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A Priori Estimates of the Prescribed Scalar Curvature on the Sphere

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Abstract

This paper considers the prescribed scalar curvature problem on the sphere for $n \geq 3$. Given a prescribed scalar curvature function $K : S^n \to \mathbb{R}$ and a centered dilation defined by $F_y = \Sigma^{-1} \circ D_\beta \circ \Sigma$, $y \in B^{n+1}$, where Σ is the stereographic projection and D_β is a dilation in $\mathbb{R}^n$, in this work we estimate the gradient of the function K near the critical point of the function $\overline{J}_p(y) = \int_{S^n} K(\zeta)\phi^{p+1} d\sigma(\zeta)$ where $\phi(y) = |(F_y^{-1})'|^{\frac{n-2}{2}}$. We will use this estimate to find L^p estimates of the first two y-derivatives of the function $K \circ F_y(\xi)$.

Keywords: metrics, scalar curvature, conformal geometry.

1 Introduction

Let (S^n, δ_{ij}) be the unitary sphere with the standard metric. A natural question in Riemannian geometry is: Given a function $K : S^n \to \mathbb{R}$, is there a metric g conformally related to the standard metric δ_{ij} such that K is the scalar curvature of S^n with respect to the metric g? This is equivalent to the problem of finding a positive smooth function $u : S^n \to \mathbb{R}$ which satisfies the equation:

$$\Delta u - \frac{n(n-2)}{4} u + \frac{n-2}{4(n-1)} K u^{\frac{n+2}{n-2}} = 0. \tag{1}$$

If we set $g = u^{\frac{4}{n-2}} \delta_{ij}$, where u is a solution of this problem, then the function K is the scalar curvature of S^n with respect to the metric g.

The problem of conformal deformation of metrics in S^n have been extensively studied by many authors (for example, see [1, 2, 3, 4, 5, 6, 7, 8] and the references therein). An important feature of this problem is that it is a conformal invariant one. More precisely, if u is a solution of equation (1), then for any conformal map $F : S^n \to S^n$ the function $\alpha_F(u) = |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$ is a solution to problem (1) with scalar curvature $K \circ F$.

The problem of conformal deformation of metrics in S^n can be approached using the so called Yamabe method, which consists in studying first the subcritical problem in the equation (1):

$$\Delta u_p - \frac{n(n-2)}{4} u_p + \frac{n-2}{4(n-1)} K u_p^p = 0, \tag{2}$$

with $p \in \left(1, \frac{n+2}{n-2}\right)$ and then consider the limit of the solutions when $p \uparrow \frac{n+2}{n-2}$.

Let $E(u)$ be the energy norm associated with the linear part of (2), that is:

$$E(u) = \int_{S^n} \left(\frac{n(n-2)}{4} u^2 + |\nabla u|^2 \right) d\sigma_g,$$

and let

$$\mathcal{S} = \{u \in H^1(S^n) \text{ with } u \geq 0 \text{ almost everywhere and with } E(u) = E(1)\}.$$

Let us consider the open unit ball B^{n+1} and the map $\Phi : B^{n+1} \to \mathcal{S}$ defined by:

$$\Phi(y) = \alpha_y := \alpha_{F_y}(1) = |(F_y^{-1})'|^{\frac{n-2}{2}},$$

where $F_y : S^n \to S^n$ is the restriction to S^n of a special conformal map $F_y : \overline{B^{n+1}} \to \overline{B^{n+1}}$ that satisfies $F_y(0) = y$ and fix the points $\pm\frac{y}{|y|}$; this function maps 0 to y and commutes with rotations about the line joining the origin and the point y. This map is referred to as a centered dilation.

For $p \in \left(1, \frac{n+2}{n-2}\right)$ and $u \in \mathcal{S}$, let $J_p(u)$ defined by:

$$J_p(u) = \int_{S^n} K u^{p+1} d\sigma.$$

If u is a critical point of $J_p(.)$ on $\mathcal{S}$, then a multiple of u satisfies problem (2). Let us define the function $\overline{J}_p = J_p \circ \Phi$.

This work is motivated by the work of Schoen and Zhang [7] on the prescribed scalar curvature problem on the n-dimensional sphere, $n \geq 3$, where they prove an existence result for $n = 3$.

To determine the nature of the critical points of J_p, they study the critical points of $\overline{J}_p$ and then make a perturbation argument. In order to understand the nature of the critical points of $\overline{J}_p$ near the border of B^{n+1}, they study the behavior of K near those critical points.

