

# On certain closed subgroups of $\mathrm{SL}(2, \mathbb{Z}_p[[X]])$

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**ABSTRACT.** Let  $p > 2$  be a prime number and let  $\Lambda = \mathbb{Z}_p[[X]]$  be the ring of power series with  $p$ -adic integer coefficients. The special linear group of matrices  $\mathrm{SL}(2, \Lambda)$  is equipped with several natural projections. In particular, let  $\pi_X: \mathrm{SL}(2, \Lambda) \rightarrow \mathrm{SL}(2, \mathbb{Z}_p)$  be the natural projection which sends  $X \mapsto 0$ . Suppose that  $G$  is a subgroup of  $\mathrm{SL}(2, \Lambda)$  such that the projection  $H = \pi_X(G)$  is known. In this note, different criteria are found which guarantee that the subgroup  $G$  of  $\mathrm{SL}(2, \Lambda)$  is “as large as possible”, i.e.  $G$  is the full inverse image of  $H$ . Criteria of this sort have interesting applications in the theory of Galois representations.

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**RESUMEN.** Sea  $p > 2$  un primo y  $\Lambda = \mathbb{Z}_p[[X]]$  el anillo de series de potencias con coeficientes enteros  $p$ -adicos. El grupo lineal de matrices especial  $\mathrm{SL}(2, \Lambda)$  es equipado con varias proyecciones naturales. En particular,  $\pi_X: \mathrm{SL}(2, \Lambda) \rightarrow \mathrm{SL}(2, \mathbb{Z}_p)$  es la proyección natural que envía  $X \mapsto 0$ . Suponga que  $G$  es un subgrupo de  $\mathrm{SL}(2, \Lambda)$  tal que la proyección  $H = \pi_X(G)$  es conocida. En este artículo se establecen diferentes criterios que garantizan que el subgrupo  $G$  de  $\mathrm{SL}(2, \Lambda)$  es “tan grande como es posible”; esto es,  $G$  es la imagen inversa total de  $H$ . Criterios de esta naturaleza tienen importantes aplicaciones a la teoría de representaciones de Galois.

## 1. Introduction

Let  $p > 2$  be a prime number and let  $\Lambda = \mathbb{Z}_p[[X]]$  be the ring of power series with  $p$ -adic integer coefficients. The special linear group of matrices  $\mathrm{SL}(2, \Lambda)$  is equipped with several natural projections. In particular, there is a natural

projection onto  $\mathrm{SL}(2, \mathbb{Z}_p)$ :

$$\pi_X : \mathrm{SL}(2, \mathbb{Z}_p[[X]]) \longrightarrow \mathrm{SL}(2, \mathbb{Z}_p),$$

induced by the natural ring homomorphism  $\Lambda \rightarrow \mathbb{Z}_p$  which sends  $X$  to 0 and  $\mathbb{Z}_p$  is fixed. Suppose that  $G$  is a subgroup of  $\mathrm{SL}(2, \Lambda)$  such that the projection  $H = \pi_X(G)$  is known. In this note we are interested in finding different criteria which guarantee that the subgroup  $G$  of  $\mathrm{SL}(2, \mathbb{Z}_p[[X]])$  is “as large as possible”, i.e.  $G$  is the full inverse image of  $H$ , or equivalently,  $\pi_X(G) = H$  and  $G$  contains the kernel of  $\pi_X$ . See Theorem 2.3, Corollary 4.2 and Proposition 5.1 for the precise statements.

Criteria of this sort have interesting applications in the theory of Galois representations (and this was the motivation for this work, [4]). Representations of the  $p$ -adic type:

$$\begin{aligned} \rho_0 & : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{SL}(2, \mathbb{Z}_p) \\ \rho_1 & : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{SL}(2, \Lambda), \end{aligned}$$

appear in several natural ways. For example, the representations associated to: the Tate module of an elliptic curve (or abelian varieties in general); modular forms; certain cohomology groups of algebraic varieties, are all of type  $\rho_0$  (see [9], I-3, for more details on examples). The image of these representations is understood in general (see [10] for the elliptic curve case). More recently, representations of type  $\rho_1$  have also been found ([2], [3], [6]). Notice that by composing  $\rho_1$  with  $\pi_X$  one obtains a representation of type  $\rho_0$ . Therefore, any previous knowledge of  $\rho_0$  and appropriate criteria about the subgroups of  $\mathrm{SL}(2, \Lambda)$ , may yield information about the image of  $\rho_1$  (see [5], [7], [4]).

