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On the Alpha Power Kumaraswamy Distribution: Properties, Simulation and Application

Distribución alpha-power-Kumaraswamy: propiedades, simulación y aplicación

Mohamed Ali Ahmed[a]

Department of Statistics, Mathematics and Insurance, Al Madina Higher Institute of Management and Technology, Giza, Egypt.

Abstract

Adding new parameters to classical distributions becomes one of the most important methods for increasing distributions flexibility, especially, in simulation studies and real data sets. In this paper, alpha power transformation (APT) is used and applied to the Kumaraswamy (K) distribution and a proposed distribution, so called the alpha power Kumaraswamy (AK) distribution, is presented. Some important mathematical properties are derived, parameters estimation of the AK distribution using maximum likelihood method is considered. A simulation study and a real data set are used to illustrate the flexibility of the AK distribution compared with other distributions.

***Key words*:** Alpha power transformation; Maximum likelihood estimation; Moments; Orders statistics; The Kumaraswamy distribution.

Resumen

Agregar nuevos parámetros a las distribuciones clásicas se convierte en uno de los métodos más importantes para aumentar la flexibilidad de las distribuciones, especialmente en estudios de simulación y conjuntos de datos reales. En este documento, se utiliza la transformación de potencia alfa (TPA) y es aplicada a la distribución de Kumaraswamy (K) y a una distribución propuesta, denominada distribución de energía alfa de Kumaraswamy (AK). Se derivan algunas propiedades matemáticas, y se muestra la estimación de parámetros de la distribución AK utilizando el método de máxima verosimilitud. Un estudio de simulación y un conjunto de datos reales se utilizan para ilustrar la flexibilidad de la distribución AK en comparación con otras distribuciones.

***Palabras clave*:** Transformación de potencia alfa; Estimación de máxima verosimilitud; Momentos; Estadísticas de pedidos; La distribución de Kumaraswamy.

[a]Ph.D. E-mail: mrmohamedali2005@yahoo.com

1. Introduction

Kumaraswamy (1980) proposed a probability distribution for double bounded random processes of hydrological applications, with the following cumulative distribution function (CDF) and probability density function (PDF), respectively,

$$F(x) = 1 - \left(1 - x^{\beta}\right)^{\theta} ; 0 < x < 1;\ \beta, \theta > 0, \tag{1}$$

and

$$f(x) = \beta\theta X^{\beta-1}\left(1 - x^{\beta}\right)^{\theta-1}. \tag{2}$$

Jones (2009) illustrated that the K distribution has some properties like the beta distribution such as both of their densities are unimodal, uniantimodal, increasing, decreasing or constant depending on its parameters, where the CDF and PDF of the beta distribution are

$$F(x, a, b) = I_x(a, b); 0 < x < 1;\ a\ > 0,\ b\ > 0,$$

and

$$f(x, a, b) = \frac{x^{a-1}(1 - x)^{b-1}}{B(a, b)},$$

where, I_x is the regularized incomplete beta function.

Also, Jones (2009) highlighted several advantages of the K distribution over beta distribution: the K distribution is very simple to use especially in simulation studies due to the simple closed form of both its quantile function and cumulative distribution function, the normalized constant of the K distribution is very simple, his explicit formulae for the quantile function, distribution function and moments of order statistics and has a simple formulae for L-moments. Mahdavi & Kundu (2017) presented, for the first time, a transformation of the parent CDF by adding a new parameter to derive a proposed family of distributions, this method is called the APT method. The main object of this study is to propose and study a new distribution called APK distribution based on the APT method. The paper is organized as follows: In section 1, the introduction of the study is presented. In section 2, the proposed model is derived, its special cases are presented and its asymptotes are studied. In section 3, some properties are obtained. In section 4, the Hazard function is given. In section 5, the Rényi entropy is obtained. In section 6, the stress strength model is given. In section 7, order statistics are studied. In Section 8, the distribution parameters are estimated by the maximum likelihood estimation (MLE) method. In Section 9, a simulation study is illustrated. Finally, in Section 10, an application is used to investigate, practically, the flexibility of the proposed distribution.

2. The New AK Distribution

Since, the CDF and PDF of the APT, Mahdavi & Kundu (2017), for a continuous random variable X, respectively, are

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1}; \alpha > 0, \alpha \neq 1, \\ F(x);\ \alpha = 1, \end{cases} \tag{3}$$

and

$$f_{APT}(x) = \begin{cases} \frac{\log\,\alpha}{\alpha-1} f(x)\, \alpha^{F(x)}; \alpha > 0, \alpha \neq 1, \\ f(x);\ \alpha = 1. \end{cases} \tag{4}$$

The AK distribution can be derived, as follows: substituting (1) into (3) gives

$$F_{\alpha K}(x) = \frac{\alpha^{1-\left(1-x^{\beta}\right)^{\theta}} - 1}{\alpha - 1}, 0 < x < 1; \alpha, \beta, \theta > 0; \alpha \neq 1, \tag{5}$$

differentiating last equation (w.r.t. x) yields

$$f_{\alpha K}(x) = \frac{\beta\theta \log \alpha}{\alpha - 1} \alpha^{1-\left(1-x^{\beta}\right)^{\theta}} x^{\beta-1} \left(1 - x^{\beta}\right)^{\theta-1}, \tag{6}$$

when $\alpha = 1$, the AK distribution reduces to K distribution, Kumaraswamy (1980), setting $\theta = 1$ gives the alpha power function (AP) distribution, and setting $\alpha = 1$, $\theta = 1$ gives the power function (P) distribution, Meniconi & Barry (1996). Some shapes of the density function for the AK distribution are illustrated in figure 1.

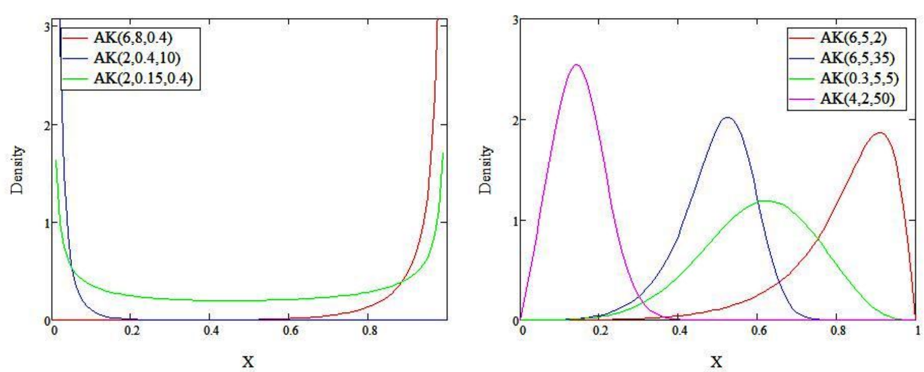

FIGURE 1: The AK density functions.

2.1. The Expansions for the CDF and PDF

In this section, expansions for the CDF and PDF of the AK distribution will be obtained

2.1.1. An Expansion for the CDF

Using exponential expansion for (5) gives

$$F_{\alpha K}(x) = \frac{1}{1-\alpha}\left(1-\sum_{i=0}^{\infty}\frac{[\log\alpha]^i}{i!}\left[1-\left(1-x^{\beta}\right)^{\theta}\right]^i\right),$$

then, using binomial expansion for last equation yields

$$F_{\alpha K}(x) = \frac{1}{1-\alpha}\left(1-\sum_{i=0}^{\infty}\frac{[\log\alpha]^i}{i!}\sum_{j=0}^{i}(-1)^j\left(\begin{array}{c} i \\ j \end{array}\right)\left(1-x^{\beta}\right)^{j\theta}\right),$$

since,

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i} = \sum_{j=0}^{\infty}\sum_{i=j}^{\infty},$$

then,

$$F_{\alpha K}(x) = \frac{1}{1-\alpha}\left(1-\sum_{j=0}^{\infty}w_j\left(1-x^{\beta}\right)^{j\theta}\right), \tag{7}$$

where,

$$w_j = \sum_{i=j}^{\infty}\frac{[\log\alpha]^i}{i!}(-1)^j\left(\begin{array}{c} i \\ j \end{array}\right).$$

