**Revista Colombiana de Matemáticas**
Volumen 39 (2005), páginas 21–35

# On the Hurewicz theorem for wedge sum of spheres

Fermin Dalmagro
Universidad Central de Venezuela, Caracas

Yamilet Quintana*
Universidad Simon Bolivar, Caracas

Abstract. This paper we provides an alternative proof of Hurewicz theorem when the topological space $X$ is a CW-complex. Indeed, we show that if $X_0 \subseteq X_1 \subseteq \cdots X_{n-1} \subseteq X_n = X$ is the CW decomposition of $X$, then the Hurewicz homomorphism $\Pi_{n+1}(X_{n+1}, X_n) \longrightarrow H_{n+1}(X_{n+1}, X_n)$ is an isomorphism, and together with a result from Homological Algebra we prove that if $X$ is $(n-1)$-connected, the Hurewicz homomorphism $\Pi_n(X) \longrightarrow H_n(X)$ is an isomorphism.

*Keywords and phrases.* Hurewicz homomorphism, CW-complexes, exact sequence, homotopic groups, homology groups.

*2000 Mathematics Subject Classification.* Primary: 55N10, 55N99, 55Q40. Secondary: 54F65.

Resumen. En este artículo damos una demostración alternativa de el teorema de Hurewicz cuando el espacio topológico $X$ es CW-complejo. En realidad probamos que si $X_0 \subseteq X_1 \subseteq \cdots X_{n-1} \subseteq X_n = X$ es una descomposición CW de $X$, el homomorfismo de Hurewicz $\Pi_{n+1}(X_{n+1}, X_n) \longrightarrow H_{n+1}(X_{n+1}, X_n)$ es un isomorfismo y usando un resultado de Álgebra Homológica demostramos que si $X$ es conexo, el homomorfismo de Hurewicz $\Pi_n(X) \longrightarrow H_n(X)$ es un isomorfismo.

## 1. Introduction

Despite the great importance of higher homotopy groups –introduced by W. Hurewicz, in 1935– they have not provided great help, from a practical view

*Research partially supported by DID-USB under Grant DI-CB-015-04.

point, since their computation is very difficult or impossible (so far). Oposite situation arises with the homology groups –introduced by H. Poincaré in 1895– since, for a diversity of topological spaces, the algebraic structure of their associated homology groups can be calculated.

There are not many algorithms to compute absolute (or relative) homotopy groups of a topological space (even when the space is given with a triangulation). One of the few main tools available for the general study of homotopy groups is the comparison between such groups and the singular homology groups. Such comparison is effected by means of a canonical homomorphism from homotopy groups to homology groups. In terms of comparison of these groups, the first significant result was the Hurewicz theorem ([16]). This theorem is an interesting subject in Mathematics and others fields: in Physical Science it has been discovered as tool to classify objects as solitons and vortices ([7], [8], [9], [17]). In Sobolev spaces Theory, (see, for example, [20]) it is used to show that given $M$ a simply connected compact manifold and $N$ a closed manifold, any map in the Sobolev space $W^{1,2}(M,N)$ can be approximated by smooth maps between $M$ and $N$. Recent papers show the use of the mentioned theorem in the study of the DNA ([12], [24]), hence it is also an useful tool in Cellular Biology. Furthermore, Quantum Field Theory needs the Hurewicz theorem to establish some results (see [2] or [19]).

In this work we use methods of Homological Algebra to provide an alternative proof of the celebrated Hurewicz theorem in the case that the topological space is a CW-complex. Formally speaking, we first show that if $X_0 \subseteq X_1 \subseteq \cdots X_{n-1} \subseteq X_n = X$ is the CW decomposition of $X$, then the Hurewicz homomorphism $\Pi_{n+1}(X_{n+1}, X_n) \longrightarrow H_{n+1}(X_{n+1}, X_n)$ is an isomorphism, and together with a lemma from Homological Algebra (lemma 2.1) we prove that the Hurewicz homomorphism $\Pi_n(X) \longrightarrow H_n(X)$ is an isomorphism, whenever $X$ is $(n-1)$-connected.

Finally, we should say that our proof is more restrictive, but it has the advantage of being simpler than the classical proofs.

## 2. Previous definitions and results of homological algebra and algebraic topology

The purpose of this section is to present the classic tools from the theories of Homological Algebra and Algebraic Topology necessary for the proof of Hurewicz's theorem. In order to prove it, we use the lemma 2.1, which is –in our case– the key result to obtain the connection between the homotopy groups and the corresponding homology groups. Let us start with the notation and some preliminary definitions.

Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ the ring of integers, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the field of rational, real and complex numbers with the usual topology.

For the Euclidean space $\mathbb{R}^{n+1}$, we denote by $\mathbb{S}^n$, the $n$-sphere; $I^{n+1}$ the $(n+1)$-closed cube, $\mathbb{B}^{n+1}$ the closed ball of center in the origin and radius 1 and $\overset{\circ}{\mathbb{B}}_{n+1}$ its interior. Also, we denote by

i) *Grup* and *AG*: the categories of all groups and all abelian groups, respectively.
ii) *Top*: the category of all topological spaces and all continuous functions.
iii) $(Top,\star)$: the category of pointed topological spaces and continuous functions preserving base point.
iv) $Top^2$: the category of pairs of topological spaces.
v) $(Top^2,\star)$: the category of pairs of pointed topological spaces and continuous functions preserving base point.
vi) *Htop*: the category of homotopy.
vii) *COM*: the category whose objects are complexes and morphisms are chain maps.

Given $(X,x_0),(Y,y_0)$ objects of $(Top,\star)$, the wedge sum of $X$ and $Y$ is the subspace $X\times\{y_0\}\cup\{x_0\}\times Y\subset X\times Y$ and we use the notation, respectively, $X\vee Y$ and $x\vee y$ for such subspace and its points. It well-know that $X\vee Y$ is also an object of $(Top,\star)$ with base point $(x_0,y_0)$. The smash product of $X$ and $Y$ is the quotient $X\wedge Y=X\times Y/X\vee Y$, with base point $p(X\vee Y)$, where $p: X\times Y\longrightarrow X\wedge Y$ is the quotient projection. For any $(x,y)\in X\times Y$ we denote $p(X\vee Y)\in X\wedge Y$ by $x\wedge y$. The suspension $(SX,*)$ of $X$ is the smash product $(S^1\wedge X,*)$ of $X$ with the 1-sphere and the cone $CX$ of $X$ is given by $CX=X\wedge I$.

In order to compute the homotopy groups of wedge sum of spheres we require the following elementary result.

