Facultad de Ciencias Naturales y Exactas
Universidad del Valle

Lower Bound for the First Steklov Eigenvalue

Óscar Andrés Montaño Carreño
Universidad del Valle

Received: October 24, 2014 Accepted: December 18, 2014
Pag. 123-130

Abstract
In this paper we find lower bounds for the first Steklov eigenvalue in Riemannian n-manifolds, $n = 2, 3$, with non-positive sectional curvature.

Keywords: First Steklov eigenvalue, sectional curvature.

1 Introduction

Let $(\overline{M}, g)$ be a n-dimensional, compact, connected Riemannian manifold with smooth boundary ∂M. The operator $L : C^\infty(\partial M) \to C^\infty(\partial M)$ defined by $Lu = \frac{\partial \hat{u}}{\partial \eta}$, where $\hat{u}$ is the harmonic extension over M of u and η is the unit outward normal to ∂M is known as Dirichlet-Neumann operator. The first non-nule eigenvalue $\nu(M)$ of L is called the first eigenvalue of the Steklov problem and it is variationaly characterized by:

$$\nu(M) = \min \left\{ R[\varphi] : \int\limits_{\partial M} \varphi d\sigma = 0, \varphi \in C^\infty(\overline{M}) \right\}, \tag{1}$$

where $R[\varphi] = \frac{\int\limits_M |\nabla \varphi|^2 dv}{\int\limits_{\partial M} \varphi^2 d\sigma}$ is called Rayleigh quotient. For the Euclidian ball of the radio $r > 0$ the first eigenvalue is $\nu = \frac{1}{r}$ with its own space generated by the coordinate functions. A function φ such that $\int\limits_{\partial M} \varphi d\sigma = 0$ is called test function. Starting from the variational characterization given by (1), if φ is a test function over M then $\nu(M) \leq R[\varphi]$.

In recent years, different authors, among them Weinstock [14], Kuttler and Sigillito [8], Payne [13], Escobar [4, 5, 6], Wang and Xia [15, 16], Illias and Makloul [7] and Montaño [9, 10, 11, 12] among others have dealt with the problem of finding geometrical estimates for the first eigenvalue of Steklov.

In this article we find lower bounds for the first eigenvalue of Steklov in geodesic balls and simply connected domain of a Riemannian n-manifold; where $n = 2, 3$, complete of non-positive sectional curvature.

Our result is extended to Riemannian n-manifolds; where $n = 2, 3$, the estimate of Kutler-Sigillito for star-shaped domains of the plane [8].

2 Preliminaries

For this article, (M, g) will be a Riemannian n-manifold; where $n = 2, 3$, complete, simply connected and with non-positive sectional K curvature. Since M is complete, then for every $p \in M$ the exponential function exp_p is defined over all T_pM. Moreover, since $K \leq 0$ over all M the Hadamard theorem [1] implies that $exp_p : T_pM \to M$ is a difeomorphism, that is, M is difeomorphic to $\mathbb{R}^n \approx T_pM$, $n = 2, 3$.

Under the given hypothesis

$$v(t, \theta) = exp_p t\xi(\theta), \ t \geq 0, \ \xi \in S^{n-1}, \tag{2}$$

it is a parameterization in geodesic coordinates for M.

When $n = 2$ the metrics can be written in the form of

$$ds^2 = dt^2 + f^2(t, \xi)d\xi^2, \tag{3}$$

where $f(0, \xi) = 0$, $\frac{\partial f}{\partial t}(0, \xi) = 1$ and $d\xi^2$ is the metric of S^1 [2].

When $n = 3$ the metrics can be written in the form of

$$ds^2 = dt^2 + h_{ij}(t, \theta)d\theta^i \otimes d\theta^j. \tag{4}$$

For every $t > 0$ the boundary of the geodesic ball with center p and radio t, $\partial\mathfrak{B}(p, t)$,is difeomorphic to S^2 and thus $h_{ij}(t, \theta)d\theta^i \otimes d\theta^j$ is a metric over S^2. The Uniformization theorem implies that the metric is conformally equivalent to the standard metric over S^2 ([6] pag 152). Therefore, we can assume that the metric over the 3-manifold is also in the form of

$$ds^2 = dt^2 + f^2(t, \xi)d\xi^2, \tag{5}$$

where $f(0, \xi) = 0$, $\frac{\partial f}{\partial t}(0, \xi) = 1$ and $d\xi^2$ is the standard metric of S^2.