2.1.2. An Expansion for the PDF

Differentiating (7) (w.r.t. x) gives

$$f_{\alpha K}(x) = \frac{\beta\theta}{1-\alpha}\sum_{j=1}^{\infty}w_j\,x^{\beta-1}\left(1-x^{\beta}\right)^{j\theta-1},$$

shifting j leads to

$$f_{\alpha K}(x) = \frac{\beta\theta}{1-\alpha}\sum_{j=0}^{\infty}w_{j+1}\,x^{\beta-1}\left(1-x^{\beta}\right)^{(j+1)\theta-1}, \tag{8}$$

where,

$$w_{j+1} = \sum_{i=j+1}^{\infty}\frac{[\log\alpha]^i}{i!}(-1)^{j+1}\left(\begin{array}{c} i \\ j+1 \end{array}\right);\ i\geq j+1.$$

Conditions of the Expansion for the PDF. Since,

$$\frac{\theta}{1-\alpha}\sum_{j=0}^{\infty}w_{j+1}\int_0^1\beta\,x^{\beta-1}\left(1-x^{\beta}\right)^{(j+1)\theta-1}dx = 1,$$

then,

$$\frac{-\theta}{1-\alpha}\sum_{j=0}^{\infty} w_{j+1}\left[\frac{\left(1-x^{\beta}\right)^{(j+1)\theta-1}}{(j+1)\,\theta}\right]_0^1 = 1,$$

moreover,

$$\frac{\sum\limits_{j=0}^{\infty} w_{j+1}}{(1-\alpha)\,(j+1)} = 1,$$

hence,

$$\sum_{j=0}^{\infty} w_{j+1} = (1-\alpha)\,(j+1)\,. \tag{9}$$

One can obtain this condition in different form as follows:

since,

$$\frac{\theta}{1-\alpha}\sum_{j=0}^{\infty} w_{j+1}\int_0^1 \beta\, x^{\beta-1}\left(1-x^{\beta}\right)^{(j+1)\theta-1} dx = 1,$$

then,

$$\frac{\theta}{1-\alpha}\sum_{j=0}^{\infty} w_{j+1}\, B\left(1,(j+1)\,\theta\right) = 1,$$

substituting (9) into last equation gives

$$\theta\,(j+1)\,B\left(1,(j+1)\,\theta\right) = 1,$$

hence,

$$B\left(1,(j+1)\,\theta\right) = \frac{1}{(j+1)\,\theta}. \tag{10}$$

2.2. The Asymptotes of the CDF and PDF

In this section, the asymptotes of the CDF and PDF of the AK distribution will be obtained.

2.2.1. The Asymptotes of the CDF

First: as x converges to zero. Using only first and second terms of exponential expansion for the CDF gives

$$F_{\alpha K}(x) \sim \frac{1}{\alpha-1}\left\{\left[1-\left(1-x^{\beta}\right)^{\theta}\right]\log\,\alpha\right\},$$

using only first and second terms of binominal expansion gives

$$F_{\alpha K}(x) \sim \frac{1}{\alpha-1}\left\{\left[1-\left(1-\theta x^{\beta}\right)\right]\log\,\alpha\right\},$$

then,

$$F_{\alpha K}(x) \sim \frac{1}{\alpha - 1} \log \, \alpha^{\theta x^{\beta}}.$$

Second: as x converges to 1. Using only first and second terms of binominal expansion leads to

$$F_{\alpha K}(x) \sim \frac{1}{\alpha - 1} \left(\alpha^{\theta x^{\beta}} - 1 \right).$$

2.2.2. The Asymptotes of the PDF

First: as x converges to zero. Since,

$$\lim_{x \to 0} \left(1 - x^{\beta}\right)^{\theta - 1} = 1, \; \lim_{x \to 0} \; \alpha^{1 - \left(1 - x^{\beta}\right)^{\theta}} = 1,$$

then,

$$f_{\alpha K}(x) \sim \frac{\beta\theta \log \alpha}{\alpha - 1} x^{\beta - 1}.$$

Second: as x converges to 1. Since,

$$\lim_{x \to 1} x^{\beta - 1} = 1, \; \lim_{x \to 1} \; \alpha^{1 - \left(1 - x^{\beta}\right)^{\theta}} = \alpha,$$

then,

$$f_{\alpha K}(x) \sim \frac{\alpha\beta\theta \log \alpha}{\alpha - 1} \left(1 - x^{\beta}\right)^{\theta - 1},$$

using only first and second terms of binominal expansion gives

$$f_{\alpha K}(x) \sim \frac{\alpha\beta\theta \log \alpha}{\alpha - 1} \left[1 - (\theta - 1)\, x^{\beta}\right].$$

3. Some Properties of the AK Distribution

In this section some properties of the AK distribution will be considered as follows:

3.1. The r-th Moment

Generally, the r-th moment of a continuous random variable X, (Johnson, Kotz & Balakrishnan 1995), is given by $E\left(X^r\right) = \int\limits_x x^r f\left(x\right) dx$,substituting (8) into last equation yields

$$E\left(X^r\right) = \int\limits_0^1 x^r \frac{\beta\theta}{1 - \alpha} \sum_{j=0}^{\infty} w_{j+1}\, x^{\beta - 1} \left(1 - x^{\beta}\right)^{(j+1)\theta - 1} dx,$$

then,

$$E\left(X^r\right)=\frac{\theta}{1-\alpha}\sum_{j=0}^{\infty}w_{j+1}\int_0^1\beta x^{r+\beta-1}\left(1-x^{\beta}\right)^{(j+1)\theta-1}dx,$$

hence,

$$E\left(X^r\right)=\frac{\theta}{1-\alpha}\sum_{j=0}^{\infty}w_{j+1}\,B\left(\frac{r+\beta}{\beta},(j+1)\,\theta\right),$$

one can see that, setting $r=0$ gives

$$E\left(X^0\right)=\frac{\theta}{1-\alpha}\sum_{j=0}^{\infty}w_{j+1}\,B\left(1,(j+1)\,\theta\right),$$

substituting (9) and (10) into last equation yields

$$E\left(X^0\right)=1.$$

Mean, variance, skewness, and kurtosis of the AK distribution can be calculated, numerically, for α in Table 1.

Table 1: Mean, variance, skewness and kurtosis of AK distribution for various values of α.

Measure	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.5$	$\alpha=0.9$	$\alpha=1.1$	$\alpha=1.5$	$\alpha=2$	$\alpha=10$
Mean	1.383	1.289	1.218	1.216	1.223	1.242	1.27	1.556
Variance	0.23	0.167	0.116	0.112	0.117	0.131	0.150	0.358
Skewness	−0.871	−0.947	−1.157	−1.206	−1.178	−1.101	−1.015	−0.767
Kurtosis	−0.332	−0.097	0.534	0.671	0.5822	0.345	0.095	−0.596

From the last table, when $0<\alpha<1$: as α increases, mean, variance and skewness decrease but kurtosis increases. On the other hand, when $\alpha>1$: as α increases, mean, variance and skewness increase but kurtosis decreases.