**Lemma 2.1.** *Let us consider the following commutative diagram of objects and morphism in the category Grup*

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \longrightarrow & 0\\
\downarrow{\scriptstyle\Psi_1} & & \downarrow{\scriptstyle\Psi_2} & & \downarrow{\scriptstyle\Psi_3} & & \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \longrightarrow & 0
\end{array}$$

*where each row is a short exact sequence. If $\Psi_1$ is an epimorphism and $\Psi_2$ is an isomorphism, then $\Psi_3$ is an isomorphism.*

*Proof.* First of all, we see that $\Psi_3$ is monomorphism: Let us consider $a_3\in A_3$ such that $\Psi_3(a_3)=0$; then there exists $a_2\in A_2$ such that $f_2(a_2)=a_3$, since $f_2$ is epimorphism. On the other hand, there is exists $b_1\in B_1$ with $g_1(b_1)=\Psi_2(a_2)$, since $g_2\circ\Psi_2(a_2)=0$. Furthermore, by our hypothesis on $\Psi_1$ there exists $a_1\in A_1$ with $\Psi_1(a_1)=b_1$. Therefore, $\Psi_2(a_2)=\Psi_2\circ f_1(a_1)$, and consequently, $a_3=f_2(a_2)=f_2\circ f_1(a_1)=0$.

Now, let us show that $\Psi_3$ is an epimorphism; to show this we will see that $Im\,(\Psi_3) = B_3$. It is clear that $Im(\Psi_3) \subset B_3$. Given $b_3 \in B_3$, there is exists $b_2 \in B_2$ satisfying $g_2\,(b_2) = b_3$, since $g_2$ is an epimorphism.

Furthermore, there exists an unique $a_2 \in A_2$ with $\Psi_2\,(a_2) = b_2$, since $\Psi_2$ is isomorphism. Putting $a_3 = f_2\,(a_2)$ we have $\Psi_3\,(a_3) = \Psi_3 \circ f_2\,(a_2) = g_2\,(b_2) = b_3$, and therefore $\Psi_3$ is an epimorphism. This finishes the proof. ☑

**Lemma 2.2.** *Let us consider the following commutative diagram of objets and morphism of Grup.*

$$\begin{array}{ccccccccc}
 & & L_1 & \xrightarrow{i_1} & L & \xleftarrow{i_2} & L_2 & & \\
 & & \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi_2} & & \\
0 & \longrightarrow & H_1 & \xrightarrow{j_1} & H & \xleftarrow{j_2} & H_2 & \longleftarrow & 0
\end{array}$$

*If $L = L_1 \bigoplus L_2$, $H = H_1 \bigoplus H_2$ and $\Psi_1$, $\Psi_2$ are isomorphisms, then $\Psi$ is an isomorphism.*

*Proof.* It is similar to the proof of previous lemma. ☑

**Theorem 2.1** (Exact Homology Sequence)**.** *If the sequence*

$$0 \longrightarrow K' \xrightarrow{T} K \xrightarrow{F} K'' \longrightarrow 0 \tag{1}$$

*is exact in the category $COM$, then for every $n$, there exists a connection homomorphism $\delta_n : H_n(K'') \longrightarrow H_{n-1}(K')$, such that the sequence*

$$\begin{aligned}
\cdots \longrightarrow \; & H_n(K') \xrightarrow{t_n} H_n(K) \xrightarrow{f_n} H_n(K'') \\
\xrightarrow{\delta_n} \; & H_{n-1}(K') \longrightarrow \cdots
\end{aligned} \tag{2}$$

*is exact.*

**Remark 2.1.** *The exact sequence* (2) *is called homology sequence of* (1)*.*

For the proof of theorem 2.1 see for example [15].

2.1. **Singular homology with coefficients $\mathbb{Z}$.** One of the reasons for which the group of the integers has been consistently considered and has played a privileged role as coefficient group for homology is that the homology and cohomology groups of a geometric complex with arbitrary coefficients or value group are completely determined by its homology groups with integer coefficients.

Therefore, it is sufficient to consider the homology (simplicial or singular) with coefficients $\mathbb{Z}$. The theorems in this paper can be applied to groups homology with arbitrary value group G.

Once said this, given $q \in \mathbb{Z}$, let us consider the free abelian group $S_q(X)$ generated by set of all singular $q$-simplices of topological space $X$. We put, as usual, $S_q(X) = 0$ for $q < 0$. The following results are classical (see [10], [14], or [15] for the proofs).

**Proposition 2.1.** *The sequence*

$$\cdots \longrightarrow S_{q+1}(X) \xrightarrow{\partial_{q+1}} S_q(X) \xrightarrow{\partial_q} S_{q-1}(X) \longrightarrow \cdots$$

*is an object of the category $COM$. It is called the singular complex of $X$ and is denoted by $S(X)$.*

**Theorem 2.2.** *If $X$ is an object of $Top$, such that $X$ is a single point, then*

$$H_q(X) = \begin{cases} 0, \text{ for } q \neq 0 \ , \\ \mathbb{Z}, \text{ for } q = 0 \ . \end{cases}$$

**Theorem 2.3** (Excision)**.** *Let $(X, A)$ be an object of $Top^2$ and $U \subset A$. If $\overline{U} \subset \overset{\circ}{A}$, then $U$ can be excised from pair $(X, A)$.*

**Theorem 2.4.** *Let $(X, A)$ be an object of $Top^2$ and $U \subset A$. Let us consider $V \subset U$. If $V$ is a deformation retract of $U$ and $V$ can be excised of pair $(X, A)$, then $U$ also can be excised from pair $(X, A)$.*

**Theorem 2.5.** *For $n \in \mathbb{N}$ and $q > 0$, we have that*

$$H_q(\mathbb{S}^n) = \begin{cases} \mathbb{Z}, \text{ si } q = n \ , \\ 0, \text{ si } q \neq n \ . \end{cases}$$

**Theorem 2.6.** *Given $m, n \in \mathbb{N}$, let us consider $\mathbb{S}^n \vee \mathbb{S}^m$, the wedge sum of spheres $\mathbb{S}^n$ and $\mathbb{S}^m$. For each $q > 0$ we have*

$$H_q(\mathbb{S}^n \vee \mathbb{S}^m) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, \text{ if } q = n = m, \\ 0 \text{ in any other case.} \end{cases}$$

**Corollary 2.1.** *For $q > 0$ :*

a)
$$H_q\left(\bigvee_{j=1}^{m} \mathbb{S}_j^n\right) = \begin{cases} \bigoplus_{j=1}^{m} \mathbb{Z}, \text{ if } q = n \ , \\ 0, \text{ if } q \neq n \ , \end{cases}$$

*where $\bigvee_{j=1}^{m} \mathbb{S}_j^n$ is the wedge sum of $\mathbb{S}^n$, $m$ times.*

b) *If $k < m$ the inclusion $\bigvee_{j=1}^{k} \mathbb{S}_j^n \hookrightarrow \bigvee_{j=1}^{m} \mathbb{S}_j^n$ induces a monomorphism in homology.*

One might want to see a concrete (relative) cycle whose homology class generates $H_n(\mathbb{B}_n, \mathbb{S}^{n-1})$, such cycles are usually called fundamental cycles. There is a very simple one, namely

**Proposition 2.2.** *The identity map $1_n : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ is a cycle mod $\mathbb{S}^{n-1}$, whose homology class $[1_n]$ is a generator of $H_n(\mathbb{B}^n, \mathbb{S}^{n-1}) \cong H_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$. Its boundary $\partial(1_n)$ is a cycle on $\mathbb{S}^{n-1}$, whose homology class is a generator of $H_{n-1}(\mathbb{B}^n, s_0)$.*

2.2. **Higher homotopy groups.** It is well known that for every continuous maps $f, g : (SX, *) \longrightarrow (Y, y_0)$, the set $[(SX, *), (Y, y_0)]$ can be given structure of a group if we define the product $[f].[g]$ by $[f].[g] := [\Delta' \circ (f \vee g) \circ \mu']$, where $\mu' : SX \longrightarrow SX \vee SX$ is given by

$$\mu'(t \wedge x) = \begin{cases} (2t \wedge x) \vee (1 \wedge x_0), \text{ if } 0 \leq t \leq \frac{1}{2} \\ (1 \wedge x_0) \vee (2t - 1 \vee x), \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

and the map $\Delta' : Y \vee Y \longrightarrow Y$, satisfies $\Delta'(y \vee y_0) = y = \Delta'(y_0 \vee y)$.