In this paper we will study more closely the behavior of the function K near the critical points of $\overline{J}_p$. Given a critical point y_0 of $\overline{J}_p$, near S^n, in this work we will find an estimate of the gradient of the function K at the point $\frac{y_0}{|y_0|}$ and we will use this to find L^p estimates of the function $f \circ F_y(\xi) = \left(K \circ F_y(\xi) - K\left(\frac{y}{|y|}\right)\right)$ and its first two y-derivatives. Our method to get the estimates parallels that of Escobar and Garcia ([3]) in the problem of prescribed mean curvature on the boundary of the ball. In a coming paper we will use the estimates found in this work to give an alternative proof for some of the results in [7].

2 Preliminaries

Let $y \in B^{n+1}$. Up to a rotation we will assume that $y = (0, ..., 0, y_{n+1})$. In this case the centered dilation function F_y is given by $F_y(x) = \Sigma^{-1} \circ D_\beta \circ \Sigma(x)$, where the functions Σ and D_β are defined as follows. The function $\Sigma : \overline{B^{n+1}} \setminus \{(0, 0, \ldots, 0, -1)\} \to \overline{\mathbb{R}^{n+1}_+}$ is defined as $\Sigma = T_{-2} \circ I_2 \circ T_1$ where $T_a(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1} + a)$, and I_R is the inversion map $x \to \frac{R^2 x}{|x|^2}$. Here $\mathbb{R}^{n+1}_+$ denotes the upper half $(n+1)$-dimensional Euclidean space.

Hence,

$$\Sigma(x) = \left(\frac{4\overline{x}}{|\overline{x}|^2 + (1 + x_{n+1})^2}, \frac{2(1 - |\overline{x}|^2 - x_{n+1}^2)}{|\overline{x}|^2 + (1 + x_{n+1})^2}\right),$$

where $\overline{x} = (x_1, \ldots, x_n)$ and $x = (\overline{x}, x_{n+1})$. Observe that if $|x| = \sqrt{|\overline{x}|^2 + x_{n+1}^2} = 1$, then

$$\Sigma(x) = \left(\frac{2\overline{x}}{1 + x_{n+1}}, 0\right).$$

Thus the map $\Sigma|_{S^n - \{(0,0,\ldots,0,-1)\}}$ is the stereographic projection. The inverse function of Σ is:

$$\Sigma^{-1}(x) = \left(\frac{4\overline{x}}{|\overline{x}|^2 + (2 + x_{n+1})^2}, \frac{4 - |x|^2}{|\overline{x}|^2 + (2 + x_{n+1})^2}\right).$$

When $x_{n+1} = 0$ we get

$$\Sigma^{-1}(\overline{x}, 0) = \left(\frac{4\overline{x}}{|\overline{x}|^2 + 4}, \frac{4 - |\overline{x}|^2}{|\overline{x}|^2 + 4}\right),$$

which is the inverse of the stereographic projection from the south pole of the sphere.

The function $D_\beta : \overline{\mathbb{R}^{n+1}_+} \to \overline{\mathbb{R}^{n+1}_+}$ is defined by $D_\beta(x) = \beta x$, where $\beta = \frac{1-|y|}{1+|y|}$.

Proposition 2.1. *If $F_y = \Sigma^{-1} \circ D_\beta \circ \Sigma$ then*

$$F_y(x) = B^{-1}(4\beta A\overline{x}, (A^2 - 4\beta^2|\overline{x}|^2 - \beta^2(1 - |x|^2)))$$

and $F_y(0) = y$, where $\beta = \frac{1-|y|}{1+|y|}$, $x = (x_1, \ldots, x_{n+1})$, $\bar{x} = (x_1, \ldots, x_n)$,

$$A = |\overline{x}|^2 + (1 + x_{n+1})^2 \quad \text{and} \quad B = 4\beta^2|\overline{x}|^2 + [A + \beta(1 - |x|^2)]^2.$$

Proof. Since

$$D_\beta \circ \Sigma(x) = \left(\frac{4\beta\overline{x}}{|\overline{x}|^2 + (1+x_{n+1})^2}, \frac{2\beta(1-x_{n+1}^2 - |\overline{x}|^2)}{|\overline{x}|^2 + (1+x_{n+1})^2} \right),$$

then

$$\begin{aligned} F_y(x) &= \Sigma^{-1}\left(\frac{4\beta\overline{x}}{|\overline{x}|^2 + (1+x_{n+1})^2}, \frac{2\beta(1-x_{n+1}^2 - |\overline{x}|^2)}{|\overline{x}|^2 + (1+x_{n+1})^2} \right) \\ &= \Sigma^{-1}(4\beta A^{-1}\overline{x}, 2\beta A^{-1}(1-x_{n+1}^2 - |\overline{x}|^2)) \\ &= B^{-1}(4\beta A\overline{x}, (A^2 - 4\beta^2|\overline{x}|^2 - \beta^2(1-|x|^2))). \end{aligned}$$

