Revista Colombiana de Matemáticas
Volumen 39 (2005), páginas 133–143

Symmetries and integration of differential equations

Gerardo Torres del Castillo
Magdalena Marciano Melchor

Universidad Autónoma de Puebla, México

Abstract. A proof of the Lie theorem which relates the symmetries of a first order differential equation (or of a linear differential form) with its integrating factors is given. It is shown that a similar result partially applies for systems of linear differential forms and ordinary differential equations of any order.

Keywords and phrases. Ordinary differential equations, symmetries.

2000 Mathematics Subject Classification. Primary: 34A26, 54H15. Secondary: 58D 19, 35F05.

Resumen. Se da una prueba del teorema de Lie que relaciona las simetrías de una ecuación diferencial de primer orden(o de una forma diferencial lineal) con su factor integrante. Se demuestra que un resultado similar parcialmente aplica para sistemas de formas diferenciales lineales y ecuaciones diferenciales ordinarias de cualquier orden.

1. Introduction

A first order ordinary differential equation can be usually expressed in the form $dy/dx = f(x, y)$, where f is some function of two variables, or, equivalently, as $Ldx+Mdy = 0$, where L and M are functions of two variables with $-L/M = f$. It may happen that the differential form, or *Pfaffian* form, $Ldx + Mdy$ is the differential of some function, that is, there exists some function of two variables, ϕ, such that $d\phi = Ldx + Mdy$, in which case it is said that $Ldx + Mdy$ is exact and the differential equation $Ldx + Mdy = 0$ amounts to $d\phi = 0$, in such a way that its solution is given simply by $\phi =$ constant.

When $Ldx + Mdy$ is not exact, there exists a function μ, called an integrating factor of $Ldx + Mdy$, such that $\mu(Ldx + Mdy)$ is exact; but finding directly the integrating factor, given L and M, can be highly involved (see, for example, [1]). Nevertheless, it turns out that knowing an integrating factor of the differential form $Ldx + Mdy$ is equivalent to knowing a one-parameter group of transformations that leaves the equation $Ldx + Mdy = 0$ invariant or, more precisely, to knowing the *infinitesimal generator* of that group. In other words, starting from the symmetries of $Ldx + Mdy$, its integrating factors can be obtained and conversely.

This correspondence is an example of the relation between the theory of groups and the methods of integration of differential equations established originally by Sophus Lie. Lie found that the methods employed in the solution of differential equations can be understood by means of the theory of groups (see, for example, [2–5]).

In this paper the relationship between the integrating factors of a differential form and the groups of transformations that leave it invariant is presented, considering the general case of a differential form in n variables, $\sum_{i=1}^{n} a_i dx^i$ (the case with $n = 2$ is considered, for example, in [3–5]). As mentioned above, the differential forms in two variables correspond to first order ordinary differential equations; the differential forms in more than two variables have application, for instance, in thermodynamics and in connection with mechanical systems with constraints. Furthermore, an ordinary differential equation of order n corresponds to a system of n differential forms in $n + 1$ variables and the integrating factors of a system of differential forms is also related with its symmetries.

In Sec. 2 the correspondence between integrating factors and one-parameter groups of invariance of a differential form in n variables is considered, including some examples. In Sec. 3 the systems of differential forms are studied and it is shown that an ordinary differential equation of order n is equivalent to a system defined by n differential forms in $n + 1$ variables (an alternative treatment of the symmetries of an ordinary differential equation of order n can be found in [3–5]).

2. Symmetries and integrating factors of a differential form

A differential form or Pfaffian form in n variables is an expression of the form $\sum_{i=1}^{n} a_i dx^i$, where $x^1, x^2, \ldots, x^n$, are n independent variables and $a_1, a_2, \ldots, a_n$ are n functions of the variables x^i. It will be assumed that the derivatives of any order of all the functions that appear in what follows exist and are continuous. Furthermore, in order to simplify the notation, it will be assumed that there exists sum over all the possible values for each index appearing twice in the same term, once as subscript and once as superscript; for example, $a_i dx^i = \sum_{i=1}^{n} a_i dx^i$.