Furthermore, the identity of this group is the class $[y_0]$ of the constant map and the inverse is given by $[f]^{-1} := [f \circ \nu']$, where $\nu' : SX \longrightarrow SX$ is given by $\nu'(t \vee x) = (1 - t) \vee x$, $t \in I$, $x \in X$.

Using that $S(\mathbb{S}^{n-1}) \cong \mathbb{S}^n$, the $n$-th homotopy group is defined by

$$\Pi_n(Y, y_0) := [(S\mathbb{S}^{n-1}, *), (Y, y_0)] \cong [(\mathbb{S}^n, s_0), (Y, y_0)].$$

With this formulation $\Pi_n(Y, y_0)$ is object of *Grup* for $n \geq 1$ and is object of *AG* for $n \geq 2$. Similarly, if $(X, A, x_0)$ is an object of $(Top^2, \star)$, the $n$-th homotopy group of $(X, A, x_0)$, $\Pi_n(X, A, x_0)$ is object of *Grup* for $n \geq 2$ and object of *AG* for $n \geq 3$. Furthermore, the homotopy class of a function $f/_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \longrightarrow A$ in $\Pi_{n-1}(A, x_0)$ only depends on $f$, thus we can define $\partial_n : \Pi_n(X, A, x_0) \longrightarrow \Pi_{n-1}(A, x_0)$ given by $\partial_n([f]) = [f/_{\mathbb{S}^{n-1}}]$.

This morphism is a homomorphism whenever $n > 1$, and if $i : (A, x_0) \longrightarrow (X, x_0)$ is the inclusion map, $\Omega X = (X, x_0)^{(\mathbb{S}^1, s_0)}$, $P(X; x_0, A)$ is the space of paths in $X$ starting at $x_0$ and ending in $A$, and $\rho' : \Omega X \longrightarrow P(X; x_0, A)$ is given by $\rho'(\omega) = (x_0, \omega)$; then $\Pi_{n-1}(\Omega X, \omega_0) \longrightarrow \Pi_{n-1}(P(X; x_0, A), \omega_0)$ is a homomorphism.

On the other hand, the map $q : P(X; x_0, A) \longrightarrow A$ defined by $q(\omega) = \omega(1)$ is continuous and if $f \in \hom_{(Top,\star)}((\mathbb{S}^{n-1}, s_0), (X, x_0))$ and we consider the space

$$P_f := \{(s, \omega) \in \mathbb{S}^{n-1} \times P(X; x_0, A) : f(s) = \omega(1)\},$$

then the continuous map $\delta : P_f \longrightarrow A$ defined by $\delta((s, \omega)) = f(s)$ induces a continuous map $\tilde{f} : (\mathbb{S}^{n-1}, s_0) \longrightarrow P(X; x_0, A)$ given by $\tilde{f}(s) = \delta((s, \omega))$, where $\omega(1) = f(s)$, for all $s \in \mathbb{S}^{n-1}$. Therefore, there exists a continuous map $g_{\tilde{f}} : (C\mathbb{S}^{n-1}, \mathbb{S}^{n-1}, s_0) \longrightarrow (X, A, x_0)$ associated to $\tilde{f}$, since there exists an one-one correspondence

$$\begin{aligned}[(C\mathbb{S}^{n-1}, \mathbb{S}^{n-1}, s_0), (X, A, x_0)] \leftrightarrow & [(\mathbb{S}^{n-1}, s_0), (P(X; x_0, A), \omega_0)] \\ & = \Pi_{n-1}(P(X; x_0, A), \omega_0).\end{aligned}$$

But we have a homeomorphism $h : C\mathbb{S}^{n-1} \longrightarrow \mathbb{B}_n$ defined by $h(s \wedge t) = ts + (1 - t)s_0$, $t \in I$, $s \in \mathbb{S}^{n-1}$, therefore $g_{\tilde{f}} \circ h^{-1} \in \hom_{(Top^2,\star)}((\mathbb{B}_n, \mathbb{S}^{n-1}, s_0), (X, A, x_0))$ and $j_* : \Pi_{n-1}(X, x_0) \longrightarrow \Pi_{n-1}\ (X, A, x_0)$ given by $j_*([f]) = [g_{\tilde{f}} \circ h^{-1}]$ is a homomorphism.

Hence we obtain that the following longer sequence

$$\cdots \longrightarrow \Pi_n(X,A,x_0) \xrightarrow{\partial_n} \Pi_{n-1}(A,x_0) \xrightarrow{i_*} \Pi_{n-1}(X,x_0)$$
$$\xrightarrow{j_*} \Pi_{n-1}(X,A,x_0) \longrightarrow \cdots$$

is exact.

Such sequence is called the homotopy exact sequence associated to $(X, A, x_0)$. In particular, putting $A = \{x_0\}$, we have $\Pi_n(X, x_0) \cong \Pi_n(X, \{x_0\}, x_0)$, for $n \geq 0$.

The reader is referred to [10] or [23] for the proofs of the above statements.

2.3. **CW-complexes.** One of the difficulties for computing homotopy groups is that given two arbitrary topological spaces $X$ and $Y$, it is difficult to construct any continuous map $f : X \to Y$. In this section we will only try with a class of spaces built up step by step out of simple building blocks, as simplicial complexes, for example. The advantage of trying with these spaces is that we might hope to construct continuous maps step by step, extending them over the building blocks one at a time. Such spaces are called CW-complexes.

Let us mention some properties relative to CW-complexes (see [10],[14], [15] or [23]) which we will need in the Section 3; we will assume that all the spaces considered are Hausdorff.

**Proposition 2.3.** *If $K$ is a CW-complex on $X$, then any compact subset $S \subset X$ meets only a finite number of interiors of cells.*

The reader is referred to [23] for the proof of this proposition.

