Revista Colombiana de Matemáticas
Volumen 39 (2005), páginas 145–153

On the Nonlinear Mechanical Response of an Arterial Wall

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Abstract. The nonlinear mechanical behavior of an arterial wall under the effect of a perivascular medium (muscular and areolar tissue) is investigated. The artery is modeled by an anisotropic, incompressible, nonlinear, elastic, two-layer, thick-walled tube under torsion, tension, and inflation. The arterial vessel is assumed to be surrounded by an elastic medium with the property of Winkler's foundation. An analytical expression for the components of the stress tensor is established. Comparative analysis of perivascular media is carried out.

Keywords and phrases. Deformation of a blood vessel, Model of an arterial wall, Nonlinear anisotropic theory of elasticity, Large deformations.

2000 Mathematics Subject Classification. Primary: 92C10. Secondary: 74B20.

Resumen. Se investiga el comportamiento mecánico no lineal de una pared arterial bajo el efecto de un medio perivascular (tejido areolar y muscular). La arteria es modelada por un tubo no-isotrópico no-comprimible, no lineal, de dos capas y de paredes gruesas bajo torsión tensión e inflación. Se asume que el vaso está rodeado por un medio elástico con soporte de tipo Winkler. Se establece una expresión analítica para las componentes de el tensor de tensión. Se lleva a cabo un análisis comparativo del medio perivacular.

There are various mechanical models of an arterial wall, within the framework of the linear isotropic and nonlinear anisotropic theories of elasticity, that describe its real properties. Reviews of such models are presented in B.A. Purinja and V.A. Kasyanov [6] and G.A. Holzapfel *et al* [2]. But the models which take into account the effects of perivascular tissue on the mechanical behavior of an arterial wall are still not widely accepted. The goal of this paper is to extend the model of the mechanical response of an arterial wall proposed in [2] when the effect of perivascular tissue is taken into account. An artery is represented as a thick-walled tube consisting of two layers: the media, a middle layer of an artery, and the adventitia, an external layer. These are its main mechanically important components. The third layer, the intima, is neglected. Each layer is represented by a non-collagenous matrix treated as an isotropic material, and by two families of collagen fibres helically wound along the arterial axis and symmetric with respect to the axis; however, the families have different orientations in the two layers. The fibres induce the anisotropy of the material response. The mechanical response is given according to the theory of fiber-reinforced solids. As the volume of an artery does not change within the physiological range of deformation, the material is considered to be incompressible.

The artery is modeled by a two-layer nonlinear elastic tube with a screw anisotropy that contains a disclination (Fig. 1). The muscular and areolar tissue environments of a vessel are represented by an elastic media having the properties of Winkler's foundation [7,8]. The tube-cylinder is subjected to stretching, inflating, and twisting. We will consider the case when the environment only resists against radial displacements of the vessel.

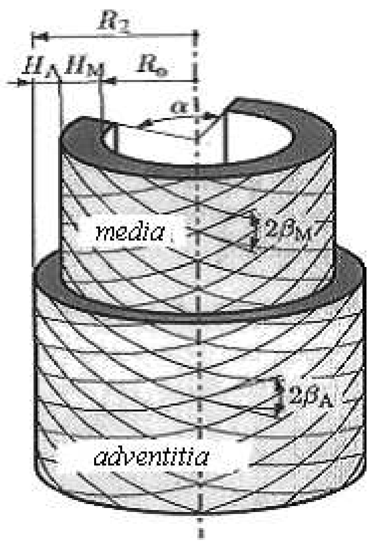

FIGURE 1. An artery as a two-layer cylinder.

The deformation of the cylinder is given by the relations

$$\begin{cases} R = R(r) \\ \Phi = \kappa\varphi + \psi z \\ Z = \lambda z \end{cases} \tag{0.1}$$
$$r_0 \le r \le r_2, 0 \le \varphi \le 200^\circ \,,$$

here (r, φ, z) and (R, Φ, Z) are cylindrical coordinates in the reference and deformed configurations respectively, ψ and λ are loading parameters, and κ is a disclination parameter. The deformation gradient is [3]

$$\mathbf{C} = R'\mathbf{e}_r\mathbf{e}_R + \frac{R}{r}\kappa\mathbf{e}_\varphi\mathbf{e}_\Phi + R\psi\mathbf{e}_z\mathbf{e}_\Phi + \lambda\mathbf{e}_z\mathbf{e}_Z\,, \tag{0.2}$$