The form $a_i dx^i$ is exact if there exists a function, ϕ, such that $a_i dx^i = d\phi$. Since the total differential of a function ϕ of n variables x^i is given by $d\phi =$

$(\partial\phi/\partial x^i)dx^i$, the fact that $a_i dx^i$ is equal to $d\phi$ is equivalent to the n equations

$$a_i = \frac{\partial \phi}{\partial x^i}\,. \tag{2.1}$$

Differentiating both sides of (2.1) with respect to x^j we have $\partial a_i/\partial x^j = \partial^2\phi/\partial x^j \partial x^i$, and since by hypothesis the derivatives of all the functions are continuous, $\partial^2\phi/\partial x^j\partial x^i = \partial^2\phi/\partial x^i\partial x^j$; hence, $a_i dx^i$ is exact if

$$\frac{\partial a_i}{\partial x^j} = \frac{\partial a_j}{\partial x^i}\,. \tag{2.2}$$

The conditions (2.2) turn out to be also sufficient for $a_i dx^i$ to be exact.

The form $a_i dx^i$ is integrable if there exists a function, μ, called an integrating factor of $a_i dx^i$, such that $\mu(a_i dx^i)$ is exact. According to (2.2), $a_i dx^i$ is integrable if and only if there exists some function μ such that

$$\frac{\partial(\mu a_i)}{\partial x^j} = \frac{\partial(\mu a_j)}{\partial x^i}\,,$$

or

$$\mu\left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i}\right) = a_j \frac{\partial \mu}{\partial x^i} - a_i \frac{\partial \mu}{\partial x^j}\,. \tag{2.3}$$

While any differential form in two variables is integrable, not all the differential forms in three or more variables are.

In order to eliminate the function μ (which is not known, if only $a_i dx^i$ is given), we multiply both sides of (2.3) by a_k and summing the resulting equation with those obtained by cyclically permuting the indices i, j, k we find that

$$a_k\left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i}\right) + a_i\left(\frac{\partial a_j}{\partial x^k} - \frac{\partial a_k}{\partial x^j}\right) + a_j\left(\frac{\partial a_k}{\partial x^i} - \frac{\partial a_i}{\partial x^k}\right) = 0\,, \tag{2.4}$$

(which only involves the functions a_i and their derivatives). Then, conditions (2.4) are necessary for $a_i dx^i$ to be integrable and it can be shown that are also sufficient (see, for example, [6]). If in (2.4) two of the indices i, j and k take the same value, the left-hand side vanishes, therefore when $n = 2$, the conditions (2.4) are satisfied for any functions a_i and any differential form is therefore integrable.

We shall consider families of transformations depending on a parameter t

$$\begin{aligned} x'^1 &= F^1(x^1, \ldots, x^n, t),\\ x'^2 &= F^2(x^1, \ldots, x^n, t),\\ &\vdots\\ x'^n &= F^n(x^1, \ldots, x^n, t), \end{aligned} \tag{2.5}$$

where the F^i are real-valued functions that depend on $n+1$ variables (throughout this paper, the symbols like x', y', $\ldots$, do not stand for derivatives but for coordinates of points after effecting a transformation). In a more compact form, the n relations (2.5) are expressed as $x'^i = F^i(x^j, t)$. The transformations (2.5) form a one-parameter group if

$$F^i(x^k, t+s) = F^i(F^j(x^k, t), s), \tag{2.6}$$

for all $t, s \in \mathbb{R}$. The relations (2.6) imply that $F^i(x^k, 0) = x^i$.

For example, if $n = 2$, using x, y in place of x^1, x^2, the transformations

$$x' = xe^{at}, \qquad y' = ye^{bt}, \tag{2.7}$$

where a and b are two fixed real numbers, form a one-parameter group of transformations since

$$xe^{a(t+s)} = (xe^{at})e^{as}, \qquad ye^{b(t+s)} = (ye^{bt})e^{bs}.$$

Other examples are given by

$$x' = x\cos t - y\sin t, \qquad y' = x\sin t + y\cos t \tag{2.8}$$

and

$$x' = x + at, \qquad y' = y + bt, \tag{2.9}$$

where a and b are arbitrary real numbers.