**Lemma 2.3.** *Let $K$ be a cell complex on $X$. If $L$ is a subcomplex of $K$, then $L$ is a cell complex of $|L|$ and if $K$ is a CW-complex on $X$, then $L$ is a CW-complex on $|L|$.*

Notice that the above lemma implies that for each skeleton $K^n$, $|K^n|$ is a subcomplex. And combining the results of Proposition 2.3 and Lemma 2.3, it is deduced that if $K$ is a CW-complex on $X$, then every compact set $S \subset X$ is contained in a finite subcomplex. It is easy to verify that

**Proposition 2.4.** *If $K$ is a CW-complex on $X$ and $L$ is a subcomplex, then $|L|$ is a closed subspace of $X$.*

So far we have taken the point of view that the space $X$ is given and $K$ is a prescription for cutting $X$ up into cells. Now, we will give an algorithm that will allow us to build $X$ cell by cell; successively "gluing" on new cells, therefore we will work in $(Top, \star)$, although the methods can be adapted for $Top$. Given $(X, x_0)$ an object of $(Top, \star)$ and $g : \mathbb{S}^{n-1} \longrightarrow X$ a continuous map, then we can form the mapping cone of $g$, $X \cup_g C\mathbb{S}^{n-1}$. The result is called $X$ with an $n$-cell attached. The map $g$ is called the attaching map of the cell. The protection $q : X \vee C\mathbb{S}^{n-1} \longrightarrow X \bigcup_g C\mathbb{S}^{n-1}$ restricts to $C\mathbb{S}^{n-1}$ gives a continuous map $f = q/_{C\mathbb{S}^{n-1}} : C\mathbb{S}^{n-1} \longrightarrow X \cup_g C\mathbb{S}^{n-1}$ which on the interior of $C\mathbb{S}^{n-1}$

is a homeomorphism. So that $f : \left(\mathbb{B}_n, \mathbb{S}^{n-1}\right) \longrightarrow \left(X \bigcup_g C\mathbb{S}^{n-1}, X\right)$ is the characteristic map of the cell $e = q\left(C\mathbb{S}^{n-1}\right) = X \bigcup_g C\mathbb{S}^{n-1}$, since $C\mathbb{S}^{n-1} \simeq \mathbb{B}_n$ and $f(\mathbb{S}^{n-1}) = q/_{C\mathbb{S}^{n-1}}\left(\mathbb{S}^{n-1}\right) = g\left(\mathbb{S}^{n-1}\right) = X$.

More generally if we have continuous map $g : \bigvee\limits_{\alpha} \mathbb{S}_{\alpha}^{n-1} \longrightarrow X$, of wedge sum of $(n-1)$-spheres into $X$, then the mapping cone $X \bigcup_g C\left(\bigvee\limits_{\alpha} \mathbb{S}_{\alpha}^{n-1}\right) = X \bigcup_g \left(\bigvee\limits_{\alpha} C\mathbb{S}_{\alpha}^{n-1}\right)$ is called $X$ with $n$-cells attached. The subset $q\left(C\mathbb{S}_{\alpha}^{n-1}\right)$ is the $n$-cell $e_{\alpha}^n$ and its attaching map $g/_{\mathbb{S}_{\alpha}^{n-1}}$. The characteristic map $f_{\alpha}^n$ of $e_{\alpha}^n$ is $f_{\alpha}^n = q/_{C\mathbb{S}_{\alpha}^{n-1}}$. Attaching a 0-cell will mean adding a disjoint point.

**Lemma 2.4** (Lemma 5.11, [23])**.** *Let $K$ be an finite CW-complex on $X$, whose cells have dimension at most $n-1$. Attach $n$-cells $e_{\alpha}^n$, $\alpha \in J_n$, by attaching maps $g_{\alpha} : \mathbb{S}_{\alpha}^{n-1} \longrightarrow X$ and let $K' = K \bigcup \{e_{\alpha}^n : \alpha \in J_n\}$. Then $K'$ is a CW-complex on $X \bigcup_{\{g_{\alpha}\}} \bigvee\limits_{\alpha} C\mathbb{S}_{\alpha}^{n-1}$ .*

**Proposition 2.5** (Proposition 5.12, [23])**.** *Let us suppose that $\{x_0\} = X^{-1} \subset X^0 \subset \cdots \subset X^n \subset X^{n+1} \subset \cdots$ is a sequence of spaces such that $X^n$ is obtained from $X^{n-1}$ by attaching $n$-cells, $n \geq 0$. If we give $X = \bigcup\limits_{n \geq -1} X^n$ the weak topology, i.e. $S \subset X$ is closed iff $S \cap X^n$ is closed in $X^n$, for each $n \geq -1$, then $K = \{$ all cells $\}$ is a CW- complex on $X$.*

Thus we have given ourselves the opportunity of building CW-complexes to our own specifications. The generalized notion of CW-complex is as follows.

**Definition 2.1.** *Let $(X, A)$ object of $Top^2$, a relative CW-complex on $(X, A)$ is a sequence of topological spaces*

$$A = (X, A)^{-1} \subset (X, A)^0 \subset \cdots \subset (X, A)^n \subset (X, A)^{n+1} \subset \cdots \subset X\,,$$

*such that $(X, A)^n$ is obtained from $(X, A)^{n-1}$ by attaching $n$-cells, $n \geq 0$; $X = \bigcup\limits_{n \geq -1} (X, A)^n$ and $X$ has the weak topology: $S \subset X$ is closed iff $S \cap (X, A)^n$ is closed in $(X, A)^n$, for all $n \geq -1$.*

*We say that the relative CW-complex on $(X, A)$ is $n$-finite if $(X, A)^n = X$ and $(X, A)^{n-1} \neq X$*

**Theorem 2.7.** *If $e_{\alpha}^n$ is a $n$-cell of a CW-complex $K$ on a topological space $X$ and $f_{\alpha}^n : \left(\mathbb{B}^n, \mathbb{S}^{n-1}\right) \longrightarrow \left(e_{\alpha}^n, \overset{\bullet}{e}{}_{\alpha}^n\right)$ is its characteristic map, then $H_n(f_{\alpha}^n) : H_n\left(\mathbb{B}^n, \mathbb{S}^{n-1}\right) \longrightarrow H_n\left(e_{\alpha}^n, \overset{\bullet}{e}{}_{\alpha}^n\right)$ is isomorphism.*

*Proof.* For any fixed $0 < r < 1$, let us consider $\mathbb{B}_r^n := \{x \in \mathbb{B}^n : r \leq \|x\| \leq 1\}$ and $Y_r^n := \{x \in \mathbb{R}^n : r \leq \|x\| \leq 2\}$.

We have that $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{B}^n_r$, $H : \mathbb{B}^n_r \times I \longrightarrow \mathbb{B}^n_r$, given by $H(x,t) = t\frac{x}{\|x\|} + (1-t)x$ is a homotopy $rel\ \mathbb{S}^{n-1}$ between the identity map $1_{\mathbb{B}^n_r}$ and $g : \mathbb{B}^n_r \longrightarrow \mathbb{S}^{n-1}$, given by $g(x) = \frac{x}{\|x\|}$. Also $\mathbb{B}^n_{\frac{r}{2}}$ and $Y^n_r$ are homeomorphic, since $\Psi^n_r : Y^n_r \longrightarrow \mathbb{B}^n_{\frac{r}{2}}$, given by $\Psi^n_r(x) = \frac{x}{2}$ is a homeomorphism.