Since the sectional curvature is non-positive, then the Bishop comparison theorem [3] implies

$$\frac{\partial}{\partial t}\left\{\frac{f(t, \xi)}{t}\right\} \geq 0, \tag{6}$$

$$f(t, \xi) \geq t,$$

and therefore

$$1 \leq \lim_{t \to 0^+} \frac{f(t, \xi)}{t} \leq \frac{f(t, \xi)}{t} \leq f(1, \xi) \leq f_o = \max_{\xi \in S^1} f(1, \xi). \tag{7}$$

3 Estimate for the First Eigenvalue over a Geodesic Ball

Theorem 3.1. *If $\mathfrak{B}_p \subset M$ is a geodesic ball with a ratio 1 and center $p \in M$, then the first eigenvalue of Steklov satisfies the inequality*

$$\nu(\mathfrak{B}_p) \geq \frac{1}{f_o^{n+1}}, \tag{8}$$

where $f_o = \max\limits_{\xi \in S^1} f(1,\xi)$. The equality is achieved only if $\mathfrak{B}_p$ is isometric to the Euclidian ball of radio 1.

Proof. From (6) we have

$$\begin{aligned}
R_g[\varphi] &= \\
\frac{\int\limits_{\mathfrak{B}_p} |\nabla\varphi|^2\, dv}{\int\limits_{\partial\mathfrak{B}_p} \varphi^2 d\sigma} &= \frac{\int\limits_{\mathfrak{B}_p} \left\{\varphi_t^2 + \frac{1}{f^2}\left|\overline{\nabla}\varphi\right|^2\right\} f^{n-1} dt d\xi}{\int\limits_{\partial\mathfrak{B}_p} \varphi^2 f^{n-1} d\xi} \\
&\geq \frac{\int\limits_{\mathfrak{B}_p} \left\{\varphi_t^2 + \frac{1}{f^2}\left|\overline{\nabla}\varphi\right|^2\right\} t^{n-1} dt d\xi}{\int\limits_{\partial\mathfrak{B}_p} \varphi^2 f^{n-1} d\xi} \\
&\geq \frac{1}{f_o^{n-1}} \frac{\int\limits_{\mathfrak{B}_p} \left\{\varphi_t^2 + \frac{1}{f^2}\left|\overline{\nabla}\varphi\right|^2\right\} t^{n-1} dt d\xi}{\int\limits_{\partial\mathfrak{B}_p} \varphi^2 d\xi},
\end{aligned}$$

and from (7) it follows that $1 \geq \frac{1}{f_o^2}$ and $\frac{t^2}{f^2} \geq \frac{1}{f_o^2}$, and therefore

$$\begin{aligned}
R_g[\varphi] &\geq \frac{1}{f_o^{n+1}} \frac{\int\limits_{\mathfrak{B}_p} \left\{\varphi_t^2 + \frac{1}{t^2}\left|\overline{\nabla}\varphi\right|^2\right\} t^{n-1} dt d\xi}{\int\limits_{\partial\mathfrak{B}_p} \varphi^2 d\xi} \\
&\geq \frac{1}{f_o^{n+1}} R_\delta[\varphi],
\end{aligned}$$

where δ is the Euclidian metric and $\overline{\nabla}$ is the gradient over S^{n-1}. If φ_1 is a eigenfunction for $\nu(\mathfrak{B}_p, g)$ and choose the constant b such that $\varphi := \varphi_1 - b$ serves as a test function for the first eigenvalue of the Euclidean ball, $\nu(\mathfrak{B}_p, \delta) = 1$, then

$$\begin{aligned}
\nu(\mathfrak{B}_p, g) &= R_g[\varphi_1] \\
&\geq R_g[\varphi] \\
&\geq \frac{1}{f_o^{n+1}} R_\delta[\varphi] \\
&\geq \frac{1}{f_o^{n+1}}.
\end{aligned}$$

The equality is given only if $f(t,\xi) = t$, in such case $\mathfrak{B}_p$ is isometric to the Euclidian ball of radio 1.