3.2. Moment Generating Function

Basically, the moment generating function (MGF) of a continuous random variable X is given by

$$M_x(t)=E\left(e^{tx}\right)=\int_x e^{tx}f\left(x\right)dx,$$

a first representation can be obtained via substituting (8) into last equation giving

$$M_x(t)=\int_0^1 e^{tx}\frac{\beta\theta}{1-\alpha}\sum_{j=0}^{\infty}w_{j+1}\,x^{\beta-1}\left(1-x^{\beta}\right)^{(j+1)\theta-1}dx,$$

then,

$$M_x(t) = \frac{\beta\theta}{1-\alpha}\sum_{j=0}^{\infty} w_{j+1}\int_0^1 e^{tx}x^{\beta-1}\left(1-x^{\beta}\right)^{(j+1)\theta-1}dx,$$

using binomial expansion for last equation yields

$$M_x(t) = \frac{\beta\theta}{1-\alpha}\sum_{j,k=0}^{\infty} w_{j+1}(-1)^k \binom{(j+1)\theta-1}{k}\int_0^1 e^{tx}x^{\beta+\beta k-1}dx,$$

then, the following integration, Gradshteyn and Ryzhik (2000), will be considered

$$\frac{{}_1F_1(a;a+1;t)}{a} = \int_0^1 e^{tz}z^{a-1}dz, \tag{11}$$

using (11) gives

$$M_x(t) = \frac{\beta\theta}{1-\alpha}\sum_{j,k=0}^{\infty} w_{j+1}(-1)^k \binom{(j+1)\theta-1}{k}\frac{{}_1F_1(\beta+\beta k;\beta+\beta k+1;t)}{\beta+\beta k},$$

moreover, the following expansion, Gradshteyn and Ryzhik (2000), will be considered

$${}_1F_1(a;b;z) = \sum_{u=0}^{\infty}\frac{\Gamma(a+u)}{\Gamma(b+u)}\frac{z^u}{u!}, \tag{12}$$

using (12) yields

$$M_x(t) = \frac{\beta\theta}{(1-\alpha)(\beta+\beta k)}\sum_{j,k,\ell=0}^{\infty} w_{j+1}(-1)^k \binom{(j+1)\theta-1}{k}\frac{\Gamma(\beta+\beta k+\ell)}{\Gamma(\beta+\beta k+1+\ell)}\frac{t^{\ell}}{\ell!},$$

hence,

$$M_x(t) = \frac{\theta}{(1-\alpha)(1+k)}\sum_{j,k,\ell=0}^{\infty} w_{j+1}(-1)^k \binom{(j+1)\theta-1}{k}\frac{t^{\ell}}{\ell!\ (\beta+\beta k+\ell)}.$$

A second representation for MGF, based on exponenatial expansion, can be given as follows:

Since,

$$M_x(t) = E\left(e^{tx}\right),$$

using exponential expansion, in last equation, leads to

$$M_x(t) = E\left(\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}\right),$$

then,

$$M_x(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}E\left(x^k\right).$$

3.3. The Quantile Function and the Median

The well-known definition of the 100 u-th is

$$u = P(X \leq x_u) = F(x_u)\ ;\ \ x_u > 0,\ 0 < u < 1,$$

equating (5) to u gives

$$x = \left\{1 - \left[1 - \frac{\log\left(1 + (\alpha - 1)\, u\right)}{\log\, \alpha}\right]^{\frac{1}{\theta}}\right\}^{\frac{1}{\beta}},$$

easily, replacing u with 1/2 (the second quartile) yields the median.

3.4. The Mean Deviation

Generally, the mean deviation about the mean and about the median for a random variable X, respectively, can be given from

$$S_1\,(x) = \int\limits_x |x - \mu|\ \ f\,(x)\ dx \text{ and } S_2\,(x) = \int\limits_x |x - M|\ f\,(x)\ dx,$$

easily, it can be given by, Ali Ahmed (2019),

$$S_1\,(x) = 2\,\mu\, F\,(\mu) - 2\, t\,(\mu) \ \text{ and } S_2\,(x) = \mu - 2\, t\,(M)\,,$$

where $T\,(q) = \int\limits_{-\infty}^{q} x\, f\,(x)\ dx$is the linear incomplete moment.

Substituting (8) into $T(.)$ gives

$$T\,(q) = \frac{\theta}{1 - \alpha} \sum_{j=0}^{\infty} w_{j+1} \int\limits_0^q \beta x^{(\beta+1)-1} \left(1 - x^{\beta}\right)^{(j+1)\theta - 1} dx,$$

then,

$$T\,(q) = \frac{\theta}{1 - \alpha} \sum_{j=0}^{\infty} w_{j+1}\, B\left(q; \frac{1 + \beta}{\beta}, (j + 1)\,\theta\right),$$

where $B\,(.;.,.)$ is the incomplete beta function.

3.5. The Mode

The natural logarithm of (6) is

$$\log f_{\alpha K}(x) = \\ \log \frac{\beta\theta \log \alpha}{\alpha - 1} + \left[1 - \left(1 - x^{\beta}\right)^{\theta}\right] \log \alpha + (\beta - 1) \log x + (\theta - 1) \log \left(1 - x^{\beta}\right),$$

differentiating the last equation (w.r.t. x) and equating it to zero yields

$$\frac{d}{dx}\log f_{\alpha K}(x)=\left[\theta\beta x^{\beta-1}\left(1-x^{\beta}\right)^{\theta-1}\right]\log\alpha+\frac{\beta-1}{x}-\frac{(\theta-1)\,\beta x^{\beta-1}}{1-x^{\beta}}=0.$$

The last equation is a nonlinear equation and it does not have an analytic solution with respect to x, therefore it have to be solved numerically, if x_0 is a root for the last equation then it must be $f^{//}\left[\log\left(x_0\right)\right]<0$.

4. The Hazard Function of the AK Distribution

Generally, the survival function of a random variable X, Meeker & Escobar (2014), can be given by

$$S\left(x\right)=1-F(x),$$

substituting (5) into last equation gives

$$\underset{\alpha K}{S}\left(x\right)=1-\frac{\alpha^{1-\left(1-x^{\beta}\right)^{\theta}}-1}{\alpha-1};\ 0<x<1;\ \alpha,\beta,\theta>0. \tag{13}$$

Simply, the Hazard function, Meeker & Escobar (2014), can be given by

$$H\left(x\right)=\frac{f(x)}{S\left(x\right)},$$

substituting (6) and (13) into last equation yields

$$\underset{\alpha K}{H}\left(x\right)=\frac{\beta\theta\left(\log\alpha\right)\,x^{\beta-1}\left(1-x^{\beta}\right)^{\theta-1}}{\alpha^{\left(1-x^{\beta}\right)^{\theta}}-1},$$

some shapes of the Hazard function for the AK distribution are illustrated in figure 2.

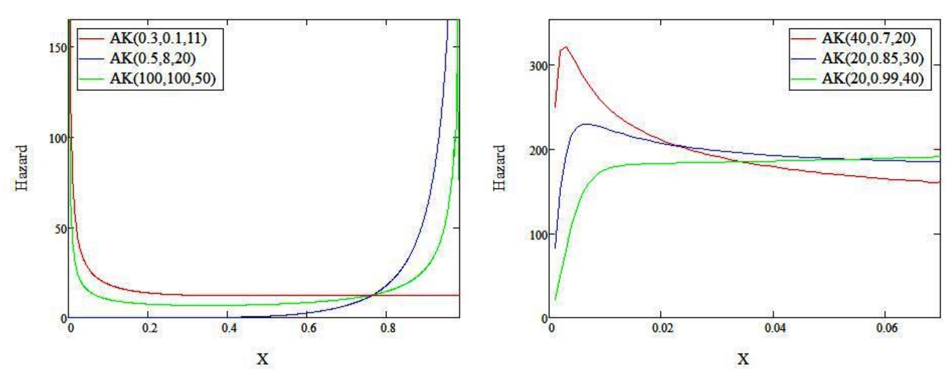

FIGURE 2: The AK Hazard functions.