Therefore

$$F_y(x) = B^{-1}(4\beta A\overline{x}, (A^2 - 4\beta^2|\overline{x}|^2 - \beta^2(1-|x|^2))).$$

If $x = 0$, then $A = 1$ and $B = (1+\beta)^2$. Hence,

$$F_y(0) = [(1+\beta)^{-2}](0, 1-\beta^2) = \left(0, \frac{1-\beta^2}{(1+\beta)^2}\right) = \left(0, \frac{1-\beta}{(1+\beta)}\right) = (0, y_{n+1}) = y.$$

Since $F_y(0) = y$ then $\beta = \frac{1-|y|}{1+|y|}$ when $y_{n+1} \geq 0$ and $\beta = \frac{1+|y|}{1-|y|}$ when $y_{n+1} \leq 0$. For this paper we will use the convention that when β is large, we call it λ and when β is small we call it μ. Observe that $F_y^{-1} = F_{-y}$. In order to get the L^p estimates of the derivatives of the function $K \circ F_y$, we need the estimates of the derivatives of the function F_y. If we rewrite the function F_y as

$$F_y(z) = \frac{(4\mu\bar{z}|z-s|^2,\ |z-s|^4 - 4\mu^2|z'|^2)}{|z-s|^4 + 4\mu^2|\bar{z}|^2},$$

where $\bar{z} = (z_1, z_2, \ldots, z_n)$, $z = (\bar{z}, z_{n+1})$, the calculations in [3] leads to

Lemma 2.2. *For* $1 \leq i, j \leq n+1$

$$\left| \frac{\partial F_y}{\partial y_i}(z) \right| \leq \frac{C}{\mu^r |z-s|^{1-r}}, \tag{3}$$

and

$$\left| \frac{\partial^2 F_y}{\partial y_j \partial y_i} \right| \leq \frac{C_1}{\mu^r|z-s|^{2-r}} + \frac{C_2}{\mu^r|z-s|^{1-r}},$$

where $z \in S^n$, $s = (\overline{0}, -1)$, $\mu = \frac{1-|y|}{1+|y|}$ *and* $0 \leq r \leq 1$.

Proposition 2.3. *For* $x \in S^n$, *we get*

$$F_y^*(\delta_{ij})_x = \left(\frac{1-|y|^2}{|y+x|^2} \right)^2 \delta_{ij},$$

where $F_y^*(\delta_{ij})$ *is the pullback of the metric* δ_{ij} *induced by the function* F_y.

Proof. Given $x \in S^n$, straightforward calculations show that

$\Sigma^*(\delta_{ij})_x(e_i, e_j) =< \frac{\partial \Sigma}{\partial x_i}, \frac{\partial \Sigma}{\partial x_j} >= \frac{4}{(1+x_{n+1})^2}\delta_{ij}$, $D^*_\beta(\delta_{ij})_{\Sigma(x)}(e_i, e_j) = \beta^2\delta_{ij}$,
and

$$(\Sigma^{-1})^*(\delta_{ij})_{\beta\Sigma(x)}(e_i, e_j) = \frac{16}{\left(\left|\frac{4\beta\overline{x}}{|\overline{x}|^2 + (1+x_{n+1})^2}\right|^2 + 4\right)^2}\delta_{ij}.$$

Hence,

$$(\Sigma^{-1})^*(\delta_{ij})_{\beta\Sigma(x)}(e_i, e_j) = \frac{(1+x_{n+1})^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}\delta_{ij}.$$

Since the metrics $\Sigma^*(\delta_{ij})$, $(\Sigma^{-1})^*$ and D^*_β are diagonal, then

$$F_y{}^*(\delta_{ij}) = \Sigma^*(\delta_{ij}).D_\beta{}^*(\delta_{ij}).\Sigma^{-1*}(\delta_{ij}).$$

Thus,

$$F^*_y(\delta_{ij})(e_i, e_j) = \frac{(1+x_{n+1})^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}.\frac{4}{(1+x_{n+1})^2}.\beta^2\delta_{ij}$$

hence,

$$F^*_y(\delta_{ij})(e_i, e_j) = \frac{4\beta^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}\delta_{ij}.$$