We define the homotopy $H' : f^n_\alpha(\mathbb{B}^n_r) \times I \longrightarrow f^n_\alpha(\mathbb{B}^n_r)$ by means of the commutative diagram

$$\begin{array}{ccc} \mathbb{B}^n_r \times I & \xrightarrow{H} & \mathbb{B}^n_r \\ \Big\downarrow{\scriptstyle f\times I} & & \Big\downarrow{\scriptstyle f} \\ f^n_\alpha(\mathbb{B}^n_r) \times I & \xrightarrow{H'} & f^n_\alpha(\mathbb{B}^n_r) \end{array}$$

$H'(y,t) = f^n_\alpha \circ H(x,t)$ if $y = f^n_\alpha(x)$, since $f^n_\alpha/_{\overset{\circ}{\mathbb{B}^n}}$ is homeomorphism. For the continuity and good definition of $H'$ see [23]. Obviously $H'$ determines that $e^n_\alpha$ is a deformation retract of $f^n_\alpha(\mathbb{B}^n_r)$. We obtain by this way the following diagram

$$\begin{array}{ccc} H_n\left(\mathbb{B}^n, \mathbb{S}^{n-1}\right) & \xrightarrow{\cong} & H_n(\mathbb{B}^n, \mathbb{B}^n_r) \\ \Big\downarrow{\scriptstyle H_n(f^n_\alpha)} & & \Big\downarrow{\scriptstyle H_n(f^n_\alpha)} \\ H_n\left(e^n_\alpha, \overset{\bullet}{e}{}^n_\alpha\right) & \xrightarrow{\cong} & H_n(e^n_\alpha, f^n_\alpha(\mathbb{B}^n_r)) \end{array} \tag{3}$$

Since $\overline{\mathbb{B}^n_r} \subset \overset{\circ}{Y}{}^n_r \cong \overset{\circ}{\mathbb{B}}{}^n_{\frac{r}{2}}$ and $\overline{f^n_\alpha(\mathbb{B}^n_r)} \subset int(f^n_\alpha \circ \Psi^n_r(Y^n_n)) \cong int(f^n_\alpha(\mathbb{B}^n_r))$, applying the theorem 2.3 we have that $\mathbb{B}^n_r$ and $f^n_\alpha(\mathbb{B}^n_r)$ can be excised, respectively, of the pairs $(\mathbb{B}^r, Y^n_r) \cong (\mathbb{B}^n, \mathbb{B}^n_{\frac{r}{2}})$ and $(e^n_\alpha, f^n_\alpha \circ \Psi^n_r(Y^n_n)) \cong \left(e^n_\alpha, f^n_\alpha\left(\mathbb{B}^n_{\frac{r}{2}}\right)\right)$. Consequently,

$$\begin{array}{ccc} H_n(\mathbb{B}^n, \mathbb{B}^n_{\frac{r}{2}}) & \xrightarrow{\cong} & H_n(\mathbb{B}^n \setminus \mathbb{B}^n_r, \mathbb{B}^n_{\frac{r}{2}} \setminus \mathbb{B}^n_r) \\ \Big\downarrow{\scriptstyle H_n(f^n_\alpha)} & & \Big\downarrow{\scriptstyle H_n(f^n_\alpha)} \\ H_n\left(e^n_\alpha, f^n_\alpha\left(\mathbb{B}^n_{\frac{r}{2}}\right)\right) & \xrightarrow{\cong} & H_n\left(e^n_\alpha \setminus f^n_\alpha(\mathbb{B}^n_r), f^n_\alpha\left(\mathbb{B}^n_{\frac{r}{2}}\right) \setminus f^n_\alpha(\mathbb{B}^n_r)\right) \end{array} \tag{4}$$

Using that $f^n_\alpha/_{\overset{\circ}{\mathbb{B}^n}}$ is homeomorphism, we have that $H_n(f^n_\alpha) : H_n(\mathbb{B}^n \setminus \mathbb{B}^n_r, \mathbb{B}^n_{\frac{r}{2}} \setminus \mathbb{B}^n_r) \longrightarrow H_n\left(e^n_\alpha \setminus f^n_\alpha(\mathbb{B}^n_r), f^n_\alpha\left(\mathbb{B}^n_{\frac{r}{2}}\right) \setminus f^n_\alpha(\mathbb{B}^n_r)\right)$ is isomorphism. Combining the diagrams (3) and (4), we obtain

$$\begin{tikzcd}
H_n\left(\mathbb{B}^n,\mathbb{S}^{n-1}\right) \arrow[r, "H_n(f^n_\alpha)"] \arrow[d, "\cong"] & H_n\left(e^n_\alpha, \overset{\bullet}{e}{}^n_\alpha\right) \arrow[d, "\cong"] \\
H_n\left(\mathbb{B}^n,\mathbb{B}^n_r\right) \arrow[r, "H_n(f^n_\alpha)"] \arrow[d, "\cong"] & H_n\left(e^n_\alpha, f^n_\alpha(\mathbb{B}^n_r)\right) \arrow[d, "\cong"] \\
H_n(\mathbb{B}^n \smallsetminus \mathbb{B}^n_r, \mathbb{B}^n_{\frac{r}{2}} \smallsetminus \mathbb{B}^n_r) \arrow[r, "\cong"] & H_n\left(e^n_\alpha \smallsetminus f^n_\alpha\left(\mathbb{B}^n_r\right), f^n_\alpha\left(\mathbb{B}^n_{\frac{r}{2}}\right) \smallsetminus f^n_\alpha\left(\mathbb{B}^n_r\right)\right)
\end{tikzcd}$$

This finishes the proof. ☑

**Theorem 2.8.** *If $K$ is a CW-complex on $X$ such that its $n$-skeleton $K^n$ is obtained attaching to $K^{n-1}$ a finite collection of $n$-cells, $n \geq 0$. Then*

$$H_q\left(|K^n|, \left|K^{n-1}\right|\right) = \begin{cases} 0, & \text{if } \; q \neq n, \\ \bigoplus\limits_{\alpha\in J_n} \mathbb{Z}, & \text{if } \; q = n, \end{cases}$$

*where $J_n \subset \mathbb{N}$ is a finite set, whose quantity of terms coincides with the quantity of $n$-cells attached to $K^n$.*

**Corollary 2.2.** *If $K$ is a CW-complex on $X$ such that its $n$-skeleton $K^n$ is obtained attaching to $K^{n-1}$ a finite collection of $n$-cells, $n \geq 0$. Then*

i) $H_q\left(|K^n|\right) = 0, \;\; \forall \;\; q > n.$
ii) $H_q\left(|K^n|\right) \cong H_q\left(X\right), \;\; \forall \;\; q < n.$
iii) *There exists an epimorphism between $H_n\left(|K^n|\right)$ and $H_n\left(\left|K^{n+1}\right|\right)$.*