5. The Rényi Entropy of the AK Distribution

The Rényi entropy of a random variable X, Meeker & Escobar (2014), is defined by

$$e_R(\rho) = \frac{1}{1-\rho} \log\left[\int_x [f(x)]^{\rho}\, dx\right],$$

substituting (8) into last equation gives

$$\underset{\alpha K}{e_R}(\rho) = \frac{1}{1-\rho} \log\left\{\left(\frac{\beta\theta}{1-\alpha}\right)^{\rho} \int_0^1 x^{\rho(\beta-1)} \left(1-x^{\beta}\right)^{\rho(\theta-1)} \left\{\sum_{j=0}^{\infty} w_j^* \left[\left(1-x^{\beta}\right)^{\theta}\right]^j\right\}^{\rho} dx\right\},$$

where $\sum_{j=0}^{\infty} w_j^* = \sum_{j=0}^{\infty} w_{j+1}$, since, $\left\{\sum_{j=0}^{\infty} w_j^* \left[\left(1-x^{\beta}\right)^{\theta}\right]^j\right\}^{\rho} = \sum_{j=0}^{\infty} m_j \left[\left(1-x^{\beta}\right)^{\theta}\right]^j$, Gradshteyn & Ryzhik (2000), where $m_0 = w_0^{*\rho}$,

$$m_n = \frac{1}{n\, w_0^*} \sum_{j=1}^{n} (j\rho - n + j)\, w_j^* m_{n-j};\; n \geq 1,$$

then,

$$\underset{\alpha K}{e_R}(\rho) = \frac{1}{1-\rho} \log\left\{\left(\frac{\beta\theta}{1-\alpha}\right)^{\rho} \sum_{j=0}^{\infty} m_j \int_0^1 x^{\rho(\beta-1)} \left(1-x^{\beta}\right)^{\rho(\theta-1)-\theta j} dx\right\},$$

moreover,

$$\underset{\alpha K}{e_R}(\rho) = \frac{1}{1-\rho} \log\left\{\frac{\beta^{\rho-1}\, \theta^{\rho}}{(1-\alpha)^{\rho}} \sum_{j=0}^{\infty} m_j \int_0^1 \beta\, x^{\rho(\beta-1)+1-1} \left(1-x^{\beta}\right)^{\rho(\theta-1)-\theta j+1-1} dx\right\},$$

then,

$$\underset{\alpha K}{e_R}(\rho) = \frac{1}{1-\rho} \log\left\{\frac{\beta^{\rho-1}\, \theta^{\rho}}{(1-\alpha)^{\rho}} \sum_{j=0}^{\infty} m_j\, B\left(\frac{\rho(\beta-1)+1}{\beta}, \rho(\theta-1)-\theta j+1\right)\right\}.$$

6. Reliability: The Stress Strength Model of the AK Distribution

Generally, the stress strength model of the distribution, Meeker & Escobar (2014), can be given by

$$R = \int_x f_1(x;\lambda_1)\, F_2(x;\lambda_2)\, dx, \tag{14}$$

where $\lambda$is the vector of parameter, β and θ are common parameters, substituting (5) and (6) into last equation yields

$$R=\int_0^1 \frac{\beta\theta\log\alpha_1}{\alpha_1-1}\alpha_1^{1-\left(1-x^\beta\right)^\theta}x^{\beta-1}\left(1-x^\beta\right)^{\theta-1}\frac{\alpha_2^{1-\left(1-x^\beta\right)^\theta}-1}{\alpha_2-1}dx,$$

setting $R=I_1-I_2$, gives

$$I_1=\int_0^1 \frac{\beta\theta\log\alpha_1}{\left(\alpha_1-1\right)\left(\alpha_2-1\right)}\alpha_1^{1-\left(1-x^\beta\right)^\theta}\alpha_2^{1-\left(1-x^\beta\right)^\theta}x^{\beta-1}\left(1-x^\beta\right)^{\theta-1}dx,$$

and

$$I_2=\int_0^1 \frac{\beta\theta\log\alpha_1}{\left(\alpha_1-1\right)\left(\alpha_2-1\right)}\alpha_1^{1-\left(1-x^\beta\right)^\theta}x^{\beta-1}\left(1-x^\beta\right)^{\theta-1}dx.$$

Firstly,

$$I_1=\frac{\beta\theta\log\alpha_1}{\left(\alpha_1-1\right)\left(\alpha_2-1\right)}\int_0^1 e^{\left[1-\left(1-x^\beta\right)^\theta\right]\log\alpha_1+\left[1-\left(1-x^\beta\right)^\theta\right]\log\alpha_2}x^{\beta-1}\left(1-x^\beta\right)^{\theta-1}dx,$$

then,

$$I_1=\frac{\log\alpha_1}{\frac{\left(\alpha_1-1\right)\left(\alpha_2-1\right)}{\left[\log\alpha_1+\log\alpha_2\right]}}\int_0^1 e^{\left[1-\left(1-x^\beta\right)^\theta\right]\left[\log\alpha_1+\log\alpha_2\right]}\left[\log\alpha_1+\log\alpha_2\right]\theta\beta x^{\beta-1}\left(1-x^\beta\right)^{\theta-1}dx,$$

hence,

$$I_1=\frac{\left(\log\alpha_1\right)\left(\alpha_1\alpha_2-1\right)}{\left(\alpha_1-1\right)\left(\alpha_2-1\right)\left[\log\alpha_1+\log\alpha_2\right]}.$$

Secondly,

$$I_2=\frac{1}{\left(\alpha_1-1\right)\left(\alpha_2-1\right)}\int_0^1 e^{\left[1-\left(1-x^\beta\right)^\theta\right]\left(\log\ \alpha_1\right)}\beta\theta\left(\log\alpha_1\right)x^{\beta-1}\left(1-x^\beta\right)^{\theta-1}dx,$$

then,

$$I_2=\frac{1}{\alpha_2-1}.$$

An Expansion for the Reliability. Substituting (7) and (8) into (14) gives

$$R=\int_0^1 \frac{\beta\theta}{1-\alpha_1}\sum_{j=0}^{\infty}w_{j+1}x^{\beta-1}\left(1-x^\beta\right)^{(j+1)\theta-1}\frac{1}{1-\alpha_2}\left(1-\sum_{j=0}^{\infty}w_j\left(1-x^\beta\right)^{j\theta}\right)dx,$$

setting

$$R=\frac{\theta}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}\left[I_1-I_2\right],$$

yields

$$I_1 = \sum_{j=0}^{\infty} w_j^* \int_0^1 \beta x^{\beta-1} \left(1 - x^{\beta}\right)^{(j+1)\theta - 1} dx,$$

and

$$I_2 = \beta \int_0^1 \sum_{j=0}^{\infty} w_j^* \, x^{\beta-1} \left(1 - x^{\beta}\right)^{(j+1)\theta-1} \sum_{j=0}^{\infty} w_j \left(1 - x^{\beta}\right)^{j\theta} dx,$$

where $\sum\limits_{j=0}^{\infty} w_j^* = \sum\limits_{j=0}^{\infty} w_{j+1}$.

Firstly,

$$I_1 = \sum_{j=0}^{\infty} w_j^* \, B\left(1, (j+1)\,\theta\right).$$

Secondly,

$$I_2 = \beta \int_0^1 \sum_{j=0}^{\infty} w_j^* \, x^{\beta-1} \left(1 - x^{\beta}\right)^{(j+1)\theta-1} \sum_{j=0}^{\infty} w_j \, \left(1 - x^{\beta}\right)^{j\theta} dx,$$

then,

$$I_2 = \beta \int_0^1 x^{\beta-1} \left(1 - x^{\beta}\right)^{\theta-1} \left\{ \sum_{j=0}^{\infty} w_j^* \left[\left(1 - x^{\beta}\right)^{\theta} \right]^j \right\} \left\{ \sum_{j=0}^{\infty} w_j \, \left[\left(1 - x^{\beta}\right)^{\theta} \right]^j \right\} dx,$$

since, Garthwaite, Jolliffe & Byron Jones (2002),

$$\left\{ \sum_{j=0}^{\infty} w_j^* \left[\left(1 - x^{\beta}\right)^{\theta} \right]^j \right\} \left\{ \sum_{j=0}^{\infty} w_j \, \left[\left(1 - x^{\beta}\right)^{\theta} \right]^j \right\} = \sum_{j=0}^{\infty} n_j \, \left[\left(1 - x^{\beta}\right)^{\theta} \right]^j,$$

where, $n_m = \sum\limits_{j=0}^{m} w_j^* \, w_{m-j}$,

then,

$$I_2 = \sum_{j=0}^{\infty} n_j \int_0^1 \beta x^{\beta-1} \left(1 - x^{\beta}\right)^{j\theta+\theta-1} \, dx,$$