From the Escobar comparison theorem [6] and from our estimate, we have

$$\frac{1}{f_o^{n+1}} \leq \nu(\mathfrak{B}_p, g) \leq 1. \tag{9}$$

4 Simply Connected Domains

In this section, D is a simply connected domain of T_pM and Ω is also a domain of M such that $\Omega = exp_p(D)$. $\upsilon(t,\theta) = exp_p t\xi(\theta)$ is a parameterization in geodesic coordinates of M, where $\xi(\theta)$ is a parameterization of S^{n-1}. We suppose that the boundary of D is smooth and it is given by $\partial D = \{R\xi : \xi \in S^{n-1}\}$, where $R : S^{n-1} \to \mathbb{R}$ is a strictly positive smooth function. In geodesic coordinates, the function $F(t,\theta) = t - R(\theta)$ is such that ∂D and $\partial\Omega$ are curves (surfaces) of 0 level of F in the parameterizations $t\xi(\theta)$ and $\upsilon(t,\theta)$ respectively. For this reason the unit normal vectors to each one of the boundaries are given by:

$$\eta_\delta = \frac{1}{W_\delta}(\partial_t - \frac{\overline{\nabla} R}{R^2}), \text{ where } W_\delta = \sqrt{1 + (\frac{|\overline{\nabla} R|}{R})^2}, \tag{10}$$

$$\eta_g = \frac{1}{W_g}(\partial_t - \frac{\overline{\nabla} R}{f^2}), \text{ where } W_g = \sqrt{1 + (\frac{|\overline{\nabla} R|}{f})^2}. \tag{11}$$

From the previous identities, if we solve the equations $\cos\gamma = \langle \eta_\delta, \partial_t \rangle = \frac{1}{W_\delta}$ and $\cos\psi = \langle \eta_g, \partial_t \rangle = \frac{1}{W_g}$ we obtain:

$$(\frac{|\overline{\nabla} R|}{R})^2 = \tan^2\gamma, \tag{12}$$

$$(\frac{|\overline{\nabla} R|}{f})^2 = \tan^2\psi. \tag{13}$$

From the inequality $\frac{1}{f} \leq \frac{1}{t}$ it is deduced that over the boundaries

$$\tan^2\psi = (\frac{|\overline{\nabla} R|}{f})^2 \leq (\frac{|\overline{\nabla} R|}{R})^2 = \tan^2\gamma. \tag{14}$$

4.1 Estimate for the Integral of the Squared Gradient Over Ω

Let us suppose that the angle γ satisfies the inequality:

$$\tan^2\gamma \leq a \tag{15}$$

for $a > 0$, that is; $(\frac{|\overline{\nabla} R|}{f})^2 \leq (\frac{|\overline{\nabla} R|}{R})^2 \leq a$.

Changing the variable $\theta = u$ and $t = \rho R(u)$ we have:

$$\int_\Omega |\nabla\varphi|^2 dv = \int_{S^{n-1}} \int_0^{R(\theta)} \left\{ \varphi_t^2 + \frac{1}{f^2} \left|\overline{\nabla}\varphi\right|^2 \right\} f^{n-1} dt d\xi$$

$$= \int_{S^{n-1}} \int_0^1 \frac{1}{f^2} \left\{ \left|\overline{\nabla}\varphi\right|^2 - 2\rho\frac{\varphi_\rho}{R} \left\langle \overline{\nabla}\varphi, \overline{\nabla}R \right\rangle + \frac{f^2 + \rho^2 \left|\overline{\nabla}R\right|^2}{R^2} \varphi_\rho^2 \right\} f^{n-1} R d\rho d\xi.$$