where $(\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z)$ and $(\mathbf{e}_R, \mathbf{e}_\Phi, \mathbf{e}_Z)$ are the unit bases associated with (r, φ, z) and (R, Φ, Z), respectively. We will also use the following relations:

$$\begin{aligned}
\mathbf{e}_R &= \mathbf{e}_r\cos(\Phi-\varphi) + \mathbf{e}_\varphi\sin(\Phi-\varphi),\\
\mathbf{e}_\Phi &= -\mathbf{e}_r\sin(\Phi-\varphi) + \mathbf{e}_\varphi\cos(\Phi-\varphi),\\
\mathbf{e}_Z &= \mathbf{e}_z.
\end{aligned}$$

The Cauchy–Green tensor is

$$\begin{aligned}
\mathbf{G} = \mathbf{C}\cdot\mathbf{C}^{\mathbf{T}} = (R')^2\mathbf{e}_r\mathbf{e}_r + \left(\frac{\kappa R}{r}\right)^2\mathbf{e}_\varphi\mathbf{e}_\varphi + \\
\frac{R^2\psi\kappa}{r}(\mathbf{e}_\varphi\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\varphi) + \lambda^2\mathbf{e}_z\mathbf{e}_z\,.
\end{aligned} \tag{0.3}$$

The components of the deformation gradient and those of the Cauchy–Green tensor depend only on the radial coordinate r.

The cylinder material is assumed to be anisotropic and incompressible. The strain energy function [2] of deformation is given by

$$\begin{aligned}
&W_j = \frac{C_j}{2}(I_1-3) + \frac{k_{1j}}{2k_{2j}}\left\{\exp\left[k_{2j}\left(I_{4j}-1\right)^2\right] + \exp\left[k_{2j}\left(I_{6j}-1\right)^2\right] - 2\right\},\\
&I_{4j} = \mathbf{A}_{1j}\circ\mathbf{G}, I_{6j} = \mathbf{A}_{2j}\circ\mathbf{G}, I_1 = \operatorname{tr}\mathbf{G},\\
&\mathbf{A}_{1j} = \mathbf{a}_{1j}\mathbf{a}_{1j}, \mathbf{A}_{2j} = \mathbf{a}_{2j}\mathbf{a}_{2j},\\
&\mathbf{a}_{1j} = \begin{pmatrix} 0 \\ \cos\beta_j \\ \sin\beta_j \end{pmatrix},\ \mathbf{a}_{2j} = \begin{pmatrix} 0 \\ \cos\beta_j \\ -\sin\beta_j \end{pmatrix},
\end{aligned} \tag{0.4}$$

where $j =$ A,M for the adventitia and the media respectively, and β_j are the angles between the collagen fibers and the circumferential direction in the media and adventitia.

In what follows we shall omit the index j, as all formulas and transformations for both layers are identical.

Using the decomposition of the first Piola–Kirchhoff stress tensor

$$\begin{aligned}
\mathbf{D} = \mathbf{D}_{rR}\mathbf{e}_r\mathbf{e}_R + \mathbf{D}_{r\Phi}\mathbf{e}_r\mathbf{e}_\Phi + \mathbf{D}_{rZ}\mathbf{e}_r\mathbf{e}_\Phi + \mathbf{D}_{\varphi R}\mathbf{e}_\varphi\mathbf{e}_R + \mathbf{D}_{\varphi\Phi}\mathbf{e}_\varphi\mathbf{e}_\Phi\\
+ \mathbf{D}_{\varphi Z}\mathbf{e}_\varphi\mathbf{e}_\Phi + \mathbf{D}_{zR}\mathbf{e}_z\mathbf{e}_R + \mathbf{D}_{z\Phi}\mathbf{e}_z\mathbf{e}_\Phi + \mathbf{D}_{zZ}\mathbf{e}_z\mathbf{e}_Z
\end{aligned}$$

we transform the equilibrium equations

$$\overset{\circ}{\nabla}\cdot\mathbf{D} = 0, \quad \overset{\circ}{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi} + \mathbf{e}_z\frac{\partial}{\partial z} \tag{0.5}$$