## 2. Statement of results

In [9], IV-23, J.-P. Serre proves the following result:

**Lemma 2.1.** *Let  $p \geq 5$  be a prime and let  $X$  be a closed subgroup of  $\mathrm{SL}(2, \mathbb{Z}_p)$  whose image in  $\mathrm{SL}(2, \mathbb{F}_p)$  is the full group  $\mathrm{SL}(2, \mathbb{F}_p)$ . Then  $X = \mathrm{SL}(2, \mathbb{Z}_p)$ .*

Nigel Boston ([1], Prop. 2) generalized Serre’s result to the following statement. From now on,  $\mathcal{M} = (p, X)$  denotes the maximal ideal of  $\Lambda = \mathbb{Z}_p[[X]]$ .

**Proposition 2.2.** *Let  $H$  be a closed subgroup of  $\mathrm{SL}(2, \Lambda)$  whose projection into  $\mathrm{SL}(2, \Lambda/\mathcal{M}^2)$  is the full group. Then  $H = \mathrm{SL}(2, \Lambda)$ .*

In this note, we intend to prove a generalization of Boston’s result. Let  $p \neq 2$  be a prime number. We define maps:

$$\begin{aligned} \pi_X & : \mathrm{SL}(2, \Lambda) \longrightarrow \mathrm{SL}(2, \mathbb{Z}_p), \quad X \mapsto 0 \\ \pi_X^i & : \mathrm{SL}(2, \Lambda/\mathcal{M}^i) \longrightarrow \mathrm{SL}(2, \mathbb{Z}/p^i\mathbb{Z}), \quad X \mapsto 0 \pmod{p^i}, \end{aligned}$$

and projections:

$$\begin{aligned}\tau_X^i &: \mathrm{SL}(2, \Lambda) \longrightarrow \mathrm{SL}(2, \Lambda/\mathcal{M}^i) \\ \tau^i &: \mathrm{SL}(2, \mathbb{Z}_p) \longrightarrow \mathrm{SL}(2, \mathbb{Z}/p^i\mathbb{Z}).\end{aligned}$$

For closed subgroups  $H \leq \mathrm{SL}(2, \Lambda)$ ,  $G \leq \mathrm{SL}(2, \mathbb{Z}_p)$  we write:

$$H_i = \tau_X^i(H) \leq \mathrm{SL}(2, \Lambda/\mathcal{M}^i), \quad G_i = \tau^i(G) \leq \mathrm{SL}(2, \mathbb{Z}/p^i\mathbb{Z}).$$

The main theorem is the following:

**Theorem 2.3.** *Let  $\mathfrak{G}$  be a closed subgroup of  $\mathrm{SL}(2, \mathbb{Z}_p)$ , and let  $\mathfrak{H}$  be a closed subgroup of  $\mathrm{SL}(2, \Lambda)$  satisfying:*

- (1)  $\pi_X(\mathfrak{H}) = \mathfrak{G}$ .
- (2) *The subgroup  $\mathfrak{H}_2$  is the full inverse image of  $\mathfrak{G}_2$  by the map  $\pi_X^2$ , i.e.*

$$\mathfrak{H}_2 = (\pi_X^2)^{-1}(\mathfrak{G}_2).$$

*Then  $\mathfrak{H}$  is the full inverse image of  $\mathfrak{G}$  by the map  $\pi_X$ , this is,  $\mathfrak{H} = (\pi_X)^{-1}(\mathfrak{G})$ .*

We present two different proofs of Theorem 2.3, which shed light on different interesting aspects. In section 3 we follow Boston's concise proof of Proposition 2.2, which makes use of Burnside's basis theorem. In section 4 we offer an alternative (longer) proof which explicitly describes the kernel of  $\pi_X$ . Moreover, several corollaries can be deduced from the description of the kernel (see Corollary 4.2). The last section is devoted to an improvement of Proposition 2.2 (see Prop. 5.1).

We end this section with the following remark. In order to prove a theorem like 2.3 one may just consider  $\mathrm{PSL}$  instead of  $\mathrm{SL}$ . We make this precise in the form of a lemma.