Since $-2\rho\frac{\varphi_\rho}{R} \left\langle \overline{\nabla}\varphi, \overline{\nabla}R \right\rangle \geq -\left\{ \alpha^2 \left|\overline{\nabla}\varphi\right|^2 + \frac{\rho^2 \left|\overline{\nabla}R\right|^2}{\alpha^2 R^2} \varphi_\rho^2 \right\}$ for every $\alpha > 0$, then

$$\int_\Omega |\nabla\varphi|^2 dv \geq \int_{S^{n-1}} \int_0^1 \left\{ \left(\frac{1}{R^2} - \frac{\rho^2 \left|\overline{\nabla}R\right|^2}{f^2 R^2} \beta^2\right) \varphi_\rho^2 + \frac{1}{f^2} \frac{\beta^2}{1+\beta^2} \left|\overline{\nabla}\varphi\right|^2 \right\} f^{n-1} R d\rho d\xi,$$

where $\beta^2 = \frac{1-\alpha^2}{\alpha^2}$. The sectional non-positive curvature implies that $f(t,\xi) \geq t$ therefore $-\frac{\rho^2}{f^2(\rho R,\xi)} \geq -\frac{1}{R^2}$ and that results in

$$\int_\Omega |\nabla\varphi|^2 dv \geq \int_{S^{n-1}} \int_0^1 \left\{ \frac{1}{R}(1 - a\beta^2)\varphi_\rho^2 + \frac{R}{f^2} \frac{\beta^2}{1+\beta^2} \left|\overline{\nabla}\varphi\right|^2 \right\} f^{n-1} d\rho d\xi.$$

If we resolve the equation $\frac{1-a\beta^2}{R} = \frac{R\beta^2}{1+\beta^2}$ for β^2 we obtain

$$\beta^2 = \frac{-R^2 - a + 1 + \sqrt{(R^2+a-1)^2 + 4a}}{2a} > 0. \tag{16}$$

Therefore

$$\begin{aligned} \frac{1-a\beta^2}{R} &= \frac{R\beta^2}{1+\beta^2} \\ &= \frac{R^2 + a + 1 - \sqrt{(R^2+a-1)^2+4a}}{2R} \\ &= \frac{2R}{R^2 + a + 1 + \sqrt{(R^2+a-1)^2+4a}} \\ &\geq \frac{2r_m}{r_M^2 + a + 1 + \sqrt{(r_M^2+a-1)^2+4a}}, \end{aligned}$$

with $r_m = \min\limits_{S^{n-1}} R$ and $r_M = \max\limits_{S^{n-1}} R$. Going back to the integral we obtain

$$\int_\Omega |\nabla\varphi|^2 dv \geq$$

$$\frac{2r_m}{r_M^2 + a + 1 + \sqrt{(r_M^2+a-1)^2+4a}} \int_{S^{n-1}} \int_0^1 \left\{ \varphi_\rho^2 + \frac{1}{f^2} \left|\overline{\nabla}\varphi\right|^2 \right\} f^{n-1} d\rho d\xi,$$

getting the estimate

$$\int_{\Omega} |\nabla\varphi|^2\, dv \geq C \int_{\mathfrak{B}_p} |\nabla\varphi|^2\, dv, \tag{17}$$

with

$$C = \frac{2r_m}{r_M^2 + a + 1 + \sqrt{(r_M^2 + a - 1)^2 + 4a}}.$$

4.2 Estimate for the Integral over the Boundary of φ^2

$$\int_{\partial\Omega} \varphi^2 d\sigma = \int_{S^{n-1}} \varphi^2 \sqrt{\frac{|\overline{\nabla} R|^2}{f^2} + 1} f^{n-1} d\xi.$$

Since $f(R,\xi) \geq R$ then

$$\begin{aligned}\int_{\partial\Omega} \varphi^2 d\sigma &\leq \int_{S^{n-1}} \varphi^2 \sqrt{\frac{|\overline{\nabla} R|^2}{R^2} + 1} f^{n-1} d\xi \\ &\leq \sqrt{a+1} \int_{S^{n-1}} \varphi^2 f^{n-1} d\xi,\end{aligned}$$

getting the estimate

$$\int_{\partial\Omega} \varphi^2 d\sigma \leq \sqrt{a+1} \int_{\partial\mathfrak{B}_p} \varphi^2 d\sigma. \tag{18}$$

