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Free k-cyclic E–lattices over a poset

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Abstract. In this note we consider a new equational class of algebras called E–lattices $\langle A, \wedge, \vee, h, 0, 1\rangle$ where $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive (0,1)–lattice and h is a lattice endomorphism. We consider the subclass $\mathbf{E}_k$ of k–cyclic E–lattices such that $h^k(x) = x$, for all x, k is a positive integer. We determine the structure of the free k–cyclic E–lattice over a poset using results obtained by L. Monteiro in [9] for the free distributive lattice over a poset.

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Resumen. En este artículo consideramos una nueva clase ecuacional de álgebras $\langle A, \wedge, \vee, h, 0, 1\rangle$ llamadas E–retículos donde $\langle A, \wedge, \vee, 0, 1\rangle$ es un retículo distributivo acotado y h es un endomorfismo de retículos. Consideramos la subclase $\mathbf{E}_k$ de E–retículos k–cíclicos tales que para cada x, $h^k(x) = x$; k es un entero positivo. Determinamos la estructura de los E–retículos k–cíclicos libres sobre un conjunto parcialmente ordenado usando resultados obtenidos por L. Monteiro en [9] para retículos distributivos libres sobre un conjunto parcialmente ordenado.

As an application to the study of switching circuits, G. Moisil introduced in [5] the symmetric Boolean algebras, that is, Bolean algebras with an automorphism of period two and later, in [6], the cyclic Boolean algebras in which the automorphism of period two is replaced by an automorphism of period k. The symmetric Boolean algebras were also studied in A. Monteiro [7], and the same author described in [8] the algebraic structure of cyclic Boolean algebras, giving in both cases the construction of the algebra with a finite set of free generators.

Our results generalize those mentioned above.

Throughout this note $\mathbf{L}$ denotes the variety of distributive (0,1)–lattices. General references for concepts and results on distributive lattices and universal algebras used in this paper are the books [2] and [3].

Definition 1. *An* E*–lattice is a pair* (A, h) *where* $A \in \mathbf{L}$ *and* h *is an* L*–endomorphism. If* $(A_1, h_1), (A_2, h_2)$ *are E–lattices, a map* $\alpha : A_1 \to A_2$ *is an* E*–homomorphism if* α *is an* L*–homomorphism such that:* $\alpha(h_1(x)) = h_2(\alpha(x))$, *for all* $x \in A_1$.

If A is an n–valued Lukasiewicz algebra ($n \neq 2$) [1], [4], then (A, s_i), $1 \leq i \leq n-1$, are E–lattices. We denote by $\mathbf{E}$ the variety of E–lattices. We study the class of k–cyclic E–lattices and we give a method to determine the free k–cyclic E–lattice over a poset using results obtained in [9].

Definition 2. *Let* k *be a fixed positive integer. We say that* (A, h) *is a* k–cyclic E*–lattice (or* E_k*–lattice) if it verifies for all* $x \in A$, $h^k(x) = x$, *where* $h^0(x) = x$ *and* $h^{n+1}(x) = h^n(h(x))$ *for every positive integer* n.

We shall denote by $\mathbf{E_k}$ the variety of E_k–lattices. In in what follows, if $\mathbf{V}$ is one of the varieties $\mathbf{L}$ or $\mathbf{E_k}$, $A \in \mathbf{V}$ and $X \subseteq A$, we shall denote by $[X]_{\mathbf{V}}$ the V–subalgebra of A generated by X. We can immediately see that:

Lemma 1. *If* $(A, h) \in \mathbf{E_k}$ *and* $X \subseteq A$ *then* $[X]_{\mathbf{E_k}} = \left[\bigcup_{j=0}^{k-1} h^j(X)\right]_{\mathbf{L}}$.

Definition 3. *Let* I *be a poset.* $\mathcal{F} \in \mathbf{V}$ *is free over* I *if the following conditions are satisfied:*

(A) *There exists an order–isomorphism* g *from* I *into* L *such that* $[g(I)]_{\mathbf{V}} = \mathcal{F}$.

(B) *Let* f *be an increasing map from* I *into* $A \in \mathbf{V}$. *Then there exists a* V*–homomorphism* h_f *from* $\mathcal{F}$ *into* A *such that* $h_f \circ g = f$.

It can be easily verified ([10], pp 24–25) that if $\mathcal{F}$ exists, it is unique up to isomorphisms and the V–homomorphism h_f in (B) is also unique.

Construction of the free k–cyclic E–lattice over a poset. Let $(I, \leq)$ be a poset. If $k = 1$, it is obvious that the free $(0,1)$–distributive lattice over the poset I together with the identity homomorphism is the free 1–cyclic E–lattice over I. Suppose $k \geq 2$ and consider the following pairwise disjoint posets $I_t = I \times t$, $1 \leq t \leq k$. The maps $a_t : I_t \to I_{t-1}$, $2 \leq t \leq k$, defined by: $a_t(i,t) = (i, t-1)$, and the map $a_1 : I_1 \to I_k$ defined by $a_1(i, 1) = (i, k)$ are order–isomorphisms.

Let $J = \sum_{t=1}^{k} I_t$ be the cardinal sum of the posets I_t, $1 \leq t \leq k$, and $a : J \to J$ the map defined by $a(j) = a_t(j)$ if $j \in I_t$, $1 \leq t \leq k$. Then a is an order–automorphism of J such that $a^k(j) = j$, for all $j \in J$, and if $j \in I_t$, $a^{t-1}(j) \in I_1$.