**Lemma 2.4.** *Let  $P_{\mathbb{Z}_p}: \mathrm{SL}(2, \mathbb{Z}_p) \rightarrow \mathrm{PSL}(2, \mathbb{Z}_p)$  be the natural projection and define similarly  $P_\Lambda: \mathrm{SL}(2, \Lambda) \rightarrow \mathrm{PSL}(2, \Lambda)$ . Let  $C$  be a closed subgroup of  $\mathrm{PSL}(2, \mathbb{Z}_p)$ , and let  $C'$  be the full inverse image of  $C$  in  $\mathrm{SL}(2, \mathbb{Z}_p)$ . Let  $\mathfrak{X}$  be the full inverse image of  $C$  in  $\mathrm{PSL}(2, \Lambda)$ , and let  $Y$  be a closed subgroup of  $\mathrm{SL}(2, \Lambda)$  such that  $P_\Lambda(Y) = \mathfrak{X}$  and  $\pi_X(Y) = C'$ . Then  $Y$  is the full inverse image of  $C'$  in  $\mathrm{SL}(2, \Lambda)$ .*

$$\begin{array}{ccc}\mathrm{SL}(2, \Lambda) \supseteq Y & \xrightarrow{\pi_X} & C' \leq \mathrm{SL}(2, \mathbb{Z}_p) \\ P_\Lambda \downarrow & & \downarrow P_{\mathbb{Z}_p} \\ \mathrm{PSL}(2, \Lambda) \supseteq \mathfrak{X} & \xrightarrow{P\pi_X} & C \leq \mathrm{PSL}(2, \mathbb{Z}_p)\end{array}$$

*Proof.* It suffices to show that  $-I$  belongs to  $Y$ . By hypothesis,  $Y$  contains an element of the form  $g = -I + X \cdot A$  with some  $2 \times 2$  matrix  $A$  over  $\Lambda$ . Since  $Y$  is closed,  $Y$  also contains  $\lim_{n \rightarrow \infty} g^{p^n} = -I$  which finishes the proof of the lemma.  $\square$

### 3. Proof using Frattini quotients

In order to prove Theorem 2.3, we follow an argument due to N. Boston ([1], p. 262, Proposition 2) which makes use of the following well known theorem.

**Theorem 3.1** (Burnside's basis theorem). *Let  $K$  be a pro- $p$  group and let  $\overline{K}$  be its Frattini quotient, i.e.  $\overline{K} = K/K^p K'$  where  $K^p$  is the subgroup of  $p$ th powers and  $K'$  is the subgroup of commutators  $(g, h) = ghg^{-1}h^{-1}$ , for all  $g, h \in K$ . If  $J$  is a closed subgroup of  $K$  and if the image of  $J$  in  $\overline{K}$  is surjective, then  $J = K$ .*

A proof of the theorem for  $p$ -groups can be found in [8], p. 274. Use an inverse limit argument to obtain the one stated here. In our case we let  $K$  be the kernel of  $\pi_X$  (which is a pro- $p$  group) and let  $J$  be the intersection of  $K$  with the subgroup  $\mathfrak{H} \leq \mathrm{SL}(2, \Lambda)$ . Before we can apply Burnside's theorem, we study the Frattini quotient of  $K$ . For every  $n \geq 2$  we define groups  $K_n$  and  $\tilde{K}$  via the following exact sequences of groups:

$$1 \longrightarrow K_n \longrightarrow \mathrm{SL}(2, \Lambda/(X^n)) \longrightarrow \mathrm{SL}(2, \mathbb{Z}_p) \longrightarrow 1$$

$$1 \longrightarrow \tilde{K} \longrightarrow \mathrm{SL}(2, \Lambda/\mathcal{M}^2) \longrightarrow \mathrm{SL}(2, \mathbb{Z}_p/(p^2)) \longrightarrow 1.$$

**Lemma 3.2.** *The kernel of the canonical surjection  $\pi_n: K_{n+1} \twoheadrightarrow K_n$  lies in  $K'_{n+1}$ , the commutator subgroup of  $K_{n+1}$ . Thus, the induced homomorphism between the Frattini quotients  $\overline{K_{n+1}}$  and  $\overline{K_n}$  is an isomorphism.*