If $y_{n+1} \geq 0$, then $\beta = \frac{1-|y|}{1+|y|}$ where $|y| = y_{n+1}$ and consequently

$$\begin{aligned}
\beta^2(1-x_{n+1}) + (1+x_{n+1}) &= \left(\frac{1-|y|}{1+|y|}\right)^2(1-x_{n+1}) + (1+x_{n+1}) \\
&= \frac{(1-|y|)^2(1-x_{n+1}) + (1+|y|)^2(1+x_{n+1})}{(1+|y|)^2} \\
&= \frac{2(1+|y|^2+2|y|x_{n+1})}{(1+|y|)^2} = \frac{2|y+x|^2}{(1+|y|)^2}.
\end{aligned}$$

Then,

$$\begin{aligned}
F^*_y(\delta_{ij})(e_i, e_j) &= \frac{4\beta^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}\delta_{ij} = \frac{((1+|y|)^2)^2\frac{(1-|y|)^2}{(1+|y|)^2}}{(|y+x|^2)^2}\delta_{ij} \\
&= \frac{(1-|y|^2)^2}{(|y+x|^2)^2}\delta_{ij} = \left(\frac{1-|y|^2}{|y+x|^2}\right)^2\delta_{ij}.
\end{aligned}$$

From now on, we will denote for $|(F_y)'|(\zeta) = \dfrac{1-|y|^2}{|y+\zeta|^2}$, $\zeta \in S^n$, the linear stretch factor of the conformal transformation F_y.

Proposition 2.4. *In stereographic coordinates*

$$|y-\zeta|^2 = \frac{4(|\overline{x}|^2+4\mu^2)}{(|\overline{x}|^2+4)(1+\mu^2)},$$

where $\mu = \frac{1-|y|}{1+|y|}$, *and* $\zeta \in S^n$.

Proof.

$$\begin{aligned}|y-\zeta|^2 = |y|^2 - 2y.\zeta + |\zeta|^2 &= 1 - 2|y|.\left(\frac{4-|\overline{x}|^2}{|\overline{x}|^2+4}\right) + |y|^2 \\ &= \frac{(1+|y|^2)(|\overline{x}|^2+4) - 2|y|(4-|\overline{x}|^2)}{|\overline{x}|^2+4} \\ &= \frac{(1+|y|)^2|\overline{x}|^2 + 4(1-|y|)^2}{|\overline{x}|^2+4},\end{aligned}$$

and therefore,

$$|y-\zeta|^2 = \frac{4(|\overline{x}|^2+4\mu^2)}{(|\overline{x}|^2+4)(1+\mu^2)},$$

where in the last equality, we have used $|y| = \frac{1-\mu}{1+\mu}$.

2 Main Result

The main purpose of this work is to give the following estimate for the gradient of the prescribed scalar curvature function K near a critical point of the function $\overline{J}_p$.

Theorem 3.1. *Let* y *be a critical point of the function* $\overline{J}_p$ *near* S^n, *then,*

$$\text{if } n=3, \quad \left|\nabla K\left(\frac{y}{|y|}\right)\right| \leq C\mu^{1-w}$$

and

$$\text{if } n\geq 4, \quad \left|\nabla K\left(\frac{y}{|y|}\right)\right| \leq C\mu^{2-w},$$

where w *is any small positive number less than one.*

Proof. Let us take rectangular coordinates in $\mathbb{R}^{n+1}$ such that $y = (0,0,\ldots,|y|)$, then $|y|^{-1}y = (0,0,\ldots,|y|^{-1}|y|) = (0,0,\ldots,1) = N$. Since

$$\overline{J}_p(y) = \int_{S^n} K(\zeta)\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)} d\sigma(\zeta),$$

then,

$$\frac{\partial \overline{J}_p}{\partial y_j} = (2-n)(p+1)\int_{S^n} K(\zeta)\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)}\left[\frac{y_j}{1-|y|^2} + \frac{(y_j-\zeta_j)}{|y-\zeta|^2}\right] d\sigma(\zeta).$$

Evaluating at the critical point y we find that for $j = 1, \dots, n$

$$0 = \frac{\partial \overline{J}_p}{\partial y_j}(y) = (2-n)(p+1) \int_{S^n} K(\zeta) \left(\frac{1-|y|^2}{|y-\zeta|^2} \right)^{\frac{n-2}{2}(p+1)} \left(\frac{-\zeta_j}{|y-\zeta|^2} \right) d\sigma(\zeta)$$

and therefore,

$$\int_{S^n} K(\zeta) \left(\frac{1-|y|^2}{|y-\zeta|^2} \right)^{\frac{n-2}{2}(p+1)} \left(\frac{\zeta_j}{|y-\zeta|^2} \right) d\sigma(\zeta) = 0.$$