In the applications to be considered here it is not necessary that the relations (2.6) hold for all the values of t and s, but it is sufficient that (2.6) be satisfied for values of t and s in some neighborhood of 0 (see (2.13) below). For example,

$$x' = \frac{x}{1-tx}\,, \qquad y' = \frac{y}{1-ty}\,,$$

satisfies (2.6) since

$$\frac{x}{1-(t+s)x} = \frac{\left(\frac{x}{1-tx}\right)}{1 - s\left(\frac{x}{1-tx}\right)}\,,$$

provided that t and s are such that all the denominators are different from zero. When the relations (2.6) hold only for some values of t and s sufficiently small, it is said that the transformations (2.5) form a local one-parameter group of transformations or a flux; however, in what follows, we shall not distinguish between these two cases.

The differential form $a_i dx^i$ is invariant under a one-parameter group of transformations $x'^i = F^i(x^j, t)$ if after substituting in it each x^i by x'^i one obtains some multiple of $a_i dx^i$, that is, $a_i\left(F^j(x^k, t)\right) dF^i(x^k, t) = \lambda a_i dx^i$, where λ is some function different from zero, which can depend on the x^i and t, and in the differentials of the functions F^i, t is treated as a parameter, that is,

$$a_i\left(F^j(x^k, t)\right)\frac{\partial F^i(x^k, t)}{\partial x^p}dx^p = \lambda a_i dx^i\,. \tag{2.10}$$

Thus, if $a_i dx^i$ is invariant under a one-parameter group of transformations, the differential equation $a_i dx^i = 0$ is transformed into $\lambda a_i dx^i = 0$ or equivalently, into $a_i dx^i = 0$.

For example, under the group of transformations (2.7), the differential form

$$8xy^4dx + (3y^2 + 4x^2y^3 - 12x^4y^4)dy \tag{2.11}$$

is transformed into

$$e^{(2a+4b)t}8xy^4dx + e^{bt}\left(3y^2e^{2bt} + 4x^2y^3e^{(2a+3b)t} - 12x^4y^4e^{(4a+4b)t}\right)dy$$
$$= e^{3bt}\left[e^{(2a+b)t}8xy^4dx + \left(3y^2 + 4x^2y^3e^{(2a+b)t} - 12x^4y^4e^{2(2a+b)t}\right)dy\right]$$

which is a multiple of (2.11) if we choose $b = -2a$. It can be seen that, by contrast, the form (2.11) is not invariant under the transformations (2.8) or (2.9), for $t \neq 0$.

Differentiating both sides of (2.10) with respect to the parameter t and evaluating then at $t = 0$, by means of the chain rule, since $F^i(x^k, 0) = x^i$, we have

$$a_i\frac{\partial \xi^i}{\partial x^p}dx^p + \frac{\partial a_i}{\partial x^j}\xi^j\delta^i_p dx^p = \left.\frac{\partial \lambda}{\partial t}\right|_{t=0} a_i dx^i\,, \tag{2.12}$$

where we have introduced

$$\xi^i(x^k) \equiv \left.\frac{\partial F^i(x^k,t)}{\partial t}\right|_{t=0}\,, \tag{2.13}$$

and we have made use of $\partial x^i/\partial x^p = \delta^i_p$, where δ^i_p is the Kronecker delta ($\delta^i_p = 1$ if $i = p$ and $\delta^i_p = 0$ if $i \neq p$). Hence, making $\nu \equiv (\partial\lambda/\partial t)|_{t=0}$, changing the names of the indices in the first term, equation (2.12) amounts to $a_j(\partial\xi^j/\partial x^i)dx^i + \xi^j(\partial a_i/\partial x^j)dx^i = \nu a_i dx^i$; therefore, if the differential form $a_i dx^i$ is invariant under the group of transformations (2.5) then the functions a_i satisfy the n conditions

$$a_j\frac{\partial \xi^j}{\partial x^i} + \xi^j\frac{\partial a_i}{\partial x^j} = \nu a_i\,. \tag{2.14}$$

The vector field with components $\xi^1, \ldots, \xi^n$ is called the infinitesimal generator of the group of transformations $x'^i = F^i(x^j, t)$.