*Proof.* One easily computes the following congruence for a commutator:

$$(1 + XA + X^n B, 1 + X^{n-1}C + X^n D) \equiv 1 + X^n(AC - CA) \pmod{X^{n+1}},$$

for arbitrary  $A, B, C, D \in M_2^0(\mathbb{Z}_p)$ , where  $M_2^0$  denotes the set of all  $2 \times 2$  trace zero matrices. Moreover, any element in  $M_2^0(\mathbb{Z}_p)$  can be written as a finite sum of commutators  $AC - CA$  using elementary matrices. Since the kernel of  $\pi_n$  is isomorphic to  $(1 + X^n M_2^0(\mathbb{Z}_p))$ , the previous argument shows that the kernel of  $\pi_n$  lies in  $K'_{n+1}$ . The isomorphism between the Frattini quotients follows immediately.  $\checkmark$

**Corollary 3.3.** *The Frattini quotient of  $K$ , the kernel of  $\pi_X$ , is isomorphic to  $\tilde{K}$ .*

*Proof.* Notice that  $K_2 \cong (1 + X M_2^0(\mathbb{Z}_p)) \cong \mathbb{Z}_p^3$ , therefore its Frattini quotient,  $\overline{K_2}$ , is isomorphic to  $\mathbb{F}_p^3$ . On the other hand,  $\tilde{K} \cong (1 + X M_2^0(\mathbb{F}_p)) \cong \mathbb{F}_p^3$ . Hence, by Lemma 3.2,  $\overline{K_n} \cong \tilde{K}$  for all  $n \geq 2$ . The corollary follows from the fact that  $K$  is the inverse limit of the  $K_n$ .  $\checkmark$

Finally, we are ready to prove the theorem.

*Proof of Theorem 2.3.* By Burnside basis theorem and Corollary 3.3, it suffices to show that if  $\mathfrak{H}$  satisfies hypotheses (1) and (2) then the subgroup  $\mathfrak{H}_2$  of  $\mathrm{SL}(2, \Lambda/\mathcal{M}^2)$  contains  $\tilde{J} = \overline{J_2} \cong \overline{K}$ . By hypothesis (2),  $\mathfrak{H}_2$  is the inverse image of  $\mathfrak{G}_2$  by  $\pi_X^2$ . Thus,  $\mathfrak{H}_2$  contains the kernel of  $\pi_X^2$ , which is  $\tilde{J}$ , by definition.  $\checkmark$

#### 4. Explicit proof

In this section we offer an alternative proof of Theorem 2.3 by analyzing,  $K$ , the kernel of  $\pi_X$ . Note that  $K = \{\gamma \in \mathrm{SL}(2, \Lambda) : \gamma \equiv \mathrm{Id} \pmod{X}\}$ . The following lemma is an easy exercise in linear algebra, which proves that  $K$  is topologically generated by three elements.

**Lemma 4.1.** *Let  $u, v, w \in \mathbb{Z}_p[[X]]^\times$  be fixed. Let  $\tilde{K}$  be the closed subgroup generated by the three matrices:*

$$T_1 = \begin{bmatrix} 1 & uX \\ 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 \\ vX & 1 \end{bmatrix}, T_3 = \begin{bmatrix} 1+wX & 0 \\ 0 & (1+wX)^{-1} \end{bmatrix}.$$

Then  $\tilde{K} = K$ .

*Proof.* In order to prove the lemma, one checks that the projection of the matrices  $T_i, i = 1, 2, 3$  in  $\mathrm{SL}(2, \Lambda/\mathcal{M}^2)$  generate the whole group. Then, an induction argument finishes the proof.  $\checkmark$

Theorem 2.3 is an immediate consequence of the previous lemma.

*Second proof of Theorem 2.3.* Let  $\mathfrak{G}$  be a closed subgroup of  $\mathrm{SL}(2, \mathbb{Z}_p)$ , and let  $\mathfrak{H}$  be a closed subgroup of  $\mathrm{SL}(2, \Lambda)$  satisfying:

- (1)  $\pi_X(\mathfrak{H}) = \mathfrak{G}$ .
- (2) The subgroup  $\mathfrak{H}_2$  is the full inverse image of  $\mathfrak{G}_2$  by the map  $\pi_X^2$ , i.e.  $\mathfrak{H}_2 = (\pi_X^2)^{-1}(\mathfrak{G}_2)$ .