**Theorem 2.9** (Theorem 6.11, [23])**.** *Let $K$ be a CW-complex on $X$ and $i : |K^n| \longrightarrow X$ the inclusion of $n$-skeleton, $n > 0$. Then $\Pi_q(i) : \Pi_q\left(|K^n|, x_0\right) \longrightarrow \Pi_q\left(X, x_0\right)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$.*

**Theorem 2.10** (Corollary 6.14, [23])**.** *If $K$ be a CW-complex on $X$ and $(X, x_0)$ is $n$-connected, then we can find a CW-complex $\tilde{K}$ with $\left(\tilde{K}\right)^n = \tilde{x}_0$ and a homotopy equivalence $f : (X, x_0) \longrightarrow \left(\left|\tilde{K}\right|, \tilde{x}_0\right)$.*

## 3. The Hurewicz theorem for wedge sum of spheres

There are always natural morphisms from homotopy groups to homology groups, defined in the following way: if $(X, A, x_0)$ is an object of $(Top^2, \star)$, $\Pi_n\left(X, A, x_0\right)$, for $n > 0$ is its homotopy group and $[f] \in \Pi_n\left(X, A, x_0\right)$, then the map $f : \left(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0\right) \longrightarrow (X, A, x_0)$ induces a homomorphism

$$H_n(f) : H_n\left(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0\right) \longrightarrow H_n\left(X, A, x_0\right)$$

between $H_n\left(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0\right)$ and $H_n(X, A, x_0)$. Since $H_n\left(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0\right) \cong \mathbb{Z}$ and $\mathbb{Z}$ is cyclic group, if we denote by $e$ a fixed generator of $H_n\left(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0\right)$, then we can define the morphism $\Psi_{n(X,A)} := \Psi_n : \Pi_n\left(X, A, x_0\right) \longrightarrow H_n\left(X, A, x_0\right)$ by $\Psi_n([f]) = H_n(f)\,(e)$.

It is clear that $\Psi_n$ is well-defined, since if we have a homotopy $f \cong g$ through maps $(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0) \longrightarrow (X, A, x_0)$ then $H_n(f) = H_n(g)$. The morphism $\Psi_n$ is called Hurewicz map.

**Proposition 3.1.** *The Hurewicz map* $\Psi_n : \Pi_n\left(X, A, x_0\right) \longrightarrow H_n\left(X, A, x_0\right)$ *is homomorphism.*

*Proof.* It suffices to show that for continuous maps $f, g : (\mathbb{B}^n, \mathbb{S}^{n-1}, s_0) \longrightarrow (X, A, x_0)$, the induced homomorphism on homology satisfy $H_n(f+g) = H_n(f) + H_n(g)$, for if this the case then $\Psi_n([f+g]) = H_n(f+g)(e) = H_n(f)(e) + H_n(g)(e) = \Psi_n([f]) + \Psi_n([g])$.

Let $r : \mathbb{B}^n \longrightarrow \mathbb{B}^n \vee \mathbb{B}^n$ be the map collapsing the equatorial $\mathbb{S}^{n-1}$ to a point, and $q_1, q_2 : \mathbb{B}^n \vee \mathbb{B}^n \longrightarrow \mathbb{B}^n$ be the quotient maps onto the two summands, collapsing the other summand to a point. We then have a diagram

$$\begin{tikzcd}
 & H_n(X, A, x_0) \\
H_n(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0) \arrow[r, "H_n(c)"] & H_n(\mathbb{B}^n \vee \mathbb{B}^n, \mathbb{S}^{n-1} \vee \mathbb{S}^{n-1}, s_0 \vee s_0) \arrow[u, "H_n(f\vee g)"'] \arrow[d, "H_n(q_1)\oplus H_n(q_2)"', "\cong"] \\
 & H_n(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0) \oplus H_n(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0)
\end{tikzcd}$$

The homomorphism $H_n(q_1) \oplus H_n(q_2)$ is an isomorphism with inverse $H_n(i_1) + H_n(i_2)$ where $i_1$ and $i_2$ are the inclusions of the two summands $\mathbb{B}^n \hookrightarrow \mathbb{B}^n \vee \mathbb{B}^n$. Since the composites $q_1 \circ r$ and $q_2 \circ r$ are homotopic to the identity map through maps $(\mathbb{B}^n, \mathbb{S}^{n-1}, s_0) \longrightarrow (\mathbb{B}^n, \mathbb{S}^{n-1}, s_0)$, the composition $H_n(q_1) \oplus H_n(q_2) \circ H_n(c)$ is the diagonal map $x \mapsto (x, x)$. From the equalities $(f \vee g) \circ i_1 = f$ and $(f \vee g) \circ i_2 = g$ we deduce that $H_n(f \vee g) \circ (H_n(i_1) + H_n(i_2))$ sends $(x, 0)$ to $H_n(f)(x)$ and $(0, x)$ to $H_n(g)(x)$, hence $(x, x)$ to $H_n(f)(x) + H_n(g)(x)$. Thus the composition across the top of the diagram is $x \mapsto H_n(f)(x) + H_n(g)(x)$. On the other hand, $f + g = (f \vee g) \circ r$, so this composition is also $H_n(f+g)$ ☑

Taking $A = \{x_0\}$, we have a special case of $\Psi_{n\,X} : \Pi_n(X, x_0) \longrightarrow H_n(X, x_0)$, since $\Pi_n(X, \{x_0\}, x_0) \cong \Pi_n(X, x_0)$. Another elementary property of Hurewicz homomorphisms is that they are natural: If $(X, A, x_0)$, $(Y, B, y_0)$ are objects of $(Top^2,\star)$ and $f\in \hom_{(Top^2,\star)}((X,A,x_0),(Y,B,y_0))$ then $f$ induces a commutative diagram as the following

$$\begin{array}{ccc} \Pi_n(X,A,x_0) & \xrightarrow{\Pi_n(f)} & \Pi_n(Y,B,y_0) \\ {\scriptstyle\Psi_n(X,A)}\big\downarrow & & \big\downarrow{\scriptstyle\Psi_n(Y,B)} \\ H_n(X,A,x_0) & \xrightarrow{H_n(f)} & H_n(Y,B,y_0) \end{array}$$

**Proposition 3.2.** *The Hurewicz homomorphism*

$$\Psi_n:\Pi_n\left(\bigvee_{j=1}^{m}\mathbb{S}_j^n\right)\longrightarrow H_n\left(\bigvee_{j=1}^{m}\mathbb{S}_j^n\right)$$

*is an isomorphism, for each $m\geq 1$.*

*Proof.* The proof is by induction on $m$. For $m=1$, we have that $\mathbb{Z}\cong \Pi_n(\mathbb{S}^n,s_0)\xrightarrow{\Psi_n} H_n(\mathbb{S}^n,s_0)\cong\mathbb{Z}$. So that, it is enough to show that $\Psi_n$ is epimorphism. In fact, if $\sigma_n\in H_n(\mathbb{S}^n,s_0)$ is a fixed generator of $H_n(\mathbb{S}^n,s_0)$ then $\sigma_n:\Delta_n\longrightarrow \Delta_n/\partial(\Delta_n)\cong\mathbb{S}^n$, putting $f=1_{\mathbb{S}^n}$ we have $\Psi_n([f])=H_n(f)(\sigma_n)=\sigma_n$. Therefore, $\Psi_n$ is epimorphism and consequently, the induced homomorphism $\widetilde{\Psi}_n:\mathbb{Z}\longrightarrow\mathbb{Z}$ is epimorphism. Since all epimorphism $\mathbb{Z}\longrightarrow\mathbb{Z}$ is isomorphism then $\Psi_n$ is isomorphism.