The last integral also vanishes when K is a constant. Thus,

$$\int_{S^n} K(N) \left(\frac{1-|y|^2}{|y-\zeta|^2} \right)^{\frac{n-2}{2}(p+1)} \left(\frac{\zeta_j}{|y-\zeta|^2} \right) d\sigma(\zeta) = 0.$$

Hence,

$$\int_{S^n} (K(\zeta) - K(N)) \left(\frac{1-|y|^2}{|y-\zeta|^2} \right)^{\frac{n-2}{2}(p+1)} \left(\frac{\zeta_j}{|y-\zeta|^2} \right) d\sigma(\zeta) = 0.$$

This equality, in the stereographic coordinates is equivalent to:

$$\int_{\mathbb{R}^n} \frac{(K(x) - K(0))x_j}{4^{n+1-\frac{n-2}{2}\delta}} \left(\frac{(1+\mu)^2(|x|^2+4)}{|x|^2+4\mu^2} \right)^{n+1-\frac{n-2}{2}\delta} \frac{2^n}{(4+|x|^2)^n} dx = 0,$$

where $\delta = \frac{n+2}{n-2} - p$ is a small positive number.

The transformation $y = \mu x$ yields

$$\int_{\mathbb{R}^n} \frac{(K(\mu x) - K(0))x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta - 1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} = 0. \tag{4}$$

By Taylor's Theorem, there exists $\delta_0 > 0$ small enough such that for $|x| \leq \mu^{-1}\delta_0$ we have:

$$K(\mu x) - K(0) = \sum_i \frac{\partial K}{\partial x_i}(0)\mu x_i + \frac{1}{2}\sum_{i,k} \frac{\partial^2 K}{\partial x_i \partial x_k}(0)\mu^2 x_i x_k + O(\mu^3|x|^3).$$

It is easy to check that

$$\left| \int_{B_{\mu^{-1}\delta_0}(0)} \frac{x_j^2 dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta - 1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} \right| \geq C,$$

and therefore

$$\left|\int_{B_{\mu^{-1}\delta_0}(0)} \mu \frac{\sum_i \frac{\partial K}{\partial x_i}(0) x_i x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}}\right| \geq C\mu\left|\frac{\partial K}{\partial x_j}(0)\right|.$$

On the other hand,

$$\frac{1}{2}\mu^2\int_{B_{\mu^{-1}\delta_0}(0)} \frac{\sum_{i,k} \frac{\partial^2 K}{\partial x_i \partial x_k}(0) x_i x_k x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} = 0$$

because of the symmetries of the ball, and a straightforward calculation yields:

$$O\left(\mu^3\int_{B_{\mu^{-1}\delta_0}(0)} \frac{|x|^3 x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}}\right) \leq C\mu^{3-w},$$

where w is a small positive number, and

$$\left|\int_{\mathbb{R}^n\setminus B_{\mu^{-1}\delta_0}(0)} \frac{(K(\mu x)-K(0))x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}}\right|$$

$$\leq C\int_{\mathbb{R}^n\setminus B_{\mu^{-1}\delta_0}(0)} \frac{|x|(4+\mu^2|x|^2)dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} \leq C\mu^{n-1-(n-2)\delta}.$$

The last inequalities and equality (4) imply:

$$C\mu\left|\frac{\partial K}{\partial x_j}(0)\right| \leq C\mu^{3-w} + C\mu^{n-1-(n-2)\delta}.$$

Then if $n=3$

$$\left|\frac{\partial K}{\partial x_j}(0)\right| \leq C\mu^{2-w} + C\mu^{1-\delta} \leq C\mu^{1-w},$$

and if $n \geq 4$,

$$\left|\frac{\partial K}{\partial x_j}(0)\right| \leq C\mu^{2-w}.$$

In the following propositions, we will use this estimate to find some estimates on the function $K(F_y((\xi)) - K\left(\frac{y}{|y|}\right)$ and the first y- derivatives of the function $K \circ F_y$.