In order to prove that  $\mathfrak{H}$  is the full inverse image of  $\mathfrak{G}$  by the map  $\pi_X$ , it suffices to show that  $K \leq \mathfrak{H}$ , where  $K = \mathrm{Ker}(\mathrm{SL}(2, \Lambda) \rightarrow \mathrm{SL}(2, \mathbb{Z}_p))$ . Let us define  $\tilde{K} = K \cap \mathfrak{H}$ . By hypothesis,  $\mathfrak{H}_2 = (\pi_X^2)^{-1}(\mathfrak{G}_2)$  which in particular implies that  $\pi_X^2(\tilde{K}) = \pi_X^2(K)$ . Hence, there exist matrices  $\tilde{T}_i \in \tilde{K}, i = 1, 2, 3$  such that:

$$\tilde{T}_1 \equiv \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}, \tilde{T}_2 \equiv \begin{bmatrix} 1 & 0 \\ X & 1 \end{bmatrix}, \tilde{T}_3 \equiv \begin{bmatrix} 1+X & 0 \\ 0 & 1-X \end{bmatrix} \pmod{(p, X)^2}.$$

Therefore, there exist  $u, v, w \in \mathbb{Z}_p[[X]]$ , with  $u, v, w \equiv 1 \pmod{(p, X)}$  (in particular  $u, v, w \in \mathbb{Z}_p[[X]]^\times$ ) such that:

$$\tilde{T}_1 = \begin{bmatrix} 1 & uX \\ 0 & 1 \end{bmatrix}, \tilde{T}_2 = \begin{bmatrix} 1 & 0 \\ vX & 1 \end{bmatrix}, \tilde{T}_3 = \begin{bmatrix} 1+wX & 0 \\ 0 & (1+wX)^{-1} \end{bmatrix}.$$

The hypothesis of Lemma 4.1 are satisfied, hence  $K = \tilde{K} \leq \mathfrak{H}$ , which concludes the proof of the theorem.  $\checkmark$

The previous proof shows that we can prove the equivalent result:

**Corollary 4.2.** *Let  $\mathfrak{G}$  be a closed subgroup of  $\mathrm{SL}(2, \mathbb{Z}_p)$ , and let  $\mathfrak{H}$  be a closed subgroup of  $\mathrm{SL}(2, \Lambda)$  satisfying:*

- (1)  $\pi_X(\mathfrak{H}) = \mathfrak{G}$ .
- (2) *There exist matrices  $T_i \in \mathfrak{H}$ ,  $i = 1, 2, 3$  such that:*

$$T_1 \equiv \begin{bmatrix} 1 & uX \\ 0 & 1 \end{bmatrix}, T_2 \equiv \begin{bmatrix} 1 & 0 \\ vX & 1 \end{bmatrix}, T_3 \equiv \begin{bmatrix} 1+wX & 0 \\ 0 & 1-wX \end{bmatrix},$$

*modulo  $(p, X)^2$ , for some  $u, v, w \in (\mathbb{Z}/p\mathbb{Z})^\times$ .*

*Then  $\mathfrak{H}$  is the full inverse image of  $\mathfrak{G}$  by the map  $\pi_X$ , this is,  $\mathfrak{H} = (\pi_X)^{-1}(\mathfrak{G})$ .*

## 5. A different improvement

In this final section, we come back to the case of the full group  $\mathrm{SL}(2, \Lambda)$ .

**Proposition 5.1.** *Let  $p \geq 5$  be a prime and let  $H$  be a closed subgroup of  $\mathrm{SL}(2, \Lambda)$ . For  $i = 1, 2$ , let  $H_i$  be the projection of  $H$  into  $\mathrm{SL}(2, \Lambda/\mathcal{M}^i)$ . If  $H_1 = \mathrm{SL}(2, \mathbb{F}_p)$  and there exist  $k \in H \cap \mathrm{Ker}(\pi_X)$  such that  $k \not\equiv \mathrm{Id} \pmod{\mathcal{M}^2}$  but  $k \equiv \mathrm{Id} \pmod{(p^2, X)}$ , then  $H = \mathrm{SL}(2, \Lambda)$ .*