Now, let us suppose that the proposition holds for $m-1$ and let us consider the following commutative diagram

$$\begin{array}{ccccccc} 0\longrightarrow & \Pi_n(\mathbb{S}^n) & \xrightarrow{\Pi_n(i)} & \Pi_n\left(\bigvee_{j=1}^{m}\mathbb{S}_j^n\right) & \xleftarrow{\Pi_n(j)} & \Pi_n\left(\bigvee_{j=1}^{m-1}\mathbb{S}_j^n\right) & \longleftarrow 0 \\ & \cong\downarrow_{\varphi_1} & & \downarrow_{\Psi_n} & & \cong\downarrow_{\varphi_2} & \\ 0\longrightarrow & H_n(\mathbb{S}^n) & \xrightarrow{H_n(i)} & H_n\left(\bigvee_{j=1}^{m}\mathbb{S}_j^n\right) & \xleftarrow{H_n(j)} & H_n\left(\bigvee_{j=1}^{m-1}\mathbb{S}_j^n\right) & \longleftarrow 0 \end{array}$$

where $i:\mathbb{S}^n\to\bigvee_{j=1}^{m}\mathbb{S}_j^n$ and $j:\bigvee_{j=1}^{m-1}\mathbb{S}_j^n\to\bigvee_{j=1}^{m}\mathbb{S}_j^n$ are respectively, the inclusion maps of $\mathbb{S}^n$ and $\bigvee_{j=1}^{m-1}\mathbb{S}_j^n$ at $\bigvee_{j=1}^{m}\mathbb{S}_j^n$. Then $\varphi_1:\Pi_n(\mathbb{S}^n)\xrightarrow{\cong}H_n(\mathbb{S}^n)$ and $\varphi_2:\Pi_n\left(\bigvee_{j=1}^{m-1}\mathbb{S}_j^n\right)\xrightarrow{\cong}H_n\left(\bigvee_{j=1}^{m-1}\mathbb{S}_j^n\right)$ are both isomorphisms. The proof is complete when we apply Lemma 2.2 to the above diagram. ☑

**Corollary 3.1.** *Given $K$ a finite CW-complex on a $(n-1)$-connected space $X$; such that its $n$-skeleton $K^n$ is obtained attaching to $K^{n-1}$ a finite collection of $n$-cells, then*

$$\Psi_{n(|K^n|)}:\Pi_n(|K^n|)\longrightarrow H_n(|K^n|)$$

*is an isomorphism.*

*Proof.* From the theorem 2.10 we can suppose that the $(n-1)$-skeleton $K^n$ is a point, consequently $|K^n| \simeq \bigvee_{\alpha \in J_n} \mathbb{S}^n_\alpha$, $H_n(|K^n|) \cong H_n\left(\bigvee_{\alpha \in J_n} \mathbb{S}^n_\alpha\right)$ and $\Pi_n(|K^n|) \cong \Pi_n\left(\bigvee_{\alpha \in J_n} \mathbb{S}^n_\alpha\right)$. Now, by the proposition 3.2 $\Pi_n\left(\bigvee_{\alpha \in J_n} \mathbb{S}^n_\alpha\right) \cong H_n\left(\bigvee_{\alpha \in J_n} \mathbb{S}^n_\alpha\right)$. Therefore, $\Psi_{n(|K^n|)} : \Pi_n(|K^n|) \xrightarrow{\cong} H_n(|K^n|)$ is an isomorphism. ☑

**Proposition 3.3.** *If $K$ a finite CW-complex on a $(n-1)$-connected space $X$; such that its $(n+1)$-skeleton $K^{n+1}$ is obtained attaching to $K^n$ a finite collection of $(n+1)$-cells, then $\Psi_{n+1} : \Pi_{n+1}\left(|K^{n+1}|, |K^n|\right) \longrightarrow H_{n+1}\left(|K^{n+1}|, |K^n|\right)$ is an epimorphism.*

*Proof.* Given $\xi_{n+1}$ a fixed generator of $H_{n+1}\left(\mathbb{B}^{n+1}, \mathbb{S}^n\right) \cong \mathbb{Z}$. As consequence of theorem 2.8 $H_{n+1}\left(|K^{n+1}|, |K^n|\right) \cong \bigoplus_{\alpha \in J_n} H_{n+1}(e^{n+1}_\alpha, \dot{e}^{n+1}_\alpha)$, then

$$\Pi_{n+1}\left(|K^{n+1}|, |K^n|\right) \xrightarrow{\Psi_{n+1}} H_{n+1}\left(|K^{n+1}|, |K^n|\right) \cong \bigoplus_{\alpha \in J_n} H_{n+1}(e^{n+1}_\alpha, \dot{e}^{n+1}_\alpha).$$

Now, given a $(n+1)$-cell $e^{n+1}_\alpha$, considering its characteristic map $f^{n+1}_\alpha$, as a consequence of Theorem 2.7 we have that

$$H_{n+1}\left(f^{n+1}_\alpha\right) : H_{n+1}\left(\mathbb{B}^{n+1}, \mathbb{S}^n\right) \xrightarrow{\cong} H_{n+1}\left(e^{n+1}_\alpha, \dot{e}^{n+1}_\alpha\right)$$

is isomorphism. We can choose $\eta_\alpha$ generators of $H_{n+1}\left(e^{n+1}_\alpha, \dot{e}^{n+1}_\alpha\right)$ such that $H_{n+1}\left(f^{n+1}_\alpha\right)(\xi_{n+1}) = \eta_\alpha$, since $H_{n+1}\left(f^{n+1}_\alpha\right)$ sends generators onto generators. But from the definition of $\Psi_{n+1}$: $\Psi_{n+1}\left[f^{n+1}_\alpha\right] = H_{n+1}\left(f^{n+1}_\alpha\right)(\xi_{n+1}) = \eta_\alpha$. Therefore, $\Psi_{n+1}$ is an epimorphism. ☑

We conclude with our alternative proof of the Hurewicz Theorem for wedge sum of spheres.