	Material	Geometry
Media	$C_M = 3$ kPa $k_{1M} = 2.3632$ kPa $k_{2M} = 0.8393$	$H_M = 0.26$ mm $\beta_M = 29°$ $r_0 = 0.71$ mm $\kappa = 1.8$
Adventitia	$C_A = 0.3$ kPa $k_{1A} = 0.5620$ kPa $k_{2A} = 0.7112$	$H_M = 0.13$ mm $\beta_M = 62°$ $r_2 = 1.1$ mm $\kappa = 1.8$

TABLE 1. Material and geometrical data for a carotid of a rabbit [2].

to the scalar form

$$\begin{gathered}\frac{\partial D_{rR}}{\partial r} + \frac{\partial D_{zR}}{\partial z} + \frac{D_{rR} - D_{\varphi\Phi}}{r} - \psi D_{z\Phi} = 0\,,\\ \frac{\partial D_{r\Phi}}{\partial r} + \frac{1}{r}\frac{\partial D_{\varphi\Phi}}{\partial \varphi} + \frac{\partial D_{z\Phi}}{\partial z} + \frac{D_{\varphi R} - D_{r\Phi}}{r} + \psi D_{zR} = 0\,,\\ \frac{\partial D_{rZ}}{\partial r} + \frac{1}{r}\frac{\partial D_{\varphi}\mathrm{Z}}{\partial \varphi} + \frac{\partial D_{zZ}}{\partial z} + \frac{1}{r}D_{rZ} = 0\,.\end{gathered} \tag{0.6}$$

These equations are supplemented by the boundary conditions on the external and internal surfaces of the vessel. On the internal surface the action of blood pressure is equivalent to a distributed normal loading. So the boundary condition is

$$R' D_{rR}|_{r=r_0} = -f, \tag{0.7}$$

where f is the intensity of pressure on the unit area of the deformed solid. The condition on the external surface represents the properties of the perivascular tissue:

$$D_{rR}|_{r=r_2} = C_A \tilde{k} \left(\frac{R(r)}{r} - 1 \right)\Bigg|_{r=r_2} \quad , \tag{0.8}$$

where $\tilde{k}$ is a dimensionless coefficient that describes the properties of the elastic foundation.

From the condition of incompressibility it follows that

$$R(r) = \sqrt{R_0^2 + \frac{1}{\kappa\lambda}(r^2 - r_0^2)}, \tag{0.9}$$

where $R_0 = R(r_0)$ is the inner radius of the vessel after deformation.

Following relations [3], let us use for the second Piola–Kirchhoff stress tensor the form

$$\mathbf{P} = 2\frac{dW}{d\mathbf{G}} - p\mathbf{G}^{-1}, \tag{0.10}$$

where p is a function of the hydrostatic pressure in the incompressible material. Since it is a function of the radial coordinate r only, it does not depend on deformation.

From the form of the energy function (0.4), the relations (0.10),

$$\mathbf{D} = \mathbf{P} \cdot \mathbf{C}\,, \tag{0.11}$$

and the restrictions on the function p, it follows that the components of the first Piola–Kirchhoff stress tensor that have the form

$$\begin{aligned}
D_{rR} &= \left(2\frac{\partial W}{\partial G_{rr}} - pG^{-1}_{rr}\right) R' \\
D_{\varphi\Phi} &= \left(2\frac{\partial W}{\partial G_{\varphi\varphi}} - pG^{-1}_{\varphi\varphi}\right) \frac{\kappa R}{r} + \left(\frac{\partial W}{\partial G_{\varphi z}} - pG^{-1}_{\varphi z}\right) R\psi \\
D_{z\Phi} &= \left(\frac{\partial W}{\partial G_{\varphi\ z}} - pG^{-1}_{\varphi z}\right) \frac{\kappa R}{r} + \left(2\frac{\partial W}{\partial G_{zz}} - pG^{-1}_{zz}\right) R\psi \\
D_{\varphi Z} &= \left(\frac{\partial W}{\partial G_{\varphi z}} - pG^{-1}_{\varphi z}\right) \lambda \\
D_{zZ} &= \left(2\frac{\partial W}{\partial G_{zz}} - pG^{-1}_{zz}\right) \lambda
\end{aligned} \tag{0.12}$$

are functions of r only. Therefore two of three equilibrium equations are satisfied identically, and the third becomes

$$\frac{dD_{rR}}{dr} + \frac{D_{rR} - D_{\varphi\Phi}}{r} - \psi D_{z\Phi} = 0. \tag{0.13}$$