Multiplying both sides of (2.14) by a_k we obtain

$$a_j a_k\frac{\partial \xi^j}{\partial x^i} + \xi^j a_k\frac{\partial a_i}{\partial x^j} = \nu a_i a_k.$$

Since the right-hand side of this equation does not change under the interchange of the indices i and k, the same must happen with the left-hand side, thus

$$a_j a_k\frac{\partial \xi^j}{\partial x^i} + \xi^j a_k\frac{\partial a_i}{\partial x^j} = a_j a_i\frac{\partial \xi^j}{\partial x^k} + \xi^j a_i\frac{\partial a_k}{\partial x^j}$$

or, equivalently,

$$\xi^j\left(a_k\frac{\partial a_i}{\partial x^j} - a_i\frac{\partial a_k}{\partial x^j}\right) = a_j\left(a_i\frac{\partial \xi^j}{\partial x^k} - a_k\frac{\partial \xi^j}{\partial x^i}\right). \tag{2.15}$$

If now we assume that $a_i dx^i$ is integrable, from (2.4) we can find the combination appearing in the left-hand side of (2.15) and substituting that expression we find

$$\xi^j \left(a_k \frac{\partial a_j}{\partial x^i} - a_i \frac{\partial a_j}{\partial x^k} - a_j \frac{\partial a_k}{\partial x^i} + a_j \frac{\partial a_i}{\partial x^k} \right) = a_j \left(a_i \frac{\partial \xi^j}{\partial x^k} - a_k \frac{\partial \xi^j}{\partial x^i} \right),$$

that can be also written as

$$\left(\xi^j a_j\right) \frac{\partial a_k}{\partial x^i} - a_k \frac{\partial \left(\xi^j a_j\right)}{\partial x^i} = \left(\xi^j a_j\right) \frac{\partial a_i}{\partial x^k} - a_i \frac{\partial \left(\xi^j a_j\right)}{\partial x^k},$$

or, equivalently, if $\xi^j a_j \neq 0$,

$$\frac{\partial}{\partial x^i} \left(\frac{a_k}{\xi^j a_j} \right) = \frac{\partial}{\partial x^k} \left(\frac{a_i}{\xi^j a_j} \right). \tag{2.16}$$

Comparing with (2.2) it follows that the form $(\xi^j a_j)^{-1} a_i dx^i$ is exact and that

$$\mu = (\xi^j a_j)^{-1} \tag{2.17}$$

is an integrating factor of $a_i dx^i$.

Conversely, if μ is an integrating factor of $a_i dx^i$ then there exist functions $\xi^1, \ldots, \xi^n$ such that μ can be expressed in the form (2.17) and equation (2.16) is satisfied. The steps leading to (2.16) are all valid and finally one deduces the existence of some function ν such that (2.14) holds. Thus we have demonstrated the validity of the following proposition.

Proposition 2.1. *Let $a_i dx^i$ be an integrable differential form, $(\xi^1, \ldots, \xi^n)$ is the infinitesimal generator of a one-parameter group of transformations under which $a_i dx^i$ is invariant if and only if $\mu = (\xi^j a_j)^{-1}$ is an integrating factor of $a_i dx^i$.*

In the case of the group (2.7), making use of ξ and η in place of ξ^1 and ξ^2, respectively, from (2.7) and (2.13) we obtain

$$\xi = \left.\frac{\partial(xe^{at})}{\partial t}\right|_{t=0} = ax, \qquad \eta = \left.\frac{\partial(ye^{bt})}{\partial t}\right|_{t=0} = by.$$