hence,

$$I_2 = \sum_{j=0}^{\infty} n_j B\left(1, j\theta + \theta\right).$$

7. Order Statistics of the AK Distribution

The density function $f(x_{u:v})$ of the u-th order statistics for $u = 1, 2, \dots, v$ from *iid* random variables $X_1, X_2, \dots, X_v$ following the AK distribution, Arnold, Balakrishnan & Nagaraja (1992), is given by

$$f(x_{u:v}) = \frac{f(x_u)}{B(u, v-u+1)} F(x_u)^{u-1} \{1 - F(x_u)\}^{v-u},$$

using binomial expansion in last equation gives

$$f(x_{u:v}) = \frac{f(x_u)}{B(u, v-u+1)} \sum_{k=0}^{v-u} (-1)^k \binom{v-u}{k} F(x_u)^{u+k-1}, \tag{15}$$

substituting (7) and (8) into (15) yields

$$f(x_{u:v}) = \frac{\beta\theta \sum_{j=0}^{\infty} w_{j+1}\, x_u^{\beta-1} \left(1 - x_u^{\beta}\right)^{(j+1)\theta-1}}{(1-\alpha)\, B(u, v-u+1)} \sum_{k=0}^{v-u} (-1)^k \binom{v-u}{k} \left(\frac{1}{1-\alpha}\right)^{u+k-1} \left(1 - \sum_{j=0}^{\infty} w_j \left(1 - x_u^{\beta}\right)^{j\theta}\right)^{u+k-1},$$

using binomial expansion in last equation leads to

$$f(x_{u:v}) = \frac{\beta\theta \sum_{j=0}^{\infty} w_{j+1}\, x_u^{\beta-1} \left(1-x_u^{\beta}\right)^{(j+1)\theta-1}}{B(u,v-u+1)} \sum_{k=0}^{v-u} (-1)^k \binom{v-u}{k} \left(\frac{1}{1-\alpha}\right)^{u+k} \times \sum_{\ell=0}^{u+k-1} (-1)^{\ell} \binom{u+k-1}{\ell} \left\{\sum_{j=0}^{\infty} w_j \left(1 - x_u^{\beta}\right)^{j\theta}\right\}^{\ell},$$

since, Garthwaite et al. (2002),

$$\left\{\sum_{j=0}^{\infty} w_j \left(1 - x_u^{\beta}\right)^{j\theta}\right\}^{\ell} = \sum_{j=0}^{\infty} m_j \left(1 - x_u^{\beta}\right)^{j\theta},$$

where $m_0 = w_0^{\ell}$, $m_n = \frac{1}{n\, w_0} \sum_{j=1}^{n} (j\ell - n + j)\, w_j m_{n-j}$; $n \geq 1$, then,

$$f(x_{u:v}) = \frac{\beta\theta x_u^{\beta-1}\left(1-x_u^{\beta}\right)^{\theta-1}}{B(u,v-u+1)} \sum_{k=0}^{v-u} (-1)^k \binom{v-u}{k} \left(\frac{1}{1-\alpha}\right)^{u+k} \sum_{\ell=0}^{u+k-1} (-1)^{\ell} \binom{u+k-1}{\ell} \times \left\{\sum_{j=0}^{\infty} w_j^* \left(1 - x_u^{\beta}\right)^{j\theta}\right\} \left\{\sum_{j=0}^{\infty} m_j \left(1 - x_u^{\beta}\right)^{j\theta}\right\},$$

since, Garthwaite et al. (2002),

$$\left\{\sum_{j=0}^{\infty} w_j^* \left(1-x_u^{\beta}\right)^{j\theta}\right\}\left\{\sum_{j=0}^{\infty} m_j \left(1-x_u^{\beta}\right)^{j\theta}\right\} = \sum_{j=0}^{\infty} q_j \left(1-x_u^{\beta}\right)^{j\theta},$$

where $\sum_{j=0}^{\infty} w_j^* = \sum_{j=0}^{\infty} w_{j+1}$, $q_s = \sum_{j=0}^{s} w_j^* m_{s-j}$, hence,

$$f\left(x_{u:v}\right) = \frac{\beta\theta}{B\left(u, v-u+1\right)} \sum_{j=0}^{\infty} p_j x_u^{\beta-1}\left(1-x_u^{\beta}\right)^{j\theta+\theta-1}, \tag{16}$$

where $p_j = q_j \sum_{k=0}^{v-u} (-1)^k \binom{v-u}{k} \left(\frac{1}{1-\alpha}\right)^{u+k} \sum_{\ell=0}^{u+k-1} (-1)^{\ell} \binom{u+k-1}{\ell}$.

The r-th Moment of Order Statistics. One, easily, finds that the r-th moment of order statistics of the AK distribution can be got by $E\left(X_{u:v}^r\right) = \int_x x_u^r f\left(x_u\right) dx_u$, substituting (16) into last equation gives

$$E\left(X_{u:v}^r\right) = \frac{\theta}{B\left(u, v-u+1\right)} \sum_{j=0}^{\infty} p_j \int_x \beta x_u^{r+\beta-1}\left(1-x_u^{\beta}\right)^{j\theta+\theta-1} dx_u,$$

then,

$$E\left(X_{u:v}^r\right) = \frac{\theta}{B\left(u, v-u+1\right)} \sum_{j=0}^{\infty} p_j\, B\left(\frac{r+\beta}{\beta}, j\theta+\theta\right).$$

8. Estimation of the AK Distribution Parameters

Let $X_1, X_2, \ldots, X_n$ be the *iid* random variables from the AK $(x;\Lambda)$ distribution, where $\Lambda = (\alpha, \beta, \theta)$, then the likelihood function for the vector of parameter $\Lambda = (\alpha, \beta, \theta)$, Garthwaite et al. (2002), can be written as

$$L = \frac{(\beta\,\theta)^n (\log\,\alpha)^n}{(\alpha-1)^n} \alpha^{\sum_{i=1}^{n}\left(1-x_i^{\beta}\right)^{\theta}} \prod_{i=1}^{n} x_i^{\beta-1} \prod_{i=1}^{n} \left[1-x_i^{\beta}\right]^{\theta-1},$$

the log likelihood function is given by

$$\ell = \frac{(\beta\,\theta)^n (\log\,\alpha)^n}{(\alpha-1)^n} + \left(\sum_{i=1}^{n}\left(1-x_i^{\beta}\right)^{\theta}\right)\log\,\alpha + (\beta-1)\sum_{i=1}^{n}\log x_i + (\theta-1)\sum_{i=1}^{n}\log\left[1-x_i^{\beta}\right],$$

the score function for the parameters α, β and θ are given by

$$\frac{\partial\ell}{\partial\alpha} = \frac{n\log(\alpha)^{n-1}\beta^n\theta^n}{\alpha\left(\alpha-1\right)^n} - \frac{n\log(\alpha)^n\beta^n\theta^n}{\left(\alpha-1\right)^{n+1}} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1-x_i^{\beta}\right)^{\theta}, \tag{17}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n\,\theta^n\,\log(\alpha)^n \beta^{n-1}}{(\alpha-1)^n} - \theta\,(\log\,\alpha) \sum_{i=1}^{n} \left(1 - x_i^{\beta}\right)^{\theta-1} x_i^{\beta}\,(\log\,x_i) + \sum_{i=1}^{n} \log x_i + (\theta-1) \sum_{i=1}^{n} \frac{x_i^{\beta}(\log x_i)}{x_i^{\beta}-1}, \tag{18}$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{n\,\beta^n\,\log(\alpha)^n \theta^{n-1}}{(\alpha-1)^n} + (\log\,\alpha) \sum_{i=1}^{n} \left(1 - x_i^{\beta}\right)^{\theta} \left[\log\left(1 - x_i^{\beta}\right)\right] + \sum_{i=1}^{n} \log\left(1 - x_i^{\beta}\right). \tag{19}$$

The unknown parameters of the AK distribution are obtained, using the maximum likelihood estimators (MLEs), by solving the nonlinear equations (17) to (19) but they cannot be solved analytically, so solving the equations, numerically, will be performed by using a statistical package. An iterative technique such as a Newton–Raphson algorithm may be performed to compute the estimates values.