*Proof.* By Boston's Proposition 2.2, it suffices to show that

$$H_2 = \mathrm{SL}(2, \Lambda/\mathcal{M}^2).$$

For simplicity, let us denote  $G = \mathrm{SL}(2, \Lambda/\mathcal{M}^2)$ ,  $G_2 = \mathrm{SL}(2, \mathbb{Z}/p^2\mathbb{Z})$  and  $G_1 = \mathrm{SL}(2, \mathbb{F}_p)$ . Also, let  $\mathcal{A} = (p^2, X)$  and define  $\Gamma(\mathcal{A})$ ,  $\Gamma(\mathcal{M})$  to be the following kernels:

$$\Gamma(\mathcal{A}) = \mathrm{Ker}(G \rightarrow G_2), \quad \Gamma(\mathcal{M}) = \mathrm{Ker}(G \rightarrow G_1).$$

In particular,  $\Gamma(\mathcal{A}) \subseteq \Gamma(\mathcal{M})$ . We claim that  $\Gamma(\mathcal{M})$  is abelian. Indeed, if  $\mathcal{F}_1, \mathcal{F}_2 \in \Gamma(\mathcal{M})$ , then there exist matrices  $F_1, F_2$  with coefficients in  $\mathcal{M}/\mathcal{M}^2$  such that  $\mathcal{F}_1 = \mathrm{Id} + F_1, \mathcal{F}_2 = \mathrm{Id} + F_2$ . Thus:  $\mathcal{F}_1 \cdot \mathcal{F}_2 = \mathrm{Id} + F_1 + F_2 = \mathcal{F}_2 \cdot \mathcal{F}_1$ . Hence,  $\Gamma(\mathcal{M})$  is an abelian normal subgroup of  $G$ , and so is  $\Gamma(\mathcal{A})$ . Furthermore the fact that  $H_1 = G_1$  implies by Lemma 2.1 that the subgroup  $H$  surjects onto  $G_2$ . Thus,  $G = H_2 \cdot \Gamma(\mathcal{A})$ , and so, in order to prove the proposition, it is enough to show that  $\Gamma(\mathcal{A})$  is included in  $H_2$ .

**Lemma 5.2.**  *$H_2 \cap \Gamma(\mathcal{A})$  is a non-trivial normal subgroup of  $G$ .*

*Proof.* The existence of an element  $k$  as in the statement of the proposition ensures that  $H_2 \cap \Gamma(\mathcal{A})$  is non-trivial. Let  $g \in G$  and  $h \in H_2 \cap \Gamma(\mathcal{A})$ . Since  $h \in \Gamma(\mathcal{A})$ ,  $ghg^{-1} \in \Gamma(\mathcal{A})$  and  $G = H_2 \cdot \Gamma(\mathcal{A})$  implies that  $g = h_0 \cdot \gamma$  for some  $h_0 \in H_2$  and  $\gamma \in \Gamma(\mathcal{A})$ . Thus:

$$ghg^{-1} = h_0 \gamma h \gamma^{-1} h_0^{-1} = h_0 h h_0^{-1} \in H_2,$$

and so  $ghg^{-1} \in H_2 \cap \Gamma(\mathcal{A})$ . □

Since  $\Gamma(\mathcal{A})$  is normal in  $G$  and  $H_2 \leq G$ , there is a well defined representation:

$$\rho: H_2 \longrightarrow \text{Aut}(\Gamma(\mathcal{A})), \quad h \mapsto (\mathcal{F} \rightarrow h\mathcal{F}h^{-1}).$$

Moreover,  $H_2 \cap \Gamma(\mathcal{M})$  is included in the kernel of  $\rho$  (because  $\Gamma(\mathcal{M})$  is abelian). Thus,  $\rho$  factors through  $H_2/H_2 \cap \Gamma(\mathcal{M}) \cong SL(2, \mathbb{F}_p) = G_1$ . The induced representation of  $SL(2, \mathbb{F}_p)$  into  $\text{Aut}(\Gamma(\mathcal{A}))$  is the adjoint representation (because  $\Gamma(\mathcal{A}) \cong M_2^0(\mathbb{F}_p)$ , the set of zero trace matrices), which is irreducible. By Lemma 5.2,  $H_2 \cap \Gamma(\mathcal{A})$  is normal and abelian, thus it is an invariant subspace for  $\rho$  and therefore for the adjoint representation of  $SL(2, \mathbb{F}_p)$ . By the irreducibility of the latter and the fact that  $H_2 \cap \Gamma(\mathcal{A})$  is non-trivial, we conclude that  $H_2 \cap \Gamma(\mathcal{A}) = \Gamma(\mathcal{A})$ , as desired.  $\square$

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