Solving the boundary-value problem

$$\begin{aligned}
&\frac{dD_{rR}}{dr} + \frac{D_{rR} - D_{\varphi\Phi}}{r} - \psi D_{z\Phi} = 0 \\
&D^{A}_{rR}\big|_{r=r_2} = C_A \tilde{k} \left(\tfrac{R(r)}{r} - 1\right)\Big|_{r=r_2}
\end{aligned} \tag{0.14}$$

with the boundary condition on the internal surface

$$R' D^{M}_{rR}\big|_{r=r_0} = -f \tag{0.15}$$

we find the internal deformed radius of the cylinder R_0 and the function of hydrostatic pressure $p(r)$ in the incompressible material. The $D_{ij}(r)$ are piecewise continuous functions of the form

$$D_{ij}(r) = \begin{cases} D^{M}_{ij}(r), r_0 < r < r_1 \\ \\ D^{A}_{ij}(r), r_1 < r < r_2 \end{cases},$$

where M denotes the media and A the adventitia.

The physiological condition of a vessel is characterized by the parameter values $f = 13.33$ kPa and $\lambda = 1.7$.

The elastic material defined by the energy function (0.4) has a screw anisotropy, and is not affected by the interaction of torsion and radial deformations. This means that at $\psi = 0$ in the cylinder the shear stresses $D_{\varphi R}$, $D_{r\Phi}$, D_{zR}, and D_{rZ} are absent. Thus, the problem of inflating and axial stretching of the cylinder can be separated from the torsion problem. Torsion will arise only if a twisting couple is applied at the cylinder faces.

Numerical solution of the problem is carried out in the absence of cylinder torsion, $\psi = 0$. Figures 2 and 3 represent the distribution of normal stresses for various values of the loading parameters λ and k.

Qualitative analysis of the distribution of stresses for various λ shows that the main contribution to the mechanical properties of a vessel wall is seen in the vicinity of the media. The increase in stretching is due mainly to the adventitia. The adventitia's elastic properties show more influence on the stress D_{zZ} than on the other two components. At $\lambda = 1.9$, the value of D_{zZ} in the adventitia takes a maximum in the media twice approximately. Thus, the external layer of a vessel protects it from breaking during stretching. This complies fully with the experimental data [2,6].

It is known that the reason for destruction of the internal surface of a vessel, as well as the occurrence of atherosclerosis, increases with an increase in the level of stress in the internal layer of the vessel wall. Is of interest to investigate the influence of the rigidity of perivascular tissue on the value of stress in the internal wall of a vessel. A comparison of the stress values shows that the effect of the external elastic environment results in a reduction in the calculated values of the normal stresses in the internal wall of the vessel (Table 2). Note

		$\lambda = 1.7$	$\lambda = 1.8$	$\lambda = 1.9$
$k = 10$	D_{rR}	2.22	0.12	1.6
	$D_{\varphi\Phi}$	4.85	2.43	4.66
	D_{zZ}	4.44	2.22	4.24
$k = 100$	D_{rR}	2.76	1.37	3.76
	$D_{\varphi\Phi}$	25.36	25.07	31.05
	D_{zZ}	23.34	23.04	28.45
$k = 200$	D_{rR}	3.51	5.5	8.384
	$D_{\varphi\Phi}$	50.64	56.25	66.34
	D_{zZ}	46.9	52.26	61.68

TABLE 2. Reduction of stresses in the internal wall of a vessel in percentage terms when $k = 0$ (absolute values).

that the presence of the elastic base results in an increase in the values of the radial components D_{rR} in the external wall of a vessel, while the values of the distribution of $D_{\varphi\Phi}$ and the axial component D_{zZ} decrease.

Considering the influence of the rigidity coefficient of the environment on the values of the stresses, it is possible to conclude that the values k that correspond to the properties of a real perivascular tissue lie in the interval $(8, 120)$.