Lemma 3.2. *Let y be a critical point of $\overline{J}_p$ near S^n and let $f = K - K\left(\frac{y}{|y|}\right)$. If $1 \leq q < n$, then, $\|f \circ F_y\|_{0,q} \leq C\mu^{2-w}$ for some $0 < w < 1$.*

Proof. Taylor's Theorem yields:

$$\begin{aligned}|f \circ F_y(\xi)| &= \left|K(F_y(\xi)) - K\left(\frac{y}{|y|}\right)\right| \\ &\leq C_1 \left|\nabla K\left(\frac{y}{|y|}\right)\right| \left|F_y(\xi) - \frac{y}{|y|}\right| + C_2 \left|F_y(\xi) - \frac{y}{|y|}\right|^2,\end{aligned}$$

in a geodesic ball of radius r and center $\frac{y}{|y|}$. Here C_1 and C_2 denote positive constants.

Then,

$$|f \circ F_y(\xi)|^q \leq C_1 \left|\nabla K\left(\frac{y}{|y|}\right)\right|^q \left|F_y(\xi) - \frac{y}{|y|}\right|^q + C_2 \left|F_y(\xi) - \frac{y}{|y|}\right|^{2q}.$$

Since

$$\|f \circ F_y\|_{0,q}^q = \int_{S^n} |f \circ F_y(\xi)|^q d\sigma_g = \int_V |f \circ F_y(\xi)|^q d\sigma_g + \int_{S^n \setminus V} |f \circ F_y(\xi)|^q d\sigma_g,$$

then it follows that

$$\begin{aligned}\left(\int_{S^n} |f \circ F_y(\xi)|^q d\sigma_g\right)^{1/q} &= \left(\int_V |f \circ F_y(\xi)|^q d\sigma_g + \int_{S^n \setminus V} |f \circ F_y(\xi)|^q d\sigma_g\right)^{1/q} \\ &\leq 2^q \left(\int_V |f \circ F_y(\xi)|^q d\sigma_g\right)^{1/q} \\ &+ 2^q \left(\int_{S^n \setminus V} |f \circ F_y(\xi)|^q d\sigma_g\right)^{1/q}.\end{aligned}$$

On the one hand,

$$\begin{aligned}\left(\int_V |f \circ F_y(\xi)|^q d\sigma_g\right)^{1/q} &\leq C_1 \mu^{1-w} \left(\int_V \left|F_y(\xi) - \frac{y}{|y|}\right|^q d\sigma_g\right)^{1/q} \\ &+ C_2 \left(\int_V \left|F_y(\xi) - \frac{y}{|y|}\right|^{2q} d\sigma_g\right)^{1/q}.\end{aligned}$$

By Holder's inequality

$$\left(\int_V \left|F_y(\xi) - \frac{y}{|y|}\right|^q d\sigma_g\right)^{1/q} \leq C \left(\int_V \left|F_y(\xi) - \frac{y}{|y|}\right|^{2q}\right)^{1/2q}.$$

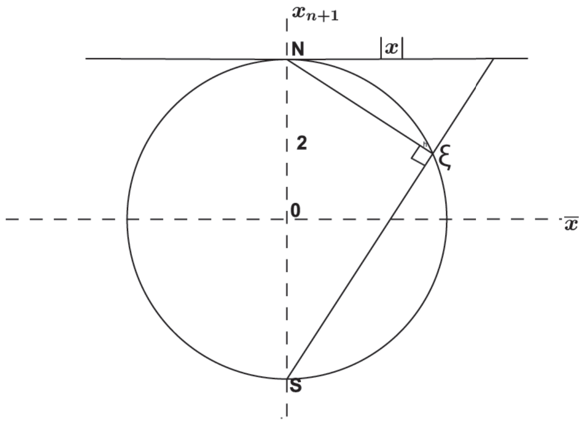
Then,

$$\left(\int_V |f\circ F_y(\xi)|^q\right)^{1/q} \leq M_1\mu^{1-w}\left[\left(\int_V \left|F_y(\xi)-\frac{y}{|y|}\right|^{2q} d\sigma_g\right)^{1/q}\right]^{1/2} + M_2\left(\int_V \left|F_y(\xi)-\frac{y}{|y|}\right|^{2q} d\sigma_g\right)^{1/q}$$

Assume that $\frac{y}{|y|}$ is the north pole N. Since $F_y(N) = N$, in stereographic coordinates $|F_y(\xi) - N| = \dfrac{4^q|\mu x|^{2q}}{(4+|\mu x|^2)^q}$ (see figure below). Therefore, the second integral on the right hand is equivalent in stereographic coordinates to:

$$\int_{B_{R_1}} \frac{4^q|\mu x|^{2q}}{(4+|\mu x|^2)^q}\frac{2^n dx}{(4+|x|^2)^n},$$

where B_{R_1} is the image of the geodesic ball V under the stereographic projection.