**Theorem 3.1** (Hurewicz Theorem)**.** *Given $K$ a finite CW-complex on a $(n-1)$-connected space $X$; such that its $n$-skeleton $K^n$ is obtained attaching to $K^{n-1}$ a finite collection of $n$-cells, then $\Psi_{n\,X} : \Pi_n(X, x_0) \longrightarrow H_n(X, x_0)$ is an isomorphism.*

*Proof.* Using Corollary 2.2 and Theorem 2.9 we have

$$\begin{array}{ccc}
\Pi_n\left(|K^{n+1}|, x_0\right) & \xrightarrow{\cong} & \Pi_n(X, x_0) \\
{\scriptstyle \Psi_{n\,|K^{n+1}|}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_{n\,X}} \\
H_n\left(|K^{n+1}|, x_0\right) & \xrightarrow{\cong} & H_n(X, x_0).
\end{array}$$

Thus our problem is reduced to show that $\Psi_{n\,|K^{n+1}|}$ is an isomorphism. From the theorems 2.8 and 2.9 we know that $H_n\left(|K^{n+1}|,|K^n|\right)=0$ and $\Pi_n(i):\Pi_n(|K^n|,x_0)\longrightarrow H_n\left(|K^{n+1}|,x_0\right)$ is an isomorphism. Hence, when we consider the homotopy and homology exact sequences of pair $\left(|K^{n+1}|,|K^n|\right)$, we obtain the following commutative diagram

$$\begin{tikzcd}
... \arrow[r] & \Pi_{n+1}\left(|K^{n+1}|,|K^n|\right) \arrow[r] \arrow[d, "\Psi_{n+1}\left(|K^{n+1}|,|K^n|\right)"] & \Pi_n(|K^n|) \arrow[r] \arrow[d, "\Psi_{n\,|K^n|}"] & \Pi_n(|K^{n+1}|) \arrow[r] \arrow[d, "\Psi_{n\,|K^{n+1}|}"] & 0 \\
... \arrow[r] & H_{n+1}\left(|K^{n+1}|,|K^n|\right) \arrow[r] & H_n(|K^n|) \arrow[r] & H_n(|K^{n+1}|) \arrow[r] & 0
\end{tikzcd}$$

By Proposition 3.3, $\Psi_{n+1}\left(|K^{n+1}|,|K^n|\right)$ is an epimorphism and by Corollary 3.1, $\Psi_{n\,|K^n|}$ is an isomorphism. Then applying Lemma 2.1 to the above commutative diagram we obtain that $\Psi_{n\,|K^{n+1}|}$ is an isomorphism; this completes the proof. ☑

**Acknowledgments.** We thank the referee for his suggestions which have improved the presentation of the paper.

## References

[1] J. F. Adams, *Algebraic Topology: A Student's Guide*, *London Mathematical Society Lecture Note Series*, Cambridge: University Press, 1972.

[2] A. P. Balachandran, V.P. Nair, N. Panchapakesan & S. G. Rajeev, Low-mass solitons from fractional charges in quantum chromodynamics, *Phys. Rev. D.* **28** no. 11 (1983), 2830–2839.

[3] H-J. Baues & A. Quintero, On the locally finite chain algebra of a proper homotopy type, *Bull. Belg. Math. Soc.* **3** (1996), 161–175.

[4] M.G. Barratt, Track groups I, II, *Proc. London Math. Soc.* **5** (1955), 71–106, 238–329.

[5] G.C. Bell, Asymptotic properties of groups acting of complexes, arXiv: math.GR/0212032v1 (2002).

[6] G.C. Bell & A.N. Dranishnikov, A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory, arXiv:math.GR/0407431 (2004).

[7] V. Benci, Solitons and bohmian machanics, *Discrete and Continuous Dynamical Systems* **8** no. 2 (2002), 303–317.

[8] A.R. Bishop, R. Carretero-G., D.J. Frantzeskakis, P.G. Kevrekidis & B.A. Malomed, Discrete solitons and vortices on anisotropic lattices., *Phys. Rev. E*, **72** (2005), 046613-1, 046613-9.

[9] M. Brewczyk, T. Karpiiuk & K. Rzążewski, Soliton and vortices in ultracold fermionic gases, *J. Phys. B: At. Mol. Opt. Phys.* **35** (2002), L135–L321.

[10] A. Dold, *Lectures on Algebraic Topology*, *Springer-Verlag*, Berlin-Heidelberg-New York, 1972.

[11] J.I. Extremiana, L.J. Hernández & M.T. Rivas, Proper CW complexes: A category for study of proper homotopy, *Collectanea Math.* **39** (1988), 149–179.

[12] D. Figeys, On proteins, DNA and topology, *TRENDS in Biotechnology* **20** no. 6 (2002), 234–235.

[13] P.G. Goerss & J.F. Jardine, *Simplicial Homotopy Theory, progress in mathematics.* Birkhäuser Verlag. Basel-Boston-Berlin, 1999.
[14] M. Greenberg, *Lectures on Algebraic Topology*, Menlo Park, California, 1966.
[15] P. J. Hilton & S. Wylie, *An Introduction to Algebraic Topology.* Cambridge University Press, London, New York, 1967.
[16] D. Kahn, The Hurewicz Theorem, *International Symposium of Algebraic Topology*, Autonomous University of Mexico and UNESCO, (1958).
[17] Chien-Hao Liu, Topological Membrane Solitons and Loop Orders of Membrane Scatterings in M-theory. *arXiv:hep-th/9610042v1* (1996).
[18] A. Lundell & S. Weingram, *Topology of CW-complexes.* The University Series in Higher Mathematics. Editorial Board. New York-Cincinnaty-Toronto, 1969.
[19] Y. Meyer & T. Rivière, A partial regularity result for a class of stationary Yang-Mills Fields in high dimension, *Rev. Math. Iberoamericana* **19** (2003), 195–219
[20] M. R. Pakzad & T. Rivière, Weak density of smooth maps for the Dirichlet energy between manifolds, submitted (2001).
[21] H. Pijeira, Y. Quintana & W. Urbina, Zero localization and asymptotic behabior of orthogonal polynomials of Jacobi-Sobolev. *Rev. Col. Mat.* **35** no. 2 (2001), 77–97.
[22] L. C. Siebenmann, Infinite simple homotopy types, *Indag. Math.* **32** (1970), 479–495.
[23] R. Switzer, *Algebraic Topology-Homotpy and Homology*, Spinger-Verlag. Berlin-Heidelberg-New York, 1975.
[24] M. Reuter, F. Wolter & N. Peinecke, Laplace-Beltrami spectra as shape-DNA, Preprint.
[25] J. W. Vick, *Homology Theory*, Academic Press, New York and London, 1973.
[26] H. Whitney, On maps of the $n$-sphere into itself, *Duke Mathematical Journal* **3** (1937), 46–50.

(Recibido en abril de 2005. Aceptado en julio de 2005)

Escuela de Matemáticas
Facultad de Ciencias
Apartado Postal: 20513, Caracas 1020 A
Universidad Central de Venezuela
Avenida Los Ilustres, Los Chaguaramos
Caracas Venezuela
*e-mail:* dalmagro@euler.ciens.ucv.ve

Departamento de Matemáticas Puras y Aplicadas
Edificio Matemáticas y Sistemas (MYS)
Universidad Simón Bolívar
Valle de Sartenejas, Baruta
Estado Miranda, Venezuela
*e-mail:* yquintana@usb.ve