Let Λ be the vector of the unknown parameter (α,β,θ), then elements of the 3×3 information matrix $I(\alpha,\beta,\theta)$ can be approximated by

$$I_{i\,j}(\hat{\Lambda}\,) = E\left[-\frac{\partial^2\,\ell(\Lambda\,)}{\partial\Lambda_i\,\partial\Lambda_j}\,|_{\Lambda=\hat{\Lambda}}\right],$$

where $I_{i\,j}^{-1}(\hat{\Lambda}\,)$is the variance covariance matrix of the unknown parameters, the asymptotic distributions of the AK parameters is

$$\sqrt{n}\,(\hat{\Lambda}_i - \Lambda_i) \approx N_3(0, I^{-1}(\hat{\Lambda}_i))\ ,\ i = 1,2,3,$$

and the approximation $100(1-\gamma)\%$ confidence intervals of the unknown parameters based on the asymptotic distribution of the $AK(\alpha,\beta,\theta)$ distribution are determined, respectively, as

$$\hat{\Lambda}_i \pm z_{\frac{\gamma}{2}} \sqrt{I^{-1}(\hat{\Lambda}_i)}\,; i = 1,2,3,$$

where $z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}\,^{th}$ percentile of a standard normal distribution.

The derivatives in the observed information matrix $I(\alpha,\beta,\theta)$ for the unknown parameters are included in appendix I.

9. A Simulation Study

In this study, MLEs for parameters of the AK distribution are obtained using random numbers to study the MLEs finite sample behavior. The algorithm of obtaining parameters estimates is detailed in the following steps:

Step (1): Generating a random sample $X_1, X_2, \ldots, X_n$ of sizes $n = (10, 20, 30$,50, 100, 300) by using the AK distribution.

Step (2): Selecting different set values of the parameters as: set(1): ($\alpha = 0.5, \beta = 2, \theta = 3$), set(2): ($\alpha = 0.9, \beta = 2, \theta = 3$), set(3): ($\alpha = 1.5, \beta = 2, \theta = 3$), set(4): ($\alpha = 0.5, \beta = 3, \theta = 3$), set(5): ($\alpha = 0.5, \beta = 4, \theta 3$), set(6): ($\alpha = 0.5, \beta = 2, \theta = 4$), set(7): ($\alpha = 0.5, \beta = 2, \theta = 5$).

Step (3): Solving (17) to (19) via iteration to compute, biases, MLEs, RMSE (the root of mean squared error) and the Pearson type of parameters estimators, Pearson (1895), of the AK distribution.

Step (4): Repeating step1 to step3, 10000 times.

In this study, samples of random numbers are generated using Mathcad package v15 where the conjugate gradient iteration method is performed. All results are included in tables and indicated in appendix II.

From the results, in appendix II, one can see that, as sample size increases, biases, estimators, and RMSEs decrease. $\hat{\beta}$ and $\hat{\theta}$ can be consistent, specially, when sample size increases. Moreover, the sampling distribution of $\hat{\alpha}$ can be the Pearson type IV distribution in all times, the sampling distribution of $\hat{\beta}$ and $\hat{\theta}$ differ according to sample size. As θ increases, for fixed values of α and β, the biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ decrease.

10. Application

A real data set is concerned to apply the empirical model, practically, using the Mathematica package version 11. In this example, some distributions are used as: the AK distribution, the AP distribution, the K distribution, the P distribution the Gumbel (mini) distribution, the beta distribution, and the McDonald (McD) distribution, McDonald (2008). The following data represents the lifetime (Hours) of T8 fluorescent lamps for 50 devices, the data are given from the UK National Physical Laboratory, for more details, one can visit: http://www.npl.co.uk/ 0.445, 0.493, 0.285, 0.564, 0.760, 0.381, 0.690, 0.579, 0.636, 0.238, 0.149, 0.244, 0.126, 0.796, 0.405, 0.553, 0.780, 0.431, 0.184, 0.375, 0.198, 0.890, 0.192, 0.463, 0.486, 0.521, 0.366, 0.486, 0.116, 0.511, 0.612, 0.117, 0.384, 0.326, 0.057, 0.412, 0.586, 0.517, 0.570, 0.588, 0.497, 0.246, 0.234, 0.228, 0.552, 0.893, 0.403, 0.458, 0.134, 0.338

The results of some goodness of fit measures are in table (2), the results of likelihood ratio tests are in table (3), figure (3) illustrates probability density functions for different distributions having similar skewness and kurtosis and figure (4) illustrates probability density functions for nested distributions.

In table 2, the MLEs of distributions parameters, parameters standard error (SEs), in parentheses, Kolmogorov-Smirnov (KS) test statistic, AIC (Akaike Information Criterion), CAIC (the consistent Akaike Information Criterion) and BIC (Bayesian information criterion), (Merovci & Puka 2014), are computed for every distribution having similar skewness and kurtosis values. The null hypothesis that the data follow the AK distribution, only, can be accepted at significance level $\alpha = 0.05$, it is clear that the AK distribution has the smallest KS, AIC, CAIC, BIC and SEs, on the other hand the AK distribution has the largest log likelihood and p-value, so that, the AK distribution can be the best fitted distribution to the data compared with other distributions having similar skewness and kurtosis.

Table 2: The MLE of the parameter(s) and the associated AIC and BIC values.

Distribution	MLE parameters			Skewness	Kurtosis	KS	P-value	Log Likelihood	AIC	BIC	CAIC
	α	β	θ								
AK	0.5	2.40	4.67	0.058	2.403	0.127	0.357	7.313	-10.627	-6.803	-10.372
	(0.137)	(0.263)	(0.654)								
beta	1.201	2.717	-	0.628	2.14	0.048	0.032	0.278	3.443	7.267	3.698
	(1.442)	(0.312)									
Gumbel (mini)	0.623	0.208	-	-1.138	1.4	0.019	0.041	-0.640	5.280	9.104	5.536
	(0.212)	(2.301)									
McD	6	4.61	0.0629	0.431	2.684	0.036	0.015	1.161	3.676	9.412	4.198
	(5.215)	(1.245)	(3.5)								

Table 3: The log-likelihood function, the likelihood ratio tests statistic and p-values.

Distribution	Parameters			Log Likelihood ℓ	Likelihood Ratio Test Statistics Λ	Degrees of Freedom DF	p-value
	α	β	θ				
AP	3	3	-	-6.728	28.076	1	1.166×10^{-7}
	(2.157)	(0.463)					
K	-	2.2	2.5	3.515	7.59	1	5.869×10^{-3}
		(0.331)	(0.954)				
P	-	2	-	-14.869	44.358	2	2.332×10^{-10}
		(0.283)					

*Note that the log likelihood of the AK distribution = 7.31.

In table 3, based on the likelihood ratio test, the null hypothesis is the data follow the nested model and the alternative is the data follow the full model, where the AP distribution, K distribution and the P distribution are nested by AK distribution, it is clear that, all null hypotheses can be rejected at significance level $\alpha = 0.05$.

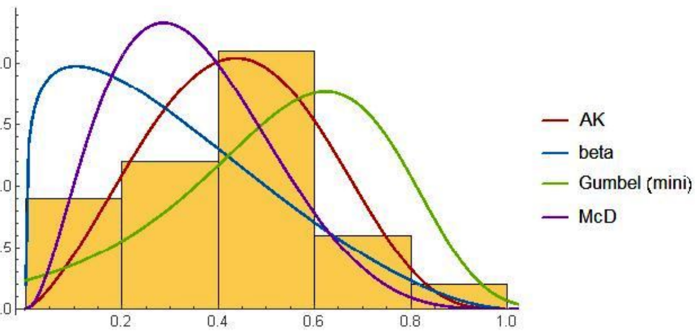

FIGURE 3: Probability density functions for different distributions having similar skewness and kurtosis.