Let $B = \{0, 1\}$ be the Boolean algebra with two elements and C the set of all increasing maps from J into B. Then C, pointwise algebrized, is a distributive (0,1)–lattice.

If $f \in C$, let $F_f(j) = f(a(j))$, for all $j \in J$, then $F_f \in C$. Let us consider the map $h_C : C \to C$ defined by $h_C(f) = F_f$, for all $f \in C$. It is easy to see that h_C is an E–automorphism of C, such that $h_C^k(f) = f$, for all $f \in C$, then $(C, h_C) \in \mathbf{E_k}$. Let $C' = \mathcal{P}(C)$ be the set of all subsets of C and $h = h_C^{-1}$. Then $\langle C', \cap, \cup, h, \emptyset, C\rangle$ is an E–lattice. As $h^k(X) = X$, for all $X \subseteq C$, then $C' \in \mathbf{E}_k$.

We are going to construct the free E_k–lattice over the poset I_1, which is isomorphic to the free E_k–lattice over I, because I and I_1 are isomorphic posets.

Let us consider the map $g^* : J \to C'$, defined by $g^*(j) = G_j$, where $G_j = \{f \in C : F_f(j) = 1\}$. Then $G_j \in C'$, for all $j \in J$. Following L. Monteiro [9] it can be easily proved that if $i, j \in J$, then $i \leq j$ iff $g^*(i) \subseteq g^*(j)$. Let $g : I_1 \to C'$ be the restriction of g^* to the poset I_1. We are going to show that $\mathcal{F} = [g(I_1)]_{\mathbf{E}_k} \subseteq C'$ is the free k–cyclic E–lattice over the poset I_1. It is obvious that the condition (A) of Definition 3 is verified.

Let us see that $h(G_j) = G_{a(j)}$, for all $j \in J$. Indeed, if $f \in h(G_j) = h_C^{-1}(G_j)$, then $F_f = h_C(f) \in G_j$, i.e. $1 = F_f(j) = f(a(j))$, so $f \in G_{a(j)}$. In a similar way we prove that $G_{a(j)} \subseteq h^{-1}(G_j)$.

From Lemma 1, $[g(I_1)]_{\mathbf{E}_k} = \left[\bigcup_{t=0}^{k-1} h^t(g(I_1))\right]_{\mathbf{L}}$. We are going to prove that $\bigcup_{t=0}^{k-1} h^t(g(I_1)) = g^*(J)$. Indeed, $h^0(g(I_1)) = g(I_1) = g^*(I_1)$ and $h^t(g(I_1)) = \{h^t(G_i) : i \in I_1\} = \left\{G_{a^t(i)} : i \in I_1\right\} = g^*(I_{k+1-t})$, for $1 \leq t \leq k-1$. Then, by results obtained by L. Monteiro [9], $\mathcal{F}$ is the free distributive (0,1)–lattice over the poset J.

Now we shall prove condition (B). Let f be an isotone map from I_1 into a k–cyclic E–lattice (A, h_A). We define a function $U : J \to A$ by:

$$U(j) = \begin{cases} h_A^{t-1}\left(f\left(a^{t-1}(j)\right)\right), & 1 \leq t < k,\ j \in I_t\,, \\ h_A\left(f\left(a^{k-1}(j)\right)\right), & \text{if } j \in I_k\,. \end{cases}$$

Since $a^{t-1}(j) \in I_1$, for every $j \in I_t$, $1 \leq t \leq k$, we have that $U(j) \in A$. We also have that U is an increasing map, because it is a composition of increasing functions. As $\mathcal{F}$ is a free distributive (0,1)–lattice over the poset J, then there exists an L–homomorphism $h_f : \mathcal{F} \to A$ such that $h_f \circ g = U$. If $j \in I_1$ then $h_f(g(j)) = h_f(g^*(j)) = U(j) = h_A^0(f(a^0(j))) = f(j)$, so in particular $h_f \circ g = f$. In order to prove that h_f is an E–homomorphism it is sufficient to prove that $h_f(h(G_j)) = h_A(h_f(G_j))$, for all $G_j \in \mathcal{F}$. If $j \in J$ then $j \in I_t$, $1 \leq t \leq k$ and we have:

$$\begin{aligned} h_f(h(G_j)) &= h_f\left(G_{a(j)}\right) = h_f(g^*(a(j))) \\ &= (h_f \circ g^*)(a(j)) = U(a(j))\,. \end{aligned}$$

If $j \in I_t$, $2 \leq t \leq k$, then $a(j) \in I_{t-1}$, $1 \leq t-1 < k$, so

$$\begin{aligned} U(a(j)) &= h_A^{t-2}\left(f\left(a^{t-2}\left(a(j)\right)\right)\right) = h_A^{t-2}\left(f\left(a^{t-1}(j)\right)\right) \\ &= h_A\left(h_A^{t-1}\left(f\left(a^{t-1}(j)\right)\right)\right) = h_A\left(U(j)\right) \\ &= h_A\left(h_f\left(g^*(j)\right)\right) = h_A\left(h_f(G_j)\right). \end{aligned}$$

If $j \in I_1$ then $a(j) \in I_k$ and

$$U\left(a(j)\right) = h_A\left(f\left(a^k(j)\right)\right) = h_A\left(f(j)\right) = h_A\left(\left(h_f \circ g\right)(j)\right),$$

so as $j \in I_1$ $g(j) = g^*(j)$, then we have

$$U(a(j)) = h_A\left(h_f\left(g(j)\right)\right) = h_A\left(h_f\left(g^*(j)\right)\right) = h_A\left(h_f\left(G_j\right)\right).$$

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