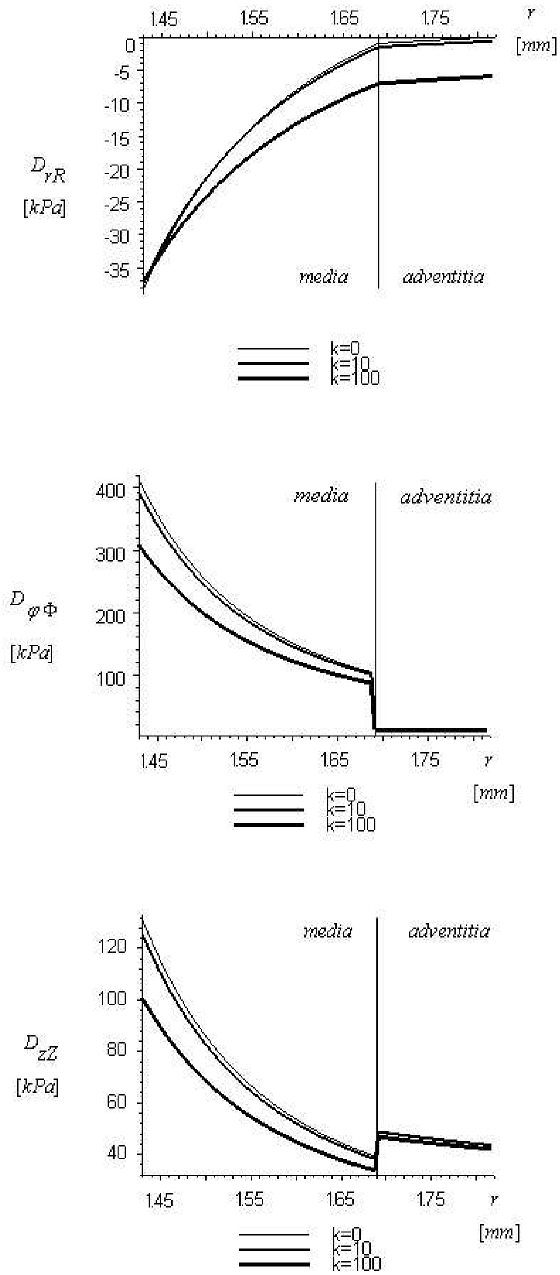

FIGURE 2. Distribution of normal stresses on the radius of the cylinder for various values of k when $\lambda = 1.7$.

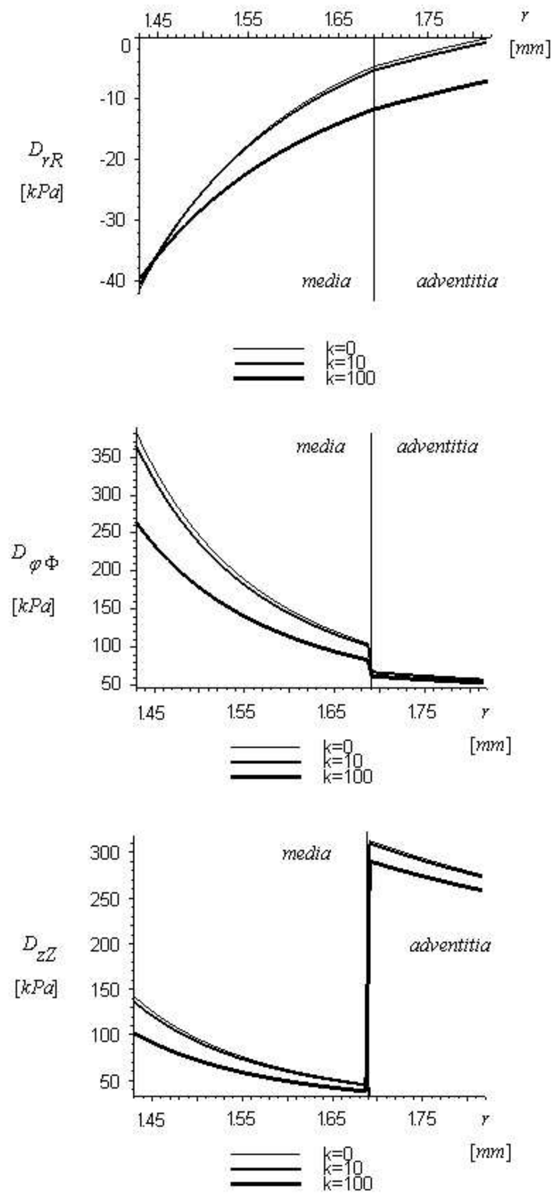

FIGURE 3. Distribution of normal stresses on the radius of the cylinder for various values of k when $\lambda = 1.9$.

Acknowledgement. I would like to thank Professor L.M. Zubov for many helpful discussions. I am indebted to Prof. Michael J. Cloud, Lawrence Technological University for assistance in preparation of this paper.

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(Recibido en septiembre de 2005. Aceptado en noviembre de 2005)

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