Recalling that the form (2.11) is invariant under the group (2.7) if $b = -2a$, from (2.11) and (2.17) one finds that $\mu = [ax(8xy^4) + by(3y^2 + 4x^2y^3 - 12x^4y^4)]^{-1} = [-6ay^3(4x^4y^2 - 1)]^{-1}$ is an integrating factor of (2.11). In effect, we have

$$\frac{8xy^4 dx + (3y^2 + 4x^2y^3 - 12x^4y^4)dy}{-6ay^3(4x^4y^2 - 1)} = d\left[-\frac{1}{6a} \ln \left(y^3 \frac{1 + 2x^2y}{1 - 2x^2y} \right) \right],$$

so that the solution of the differential equation $8xy^4 dx + (3y^2 + 4x^2y^3 - 12x^4y^4)dy = 0$ is given by $y^3(1 + 2x^2y)/(1 - 2x^2y) = \text{constant}$.

A second example, for an arbitrary value of n, is given in the case where the coefficients $a_1, \ldots, a_n$ of the differential form $a_i dx^i$ are homogeneous functions of the same degree k (that is, $a_i(\lambda x^1, \ldots, \lambda x^n) = \lambda^k a_i(x^1, \ldots, x^n)$ para all

$\lambda \in \mathbb{R}$). Any form of this class is invariant under the transformations $x'^i = x^i e^t$ since $a_i(x'^j)dx'^i = a_i(x^j e^t)d(x^i e^t) = e^{(k+1)t} a_i dx^i$. The components of the infinitesimal generator of this group are $\xi^i = \partial(x^i e^t)/\partial t|_{t=0} = x^i$, hence $\mu = (x^j a_j)^{-1}$ is an integrating factor of $a_i dx^i$, if it is integrable. Accordingly, the form $(z-y)zdx+(x+z)zdy+x(x+y)dz$, which is integrable as can be seen verifying that conditions (2.4) are satisfied, has an integrating factor given by $\mu = [x(z-y)z + y(x+z)z + zx(x+y)]^{-1} = [(x+y)(x+z)z]^{-1}$. In effect, one can verify that

$$\frac{(z-y)zdx + (x+z)zdy + x(x+y)dz}{(x+y)(x+z)z} = d\ln\frac{(x+y)z}{x+z}.$$

As an example of the application of the foregoing Proposition to find the symmetries of a differential form we shall consider the differential form

$$[P(x)y - Q(x)]dx + dy, \tag{2.18}$$

which corresponds to the linear inhomogeneous differential equation of first order $dy/dx + P(x)y = Q(x)$. By inspection one finds that

$$[P(x)y - Q(x)]dx + dy = e^{-\int^x P(u)du} d\left[ye^{\int^x P(u)du} - \int^x Q(u)e^{\int^u P(v)dv}du\right]$$

hence, an integrating factor of (2.18) is $\mu = e^{\int^x P(u)du}$ hence $e^{-\int^x P(u)du} = \xi^j a_j = \xi^1[P(x)y - Q(x)] + \xi^2$, which implies that

$$(\xi^1, \xi^2) = (\xi^1, e^{-\int^x P(u)du} - \xi^1[P(x)y - Q(x)])$$

is the infinitesimal generator of transformations that leave invariant the form (2.18). Taking, for simplicity, $\xi^1 = 0$, and comparing with (2.13), one finds that the group generated by $(\xi^1, \xi^2) = (0, e^{-\int^x P(u)du})$, is

$$x' = x, \qquad y' = te^{-\int^x P(u)du} + y.$$

3. Symmetries and integrating factors of systems of differential forms

Now we shall consider systems formed by m differential forms in n variables ($m < n$), $a_i^{(1)}dx^i, \ldots, a_i^{(m)}dx^i$ ($a_i^{(j)}$ is the i-th coefficient of the j-th form of the system) such that at each point the matrix formed by the functions $a_j^{(i)}$ has rank m. The system of forms $a_i^{(1)}dx^i, \ldots, a_i^{(m)}dx^i$ is integrable if there exist m^2 functions, $M_{(j)}^{(i)}$, $(i, j = 1, \ldots, m)$ such that $\det(M_{(j)}^{(i)}) \neq 0$ and the forms $M_{(j)}^{(k)} a_i^{(j)} dx^i$ are exact, that is, there exist m functions, $\phi^{(1)}, \ldots, \phi^{(m)}$, such that