4.3 Estimate for the rst eigenvalue of Steklov over Ω

From the estimates calculated in the previous sections (17), (18), we have the following estimate for the Rayleigh quotient:

$$\begin{aligned}R[\varphi] &= \frac{\int_{\Omega} |\nabla\varphi|^2\, dv}{\int_{\partial\Omega} \varphi^2 d\sigma} \\ &\geq \frac{C}{\sqrt{a+1}} \frac{\int_{\mathfrak{B}_p} |\nabla\varphi|^2\, dv}{\int_{\partial\mathfrak{B}_p} \varphi^2 d\sigma},\end{aligned}$$

where $C = \frac{2r_m}{r_M^2+a+1+\sqrt{(r_M^2+a-1)^2+4a}}$ and $\mathfrak{B}_p$ is the geodesic ball over M with center in p and radio 1.

If φ_1 is a eigenfunction for the first eigenvalue over Ω and chose the constant b such that $\varphi = \varphi_1 - b$ serves as a test function for the first eigenvalue of the geodesic ball $\mathfrak{B}_p$, then

$$\begin{aligned}
\nu(\Omega) &= R[\varphi_1] \\
&\geq R[\varphi] \\
&\geq \frac{C}{\sqrt{a+1}}\nu(\mathfrak{B}_p) \\
&\geq \frac{C}{\sqrt{a+1}}\frac{1}{f_o^{n-1}}.
\end{aligned}$$

References

[1] Docarmo M. P. (1992). Riemannian Geometry, Birkhauser.

[2] Chavel I. (2005). Riemannian Geometry A Modern Introduction, Cambridge University Press.

[3] Chavel I. (1984). Eigenvalues In Riemannian Geometry, Academic Press, Inc.

[4] Escobar J. F. (1997). The Geometry of the first Non-Zero Steklov Eigenvalue, Journal of Functional Analysis, 150, 544-556.

[5] Escobar J. F. (1999). An Isoperimetric Inequality and the First Steklov Eigenvalue, Journal of Functional Analysis, 165, 101-116.

[6] Escobar J. F. (2000). A comparison Theorem for the First non-zero Steklov Eigenvalue, Journal of Functional Analysis, 178, 143-155.

[7] Ilias S. and Makhoul O. (2011). A Reilly inequality for the first Steklov eigenvalue, Differential Geom. Appl. 5, 699 - 708.

[8] Kuttler J. R. and Sigillito V. G. (1968). Lower Bounds for Stekloff and free membrane eigenvalues, SIAM Review, 10, 368-370-

[9] Montaño O. A. (2013). The Stekloff problem for rotationally invariant metrics on the ball Revista Colombiana de Matemáticas, 47, 2, 181-190.

[10] Montaño O. A. (2013). Cota superior para el primer valor propio del problema de Steklov, Revista Integración, 31, 1, 53-58.

[11] Montaño O. A. (2013). Cota superior para el primer valor propio de Steklov en el espacio euclídeo, Revista de Ciencias, 17, 2, 95-103.

[12] Montaño O. A. (2014). Métricas rotacionalmente invariantes y el problema de Steklov, Revista Integración, 32, 2, 117-128.

[13] Payne L. E. (1970). Some isoperimetric inequalities for harmonic functions, SIAM J. Math. Anal. 1, 354-359.

[14] Weinstock R. (1954). Inequalities for a classical eigenvalue problem, J. Rational Mech. Anal. 3, 745 - 753.

[15] Wang Q. and Xia C. (2009). Sharp bounds for the first non-zero Steklov eigenvalues, J. Funct. Anal, 257, 2635 - 2654.

[16] Wang Q. and Xia C. (2010). Inequalities for the Steklov Eigenvalues, arXiv:1006.1154.

Autor's address

Óscar Andrés Montaño Carreño
Departamento de Matemáticas, Universidad del Valle, Cali - Colombia
oscar.montano@correounivalle.edu.co