Hence,

$$\int_V |F_y(\xi)-N|^{2q}\, d\sigma_g \leq C\int_{B_{R_1}} \mu^{2q}\left(\frac{|x|^2}{4+|x|^2}\right)^q dx \leq C\mu^{2q}.$$

Consequently,

$$\left(\int_V |f\circ F_y(\xi)|^q\right)^{1/q} \leq C\mu^{2-w}.$$

On the other hand

$$\int_{S^n\setminus V} |f\circ F_y(\xi)|^q d\sigma_g = \int_{F_y(S^n\setminus V)} |f(\zeta)|^q|(F_y^{-1})'|^n d\sigma_g(\zeta) \leq C\int_{\mathbb{R}^n\setminus B_{R_1}} \frac{\lambda^n}{(1+\lambda^2|x|^2)^n}dx,$$

where $\lambda = \frac{1+|y|}{1-|y|} = \mu^{-1}$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{R_1}} \frac{\lambda^n}{(1+\lambda^2|x|^2)^n}dx &= \lambda^n \int_{R_1}^{\infty} \frac{r^{n-1}dr}{(1+\lambda^2 r^2)^n} \\ &\leq C\lambda^n \int_{R_1}^{\infty} \frac{r^{n-1}dr}{\lambda^{2n}r^{2n}} \\ &\leq C\lambda^{-n}\int_{R_1}^{\infty} r^{-n-1}dr.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n\setminus B_{R_1}} \frac{\lambda^n}{(1+\lambda^2|x|^2)^n}dx \leq C\lambda^{-n} = C\mu^n,$$

and

$$\left(\int_{S^n\setminus V} |f\circ F_y(\xi)|^q d\sigma_g\right)^{1/q} \leq C\mu^{n/q} = C\mu^{2-\gamma}.$$

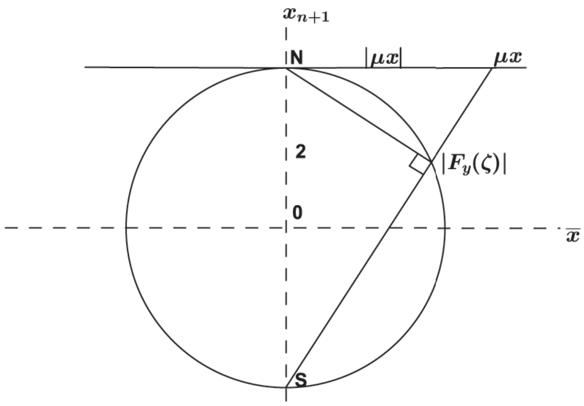
It follows that:

$$\|f\circ F_y(\xi)\|_{0,q} \leq C\mu^{2-w'}.$$

Proposition 3.3. *Let y_0 be a critical point of $\overline{J}_p$ near S^n, and let a small ball $B_{y_0} \subseteq B^{n+1}$, if $y \in B_{y_0}$ and $q \in \left(\frac{n}{2}, n\right)$ then, $\|\nabla_y(K\circ F_y)\|_{0,q} \leq C\mu^{1-w^*}$, where $\mu = \frac{1-|y_0|}{1+|y_0|}$ and w^* is a positive number less than one.*

Proof. Taking $r = 0$ in the last lemma, we get

$$\begin{aligned}I = \int_{S^n} |\nabla_y(K\circ F_y)|^q d\sigma &\leq \int_{S^n} |\nabla K(F_y)|^q |\nabla_y F_y|^q d\sigma \\ \leq C\int_{S^n} \frac{|\nabla K(F_y)|^q}{|z-s|^q}d\sigma(z),\end{aligned}$$



Using in the above figure similarity of triangles, we find that $|z-s| = \dfrac{4}{\sqrt{4+|x|^2}}$. Consequently, the integral in the right is equivalent in stereographic coordinates to

$$\int_{R^n} \frac{(4+|x|^2)^{q/2}|\nabla K(\mu x)|^q}{4^q} \frac{2^n}{(4+|x|^2)^n} dx.$$

By Taylor's Theorem there exists $D>0$ such that if $|x| \leq R = D\mu^{-1}$, then, we have

$$|\nabla K(\mu x)|^q \leq C|\nabla K(0)|^q + C|\mu x|^q.$$

Therefore,

$$I \leq C\left(\int_{B_R(0)} \frac{|\nabla K(0)|^q + |\mu x|^q}{(4+|x|^2)^{n-q/2}} dx + \int_{\mathbb{R}^{n-1}\setminus B_R(0)} \frac{dx}{(4+|x|^2)^{n-q/2}}\right).$$