$$M_{(j)}^{(k)} a_i^{(j)} dx^i = d\phi^{(k)}. \tag{3.1}$$

Then, the solution of the system of equations $a_i^{(1)}dx^i = 0, \ldots, a_i^{(m)}dx^i = 0$ is given by $\phi^{(1)} = \text{constant}, \ldots, \phi^{(m)} = \text{constant}$. As in the case where we have a single differential form, not all systems of differential forms are integrable, but any system of $n-1$ differential forms in n variables is integrable.

The system $a_i^{(1)}dx^i, \ldots, a_i^{(m)}dx^i$ is invariant under a one-parameter group of transformations if there exist functions $\Lambda^{(j)}_{(p)}$ such that

$$a_i^{(j)}\left(F^j(x^k,t)\right) dF^i(x^k,t) = \Lambda^{(j)}_{(p)} a_i^{(p)} dx^i\,, \tag{3.2}$$

(recall that there is summation over repeated indices). Thus, proceeding as in the previous section, from (3.2) it follows that

$$a_j^{(k)}\frac{\partial \xi^j}{\partial x^i} + \xi^j \frac{\partial a_i^{(k)}}{\partial x^j} = N^{(k)}_{(p)} a_i^{(p)}, \tag{3.3}$$

where $\xi^1, \ldots, \xi^n$ are the components of the infinitesimal generator of the group of transformations and the function $N^{(j)}_{(p)}$ is the partial derivative of $\Lambda^{(j)}_{(p)}$ with respect to t, evaluated at $t = 0$.

An ordinary differential equation of order n, $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$, is equivalent to a system of differential equations given by n differential forms in $n+1$ variables which, as mentioned above, is integrable. Making $x^1 = x$, $x^2 = y$, $x^3 = dy/dx$, $\ldots$, $x^{n+1} = y^{(n-1)}$, the equation $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$ is equivalent to the system

$$\begin{aligned} dx^2 - x^3 dx^1 &= 0, \\ dx^3 - x^4 dx^1 &= 0, \\ &\vdots \\ dx^n - x^{n+1} dx^1 &= 0, \\ dx^{n+1} - f dx^1 &= 0. \end{aligned} \tag{3.4}$$

For example, the system of differential forms in three variables

$$\begin{aligned} a_i^{(1)} dx^i &= dy - z dx, \\ a_i^{(2)} dx^i &= y dz + z^2 dx, \end{aligned} \tag{3.5}$$

which corresponds to the second-order ordinary differential equation $y(d^2y/dx^2) + (dy/dx)^2 = 0$, is invariant under the one-parameter group of transformations

$$x' = x + t, \qquad y' = y, \qquad z' = z\,, \tag{3.6}$$

(for convenience we use here the notation $x = x^1$, $y = x^2$, $z = x^3$), whose infinitesimal generator is $(\xi^1_{(1)}, \xi^2_{(1)}, \xi^3_{(1)}) = (1, 0, 0)$, and is also invariant under the group of transformations

$$x' = xe^t, \qquad y' = y, \qquad z' = ze^{-t}, \tag{3.7}$$

whose infinitesimal generator is $(\xi^1_{(2)}, \xi^2_{(2)}, \xi^3_{(2)}) = (x, 0, -z)$.

In the case of a system of differential forms the analog of the Proposition of the preceding section holds partially. If the system of forms $a_i^{(1)}dx^i, \ldots, a_i^{(m)}dx^i$ is integrable then the inverse of the matrix $(M^{(i)}_{(j)})$ appearing in (3.1) can be expressed in the form

$$(M^{-1})^{(i)}_{(j)} = \xi^k_{(j)} a^{(i)}_k \tag{3.8}$$

[cf. (2.17)]. For each value of j, with $1 \leqslant j \leqslant m$, the functions $\xi^1_{(j)}, \ldots, \xi^n_{(j)}$ are the components of the infinitesimal generator of a one-parameter group of transformations that leave invariant the system $a_i^{(1)}dx^i, \ldots, a_i^{(m)}dx^i$.

