = par0[1]
b = par0[2]
c = par0[3]

u = rand(LOCAL_R[id],n)

p = (1 - (1 - u.^(1/c)).^(1/b)).^(1/a)

QG(p, par1...)
end

function cdf_ekwg(cdf, x, par0, par1...)
a = par0[1]
b = par0[2]
c = par0[3]

(1 - (1 - cdf.(x,par1...).^a).^b).^c
end


function pdf_ekwg(cdf, pdf, x, par0, par1...)
a = par0[1]
b = par0[2]
c = par0[3]

g = pdf(x, par1...)
G = cdf(x, par1...)

a * b * c * g * G.^(a-1) * (1-G.^a).^(b-1) *
1 - (1-G.^a).^b).^(c-1)

end

function loglike(cdf, pdf, x, par0, par1...)
n = length(x)
soma = 0
for i = 1:n
soma += log(pdf_ekwg(cdf, pdf, x[i], par0, par1...))
end
return -soma
end
```

```
function myoptimize(sample_boot)
try
optimize(par0 -> loglike(G, g, sample_boot, par0,
par1...), starts,
Optim.Options(g_tol = 1e-2))
catch
0.0
end
end

function ekwg_bootstrap_bias(B, G, g, data,
original_estimates,
starts, par1...)

result_boot = Vector(length(original_estimates) * B)

j = 1
while j <= B
sample_boot = sample(data, length(data), replace = true)

result = myoptimize(sample_boot)

if (result == 0.0) || (result.g_converged == false)
continue
end

result_boot[(3*j-2):3*j] = result.minimizer

j = j+1
end # Here ends the while.

estimates_matrix = convert.(Float64,reshape(result_boot,
length(starts),B))'
error  = std(estimates_matrix,1)
return error, (2.*original_estimates'
.- mean(estimates_matrix,1))'
end

function ekwg_monte_carlo_bias(M, B, n, true_parameters, par1...)

result_mc_correct_vector = Vector(length(true_parameters)*M)
result_mc_vector = Vector(length(true_parameters)*M)
result_error_boot = Vector(length(true_parameters)*M)

Threads.@threads for i in 1:M
true_sample = sample_ekwg(QGexp, n, true_parameters,
Threads.threadid(), par1...)
result_mc = myoptimize(true_sample)

if result_mc != 0.0
```

```
result_mc_vector[(3*i-2):3*i] = result_mc.minimizer
result_error_boot[(3*i-2):3*i],
result_mc_correct_vector[(3*i-2):3*i] =
ekwg_bootstrap_bias(b, G, g, true_sample,
result_mc.minimizer, true_parameters, par1...)
end
end

output1 = convert.(Float64,reshape(result_mc_vector,
length(true_parameters),M))'
output2 = convert.(Float64,reshape(result_mc_correct_vector,
length(true_parameters),M))'
output3 = convert.(Float64,reshape(result_error_boot,
length(true_parameters),M))'
return (mean(output1,1),mean(output2,1),mean(output3,1))
end

g = gexp;
G = Gexp;
qgexp = QGexp;
global starts = [1.0,1.0,1.0];
global true_parameters = [1.0,1.0,1.0];
global par1 = 1.5;
m = 5_000;
b = 250;
n = 20;

global const LOCAL_R = randjump(MersenneTwister(1),Threads.nthreads());

@time mc_estimates, mc_estimates_boot, mc_error_boot =
ekwg_monte_carlo_bias(m, b, n, true_parameters, par1)
print("\n---> Average uncorrected estimates: ", mc_estimates,"\n")
print("\n---> Mean of the bootstrap-corrected estimates: ",
mc_estimates_boot,"\n")
print("\n---> Average bootstrap error estimates: ",
mc_error_boot)
```