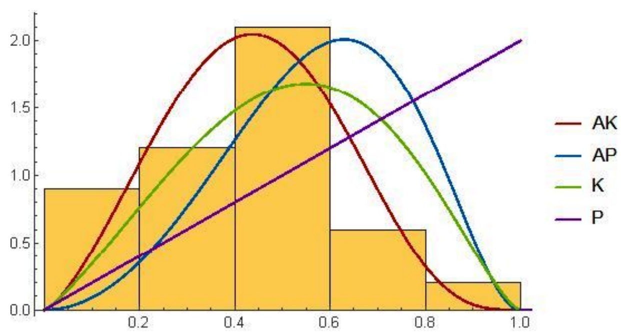

FIGURE 4: Probability density functions for the nested distribution by the AK distribution.

11. Conclusions

The alpha power Kumarasumay distribution is a flexible distribution having several advantages as it does not have any special function, has flexible mathematical properties and generalizes three important distributions. The AK distribution works practically, well, when it is compared with other distributions, specially, in simulation studies and real data sets. The author encourages researchers to do more researches and applications on the alpha power Kumarasumay distribution.

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Appendix A.

The derivatives in the observed information matrix $I(\alpha,\beta,\theta)$ for the unknown parameters

$$\frac{\partial^2\ell}{\partial\alpha^2}=\frac{n\log(\alpha)^n(n+1)\beta^n\theta^n}{(\alpha-1)^{n+2}}-\frac{2n^2\log(\alpha)^{n-1}\beta^n\theta^n}{\alpha(\alpha-1)^{n+1}}-\frac{n\log(\alpha)^{n-1}\beta^n\theta^n}{\alpha^2(\alpha-1)^n}-\frac{\sum_{i=1}^{n}\left(1-x_i^\beta\right)^\theta}{\alpha^2}+\frac{n\log(\alpha)^{n-2}(n-1)\beta^n\theta^n}{\alpha^2(\alpha-1)^n},$$

$$\frac{\partial^2\ell}{\partial\beta^2}=\sum_{i=1}^{n}\left\{\frac{\left[\log(x_i^2)\right]x_i^\beta}{x_i^\beta-1}-\frac{\left[\log(x_i^2)\right]x_i^{2\beta}}{\left[x_i^\beta-1\right]^2}\right\}(\theta-1)+\frac{\theta^n n\log(\alpha^n)(n-1)\beta^{n-2}}{(\alpha-1)^n}$$
$$-\theta\log(\alpha)\sum_{i=1}^{n}\left\{\left[\log\left(x_i^2\right)\right]\left[1-x_i^\beta\right]^{\theta-1}x_i^\beta-\left[\log\left(x_i^2\right)\right]\left[1-x_i^\beta\right]^{\theta-2}(\theta-1)x_i^{2\beta}\right\},$$

$$\frac{\partial^2\ell}{\partial\theta^2}=\log(\alpha)\sum_{i=1}^{n}\left[\log\left(1-x_i^\beta\right)^2\right]\left(1-x_i^\beta\right)^\theta+\frac{n\left[\log(\alpha)^n\right](n-1)\beta^n\theta^{n-2}}{(\alpha-1)^n},$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\beta}=\frac{n^2\left[\log(\alpha)^{n-1}\right]\beta^{n-1}\theta^n}{\alpha(\alpha-1)^n}-\frac{n^2\left[\log(\alpha)^n\right]\beta^{n-1}\theta^n}{(\alpha-1)^{n+1}}-\theta\sum_{i=1}^{n}\frac{\left[\log(x_i)\right]x_i^\beta\left[1-x_i^\beta\right]^{\theta-1}}{\alpha},$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\theta}=\frac{n^2\left[\log(\alpha)^{n-1}\right]\beta^n\theta^{n-1}}{\alpha(\alpha-1)^n}-\frac{n^2\left[\log(\alpha)^n\right]\beta^n\theta^{n-1}}{(\alpha-1)^{n+1}}+\theta\sum_{i=1}^{n}\frac{\left[\log(1-x_i^\beta)\right]\left[1-x_i^\beta\right]^\theta}{\alpha},$$

and

$$\frac{\partial^2 \ell}{\partial \beta \partial \theta} =$$

$$\log(\alpha) \sum_{i=1}^{n} \left\{ \frac{[\log(x_i)]\, x_i^{\beta} \left(1 - x_i^{\beta}\right)^{\theta}}{x_i^{\beta} - 1} - \theta\, [\log(x_i)] \left[\log(1 - x_i^{\beta})\right] x_i^{\beta} \left(1 - x_i^{\beta}\right)^{\theta-1} \right\}$$

$$+ \frac{\theta^{n-1}\beta^{n-1} n\,(n-1)\,[\log(\alpha^n)]}{(\alpha-1)^n} + \frac{\theta^{n-1}\beta^{n-1} n\,[\log(\alpha^n)]}{(\alpha-1)^n} + \sum_{i=1}^{n} \frac{[\log(x_i)]\, x_i^{\beta}}{x_i^{\beta} - 1}.$$

Appendix B.

Results of the simulation study for different data sets:

Table A1:

Set(1):($\alpha = 0.5, \beta = 2, \theta = 3$)

Sample Size	Parameters	Mean of estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 0.5$	1.789	1.289	2.277	3.011	5.149	0.261	IV
	$\beta = 2$	2.31	0.31		0.919		−0.273	I
	$\theta = 3$	4.852	1.852		4.075		−4.79	I
20	$\alpha = 0.5$	1.787	1.287	1.465	3.168	3.914	0.221	IV
	$\beta = 2$	2.09	0.09		0.693		−0.425	I
	$\alpha = 3$	3.695	0.695		2.193		0.384	IV
30	$\alpha = 0.5$	1.636	1.136	1.179	2.923	3.311	0.203	IV
	$\beta = 2$	2.031	0.039		0.479		0.12	IV
	$\alpha = 3$	3.316	0.316		1.48		−2.552	I
50	$\alpha = 0.5$	1.476	0.976	0.983	2.647	2.895	0.217	IV
	$\beta = 2$	1.995	−0.035		0.401		0.014	IV
	$\alpha = 3$	3.115	0.115		1.103		−0.219	I
100	$\alpha = 0.5$	1.162	0.662	0.664	2.107	2.283	0.265	IV
	$\beta = 2$	1.973	−0.027		0.311		−0.017	I
	$\alpha = 3$	2.956	−0.044		0.823		0.175	IV
300	$\alpha = 0.5$	0.717	0.217	0.221	0.814	0.982	0.241	IV
	$\beta = 2$	1.989	−0.014		0.199		−0.085	I
	$\alpha = 3$	2.96	−0.04		0.513		−0.345	I

Table A2:

Sample Size	Parameters	Mean of estimators	Set(2): ($\alpha = 0.9, \beta = 2, c = 3$) Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 0.9$	2.307	1.407	2.352	3.795	6.259	0.584	IV
	$\beta = 2$	2.437	0.437		1.07		-0.292	I
	$\alpha = 3$	4.834	1.834		4.861		0.223	IV
20	$\alpha = 0.9$	2.569	1.669	1.750	4.522	4.977	0.271	IV
	$\beta = 2$	2.164	0.164		0.665		-0.496	I
	$\alpha = 3$	3.503	0.503		1.971		0.194	IV
30	$\alpha = 0.9$	2.525	1.625	1.640	4.386	4.620	0.19	IV
	$\beta = 2$	2.094	0.094		0.567		0.219	IV
	$\alpha = 3$	3.207	0.207		1.338		0.456	IV
50	$\alpha = 0.9$	2.366	1.466	1.466	4.047	4.189	0.243	IV
	$\beta = 2$	2.034	0.034		0.466		0.02	IV
	$\alpha = 3$	3.023	0.023		0.979		8.167	VI
100	$\alpha = 0.9$	2.074	1.174	1.175	3.565	3.648	0.236	IV
	$\beta = 2$	1.991	-0.008		0.37		-0.007	I
	$\alpha = 3$	2.937	-0.063		0.682		0.165	IV
300	$\alpha = 0.9$	1.401	0.501	0.503	2.044	2.096	0.286	IV
	$\beta = 2$	1.987	-0.013		0.243		-0.209	I
	$c = 3$	2.955	-0.045		0.4		-0.253	I

TABLE A3:

Set(3):($\alpha = 1.5, \beta = 2, \theta = 3$)