For example, the system of differential forms in three variables

$$\begin{aligned} a_i^{(1)}dx^i &= dy - zdx, \\ a_i^{(2)}dx^i &= dz - ydx, \end{aligned} \tag{3.9}$$

which corresponds to the equation $d^2y/dx^2 = y$, is integrable and it can be verified that

$$\begin{bmatrix} a_i^{(1)}dx^i \\ a_i^{(2)}dx^i \end{bmatrix} = \begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix} \begin{bmatrix} d\big(e^{-x}(y+z)/2\big) \\ d\big(e^{x}(y-z)/2\big) \end{bmatrix}. \tag{3.10}$$

The elements of the matrix 2×2 in this last equation have the form (3.8) with $\left(\xi^1_{(1)}, \xi^2_{(1)}, \xi^3_{(1)}\right) = (0, e^x, e^x)$, which is the infinitesimal generator of the group of transformations $x' = x$, $y' = y + e^x t$, and $z' = z + e^x t$ and $\left(\xi^1_{(2)}, \xi^2_{(2)}, \xi^3_{(2)}\right) = (0, e^{-x}, -e^{-x})$ that generates the transformations $x' = x$, $y' = y + e^{-x}t$, and $z' = z - e^{-x}t$. It can be seen directly that the system (3.9) is, in effect, invariant under these two groups of transformations. From (3.10) it also follows that the solution of the equations $dy - zdx = 0$, $dz - ydx = 0$ [see (3.9)] is given by $e^{-x}(y+z)/2 = c_1$, $e^x(y-z)/2 = c_2$, where c_1 and c_2 are two constants; hence $y = c_1 e^x + c_2 e^{-x}$.

The assertion above can be demonstrated in the following manner. From (3.1) and (3.8) we have

$$a_i^{(j)} = (M^{-1})^{(j)}_{(k)} \frac{\partial \phi^{(k)}}{\partial x^i} = \xi^p_{(k)} a_p^{(j)} \frac{\partial \phi^{(k)}}{\partial x^i}\,, \tag{3.11}$$

hence

$$\xi^i_{(q)} a_i^{(j)} = \xi^p_{(k)} a_p^{(j)} \xi^i_{(q)} \frac{\partial \phi^{(k)}}{\partial x^i}\,,$$

which is equivalent to

$$\xi^i_{(q)} \frac{\partial \phi^{(k)}}{\partial x^i} = \delta^{(k)}_{(q)}. \tag{3.12}$$

Substituting (3.11) into the left-hand side of the condition (3.3) we have

$$\begin{aligned} a_j^{(k)}\frac{\partial \xi^j_{(q)}}{\partial x^i} + \xi^j_{(q)}\frac{\partial a_i^{(k)}}{\partial x^j} &= \xi^r_{(p)}a_r^{(k)}\frac{\partial \phi^{(p)}}{\partial x^j}\frac{\partial \xi^j_{(q)}}{\partial x^i} + \xi^j_{(q)}\frac{\partial}{\partial x^j}\left(\xi^r_{(p)}a_r^{(k)}\frac{\partial \phi^{(p)}}{\partial x^i}\right) \\ &= \xi^j_{(q)}\frac{\partial}{\partial x^j}\left(\xi^r_{(p)}a_r^{(k)}\right)\frac{\partial \phi^{(p)}}{\partial x^i} + \xi^j_{(q)}\xi^r_{(p)}a_r^{(k)}\frac{\partial^2 \phi^{(p)}}{\partial x^j \partial x^i} + \\ &\quad \xi^r_{(p)}a_r^{(k)}\frac{\partial \phi^{(p)}}{\partial x^j}\frac{\partial \xi^j_{(q)}}{\partial x^i} \qquad (3.13) \\ &= \xi^j_{(q)}\frac{\partial}{\partial x^j}\left(\xi^r_{(p)}a_r^{(k)}\right)\frac{\partial \phi^{(p)}}{\partial x^i} + \xi^r_{(p)}a_r^{(k)}\frac{\partial}{\partial x^i}\left(\xi^j_{(q)}\frac{\partial \phi^{(p)}}{\partial x^j}\right). \end{aligned}$$