On the one hand, taking α such that $n-\alpha$ is a very small positive number and using $q<n$ we get:

$$\begin{aligned}
\int_{B_R(0)} \frac{|\nabla K(0)|^q}{(4+|x|^2)^{n-q/2}} dx &\leq C\mu^{q(1-w)} \int_{B_R(0)} \frac{dx}{(4+|x|^2)^{n-q/2}} \\
&\leq C\mu^{q(1-w)} \int_0^{D\mu^{-1}} r^{n-1-\alpha}\left(\frac{(r^2)^{\alpha/2}}{(4+r^2)^{n-q/2}}\right) dr \\
&\leq C\mu^{q-w_0}
\end{aligned}$$

where $w_0 = -wq+\alpha-n$. It's easy check

$$\int_{B_R(0)} \frac{2^n(4+|x|^2)^{q/2}|\mu x|^q}{4^q(4+|x|^2)^n} dx \leq C\mu^q \int_0^{D\mu^{-1}} r^{-n-1+2q} dr \leq \frac{C}{2q-n}\mu^{n-q}.$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}\setminus B_R(0)} \frac{dx}{(4+|x|^2)^{n-q/2}} &\leq \lim_{b\to\infty} \int_{C\mu^{-1}}^b \frac{dx}{(4+|x|^2)^{n-q/2}} \\
&\leq \lim_{b\to\infty} \int_{C\mu^{-1}}^b \frac{r^{n-1}dr}{r^{2(n-q/2)}} = \lim_{b\to\infty} \int_{C\mu^{-1}}^b r^{-1-n+q} dr \\
&\leq \frac{C}{n-q}\mu^{n-q}.
\end{aligned}$$

Letting $w^* = \min\{\frac{w_0}{q}, 1-\frac{n-q}{q}\}$, the estimate follows from the above inequalities.

Proposition 3.4. *If $q < n$ and $1 - \frac{n}{2q} < r < \frac{1}{2}$, then the following estimate holds:*

$$\|\nabla_y \nabla_y (K \circ F_y)\|_{0,q} \leq C\mu^{-2r}.$$

Proof. By the previous estimates of $\|\nabla_y (K \circ F_y)$ and $\|\nabla_y \nabla_y (K \circ F_y)\|$, we get

$$\begin{aligned} |\nabla_y \nabla_y (K \circ F_y)|^q &\leq C(|\nabla_y F_y|^2 + |\nabla_y \nabla_y F_y|)^q \\ &\leq \left[\left(\frac{C_1}{\mu^r |z-s|^{1-r}} \right)^2 + \left(\frac{C_2}{\mu^r |z-s|^{1-r}} \right) \right]^q. \end{aligned}$$

Using Holder's inequality, to get the desired estimate is enough to estimate the integral

$$\int_{S^n} \frac{d\sigma(z)}{\mu^{2rq} |z-s|^{(2-2r)q}} = \frac{C}{\mu^{2rq}} \int_{R^n} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx.$$

But

$$\int_{R^n} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx = \int_{B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx + \int_{R^n \setminus B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx.$$

Let us estimate the first integral in the right side

$$\int_{B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx = C \int_0^1 \frac{t^{n-1}}{(4+t^2)^{n-(1-r)q}} dt = C \int_0^1 \frac{t^{n-1}}{(4+t^2)^{n+(r-1)q}} dt.$$

Since $q < n$ and $1 - \frac{n}{2q} < r$ taking $\nu > 0$ such that $0 < n - \nu < 1$, then we have:

$$\int_0^1 \frac{t^{n-1-\nu}(t^2)^{\nu/2}}{(4+t^2)^{n+(r-1)q}} dt \leq C \int_0^1 t^{n-\nu-1} dt = \frac{C}{n-\nu},$$

where we have used that $\dfrac{\nu}{2} < \dfrac{n}{2} < n - (1-r)q$.
On the other hand,

$$\int_{R^n \setminus B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx = C \int_1^\infty \frac{t^{n-1}}{(4+t^2)^{n+(r-1)q}} dt = \frac{C}{2(r-1)q+n}$$

Consequently,

$$\left(\int_{S^n} |\nabla_y \nabla_y (K \circ F_y)|^q d\sigma(z) \right)^{1/q} \leq C\mu^{-2r}.$$

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