Sample Size	Parameters	Mean of estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 1.5$	3.939	2.439	3.107	6.983	7.639	0.191	IV
	$\beta = 2$	2.566	0.566		1.214		−0.341	I
	$\alpha = 3$	4.841	1.841		2.85		0.29	IV
20	$\alpha = 1.5$	3.676	2.176	2.224	6.004	6.320	0.27	IV
	$\beta = 2$	2.256	0.256		0.774		−0.522	I
	$\alpha = 3$	3.385	0.385		1.816		0.201	IV
30	$\alpha = 1.5$	3.586	2.086	2.095	6.074	6.226	0.365	IV
	$\beta = 2$	2.169	0.169		0.653		0.134	IV
	$\alpha = 3$	3.11	0.11		1.202		0.527	IV
50	$\alpha = 1.5$	3.446	1.946	1.947	5.973	6.059	0.28	IV
	$\beta = 2$	2.081	0.081		0.53		0.024	IV
	$\alpha = 3$	2.967	−0.033		0.869		−0.264	I
100	$\alpha = 1.5$	3.417	1.917	1.918	5.766	5.814	0.211	IV
	$\beta = 2$	2.018	0.018		0.426		0.004	IV
	$\alpha = 3$	2.915	−0.085		0.62		0.151	IV
300	$\alpha = 1.5$	2.608	1.108	1.109	4.286	4.308	0.272	IV
	$\beta = 2$	1.987	−0.013		0.294		−0.088	I
	$\alpha = 3$	2.95	−0.05		0.324		0.78	IV

TABLE A4:

Set(4): ($\alpha = 0.5, \beta = 3, \theta = 3$)

Sample Size	Parameters	Mean of estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 0.5$	1.816	1.316	2.392	2.995	5.375	0.316	IV
	$\beta = 3$	3.46	0.46		1.389		−0.296	I
	$\alpha = 3$	4.944	1.944		4.242		2.282	VI
20	$\alpha = 0.5$	1.745	1.245	1.424	2.991	3.792	0.289	IV
	$\beta = 3$	3.139	0.139		0.887		−1.091	I
	$\alpha = 3$	3.678	0.678		2.156		0.423	IV
30	$\alpha = 0.5$	1.637	1.137	1.176	2.815	3.257	0.225	IV
	$\beta = 3$	3.03	0.03		0.723		0.107	IV
	$\alpha = 3$	3.302	0.302		1.472		1.336	VI
50	$\alpha = 0.5$	1.505	1.005	1.00	2.81	3.067	0.237	IV
	$\beta = 3$	2.985	−0.015		0.606		0.015	IV
	$\alpha = 3$	3.098	0.098		1.071		−0.228	I
100	$\alpha = 0.5$	1.181	0.681	0.682	2.204	2.398	0.277	IV
	$\beta = 3$	2.964	−0.036		0.47		−0.016	I
	$\alpha = 3$	2.964	−0.036		0.822		−0.591	I
300	$\alpha = 0.5$	0.751	0.251	0.254	0.875	1.049	0.241	IV
	$\beta = 3$	2.967	−0.033		0.297		−0.319	I
	$\alpha = 3$	2.979	−0.021		0.498		0.533	IV

Table A5:
Set(5): ($\alpha = 0.5, \beta = 4, \theta = 3$)

Sample Size	Parameters	Mean of estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 0.5$	2.772	2.272	2.365	3.982	5.420	0.241	IV
	$\beta = 4$	4.62	0.62		1.871		−0.314	I
	$\alpha = 3$	3.895	0.895		3.166		2.572	VI
20	$\alpha = 0.5$	1.785	1.285	1.482	3.053	3.959	0.215	IV
	$\beta = 4$	4.182	0.182		1.18		−0.492	I
	$\alpha = 3$	3.717	0.717		2.228		0.269	IV
30	$\alpha = 0.5$	1.678	1.178	1.225	3.041	3.540	0.211	IV
	$\beta = 4$	4.052	0.052		0.98		0.167	IV
	$\alpha = 3$	3.333	0.333		1.526		0.283	IV
50	$\alpha = 0.5$	1.526	1.026	1.031	2.777	3.092	0.229	IV
	$\beta = 4$	3.968	−0.032		0.811		0.019	IV
	$\alpha = 3$	3.097	0.097		1.094		−0.228	I
100	$\alpha = 0.5$	1.171	0.671	0.673	2.129	2.368	0.251	IV
	$\beta = 4$	3.959	−0.041		0.626		−0.02	I
	$\alpha = 3$	2.967	−0.033		0.827		0.064	IV
300	$\alpha = 0.5$	0.743	0.243	0.247	0.8	1.020	0.323	IV
	$\beta = 4$	3.957	−0.043		0.394		−0.145	I
	$\alpha = 3$	2.98	−0.02		0.496		0.509	IV

Table A6:

Set(6): ($\alpha = 0.5, \beta = 2, \theta = 4$)

Sample Size	Parameters	Mean of estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 0.5$	1.749	1.249	3.023	2.93	6.762	0.256	IV
	$\beta = 2$	2.292	0.292		0.884		−0.263	I
	$\alpha = 4$	6.738	2.738		6.03		3.163	VI
20	$\alpha = 0.5$	1.1712	1.212	1.563	2.924	4.308	0.244	IV
	$\beta = 2$	2.082	0.082		0.602		0.47	IV
	$\alpha = 4$	4.984	0.984		3.107		0.357	IV
30	$\alpha = 0.5$	1.628	1.128	1.228	2.912	3.599	0.22	IV
	$\beta = 2$	2.034	0.034		0.47		0.08	IV
	$\alpha = 4$	4.485	0.485		2.115		0.385	IV
50	$\alpha = 0.5$	1.475	0.975	0.983	2.575	3.00	0.237	IV
	$\beta = 2$	1.988	−0.033		0.388		0.015	IV
	$\alpha = 4$	4.122	0.122		1.501		−0.328	I
100	$\alpha = 0.5$	1.106	0.606	0.610	2.105	2.386	0.241	IV
	$\beta = 2$	1.967	−0.025		0.306		−0.012	I
	$\alpha = 4$	3.934	−0.066		1.083		−0.136	I
300	$\alpha = 0.5$	0.733	0.213	0.221	0.7	1.014	0.255	IV
	$\beta = 2$	1.99	−0.011		0.192		−0.108	I
	$\alpha = 4$	3.941	−0.059		0.709		−0.268	I

Table A7:

Set(7): $(\alpha = 0.5, \beta = 2, \theta = 5)$

Sample Size	Parameters	Mean ofl estimators	Biases	Tota Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha = 0.5$	1.688	1.188	4.00	2.908	8.918	0.65	IV
	$\beta = 2$	2.290	0.290		0.659		−0.258	I
	$\alpha = 5$	8.815	3.815		8.405		0.273	IV
20	$\alpha = 0.5$	1.627	1.127	1.771	2.853	5.162	0.242	IV
	$\beta = 2$	2.094	0.094		0.507		−0.715	I
	$\alpha = 5$	6.363	1.363		4.273		0.232	IV
30	$\alpha = 0.5$	1.559	1.059	1.231	2.804	4.028	0.234	IV
	$\beta = 2$	2.039	0.030		0.46		0.071	IV
	$\alpha = 5$	5.628	0.628		2.856		0.294	IV
50	$\alpha = 0.5$	1.415	0.915	0.928	2.377	3.076	0.254	IV
	$\beta = 2$	1.99	−0.009		0.377		0.021	IV
	$\alpha = 5$	5.159	0.159		1.917		0.219	IV
100	$\alpha = 0.5$	1.146	0.600	0.605	1.887	2.347	0.238	IV
	$\beta = 2$	1.976	−0.024		0.297		−0.011	I
	$\alpha = 5$	4.926	−0.074		1.365		−0.087	I
300	$\alpha = 0.5$	0.765	0.200	0.207	0.634	1.044	0.283	IV
	$\beta = 2$	1.989	−0.01		0.188		−0.234	I
	$\alpha = 5$	4.944	−0.056		0.809		−0.462	I

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