The last term of (3.13) vanishes as a consequence of (3.12), therefore, using again (3.1), we obtain

$$a_j^{(k)}\frac{\partial \xi^j_{(q)}}{\partial x^i} + \xi^j_{(q)}\frac{\partial a_i^{(k)}}{\partial x^j} = \xi^j_{(q)}\frac{\partial}{\partial x^j}\left(\xi^r_{(p)}a_r^{(k)}\right)M^{(p)}_{(s)}a_i^{(s)},$$

and comparing with (3.3) one concludes that, effectively, $\xi^1_{(q)}, \ldots, \xi^n_{(q)}$ are the components of the infinitesimal generator of a one-parameter group of transformations that leave invariant the system $a_i^{(1)}dx^i, \ldots, a_i^{(m)}dx^i$.

By contrast with the case where m is equal a 1, the converse of the preceding result is not valid in general. For example, the system (3.5) is invariant under the groups of transformations (3.6) and (3.7). Calculating the matrix (3.8) one finds that

$$\begin{bmatrix} (M^{-1})^{(1)}_{(1)} & (M^{-1})^{(1)}_{(2)} \\ (M^{-1})^{(2)}_{(1)} & (M^{-1})^{(2)}_{(2)} \end{bmatrix} = \begin{bmatrix} -z & -xz \\ z^2 & xz^2 - yz \end{bmatrix},$$

and a straightforward computation shows that the inverse of this matrix is

$$(M^{(i)}_{(j)}) = \frac{1}{yz}\begin{bmatrix} xz - y & x \\ -z & -1 \end{bmatrix}.$$

Then, substituting the expressions (3.5),

$$\begin{aligned} \frac{1}{yz}\begin{bmatrix} xz - y & x \\ -z & -1 \end{bmatrix}\begin{bmatrix} a_i^{(1)}dx^i \\ a_i^{(2)}dx^i \end{bmatrix} &= \frac{1}{yz}\begin{bmatrix} yzdx + (xz-y)dy + xydz \\ -zdy - ydz \end{bmatrix} \\ &= \frac{1}{yz}\begin{bmatrix} d(xyz - y^2/2) \\ d(-yz) \end{bmatrix} \qquad (3.14) \end{aligned}$$

[cf. (3.1)]. While the second of the entries in the last expression is an exact differential, $\frac{1}{yz}d(-yz) = d\ln(yz)^{-1}$, the first entry is not an exact differential, but only integrable.

From (3.14) it follows that the solution of the differential equation $y(d^2y/dx^2) + (dy/dx)^2 = 0$ is given by $xyz - y^2/2 = c_1$, $-yz = c_2$, where c_1 and c_2 are two constants. Substituting the second of these last equalities in the first one we have, $-c_2x - y^2/2 = c_1$, therefore, finally, $y = [-2(c_1 + c_2x)]^{1/2}$.

4. Concluding remarks

Usually the subject of integrable equations and integrating factors is only briefly considered in the texts on differential equations, in view of the difficulty to find the integrating factors in a direct manner. However, from the results of Sec. 2 it follows that for any first-order ordinary differential equation an integrating factor can be found if some one-parameter group of transformations that leave invariant the corresponding differential form is identified. As shown in this paper, any integrable differential form, or system of m differential forms, always has symmetries, with m one-parameter groups of transformations that leave invariant the system.

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(Recibido en agosto de 2005. Aceptado en noviembre de 2005)

Departamento de Física Matemática
Instituto de Ciencias
Universidad Autónoma de Puebla
Apartado postal 1152
72001 Puebla, México.
e-mail: gtorres@fcfm.buap.mx

Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Apartado postal 1152
72001 Puebla, México
e-mail: est068@fcfm.buap.mx