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The analytic fixed point function II

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Abstract. Let φ be analytic in the unit disk $\mathbb{D}$ and let $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) \neq 0$. Then $w = z/\varphi(z)$ has an analytic inverse $z = f(w)$ for $w \in \mathbb{D}$, the fixed point function. This paper studies the case that $\varphi(1) = \varphi'(1) = 1$ with a growth condition for $\varphi''(x)$ and determines the asymptotic behaviour of various combinations of the coefficients of φ connected with f. The results can be interpreted in various contexts of probability theory.

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Resumen. Sea φ analítica en el disco unitario $\mathbb{D}$ y $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) \neq 0$. Entonces $w = z/\varphi(z)$ tiene una inversa analítica $z = f(w)$ para $w \in \mathbb{D}$, la función de punto fijo. Este artículo estudia el caso en que $\varphi(1) = \varphi'(1) = 1$ con una condición de crecimiento para $\varphi''(x)$ y determina el comportamiento asintótico de varias combinaciones de los coeficientes de φ conectados con f. Los resultados se pueden interpretar en varios contextos de la teoría de la probabilidad.

1. Introduction

Let the function φ be analytic in the unit disk $\mathbb{D}$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) \neq 0$. In [MePo05, Sec. 3] it was shown that there is a unique function f that maps $\mathbb{D}$ conformally onto a starlike domain F in $\mathbb{D}$ and satisfies $f(0) = 0$,

$$w\,\varphi(f(w)) = f(w) \qquad \text{for } w \in \mathbb{D}. \tag{1.1}$$

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Thus $z = f(w)$ is the inverse function of $w = z/\varphi(z)$. We call f the *fixed point function* of φ because $f(w)$ is the unique fixed point of $w\varphi$ in $\mathbb{D}$.

The fixed point function f has a continuous and injective extension to $\overline{\mathbb{D}}$, see [MePo05, Th. 3.2]. Furthermore [MePo05, Th. 2.2] we have

$$\partial\mathbb{D} \cap \partial F = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1,\ |\varphi'(\zeta)| \le 1\} \tag{1.2}$$

where $\varphi(\zeta)$ and $\varphi'(\zeta)$ are angular limits [Po92, Sect. 4.3]. It follows from (1.1) by differentiation that

$$w\,\frac{f'(w)}{f(w)} = \frac{1}{1 - w\,\varphi'(f(w))} = \frac{1}{1 - z\,\varphi'(z)/\varphi(z)} \tag{1.3}$$

for $z = f(w)$, $w \in \mathbb{D}$.

We shall restrict ourselves to the case that $\varphi(1) = 1$ and $\varphi'(1) \le 1$; since $\varphi(\mathbb{D}) \subset \mathbb{D}$ the Julia-Wolff lemma [Po92, Prop. 4.13] shows that $\varphi(1) = 1$ implies that the angular derivative $\varphi'(1)$ is positive real or infinite. The case $\varphi'(1) < 1$ will be considered only in the last section.

In Section 4 we study the condition

$$\varphi(x) = x + b(1-x)^\beta + o\big((1-x)^\beta\big) \qquad \text{as } x \to 1- \tag{1.4}$$

where $1 < \beta \le 2$ and $0 < b < \infty$. Then $\varphi''(1)$ is finite if and only if $\beta = 2$. Our main result is Theorem 4.3 about coefficients.

The results about the coefficients can be interpreted as results about probabilities. Let X denote a random variable with values in $\mathbb{N}_0$ and the distribution $a_k = \mathbb{P}(X = k)$ for $k = 0, 1, \dots$. Then

$$\varphi(z) = \sum_{k=0}^{\infty} a_k\, z^k \qquad (z \in \overline{\mathbb{D}}) \tag{1.5}$$

is the generating function of X and satisfies $\varphi(1) = 1$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. We assume that $\varphi(0) = \mathbb{P}(X = 0) > 0$.

Let S_n be the sum of n independent random variables all distributed like X. The Bürmann-Lagrange formula (Theorem 2.1) shows that the fixed point function f has a special affinity to probabilities of the form $\mathbb{P}(S_n = n - k)$.

The study of S_n is a classical chapter of probability theory, see e.g. the book of V.V. Petrov [Pe75]. Most of our results on probability are known, at least, in the case $\beta = 2$ of finite variance.

2. **The Bürmann-Lagrange formula**

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic with $\varphi(0) \neq 0$ and let $z = f(w)$ be the inverse function of $w = z/\varphi(z)$. We define $a_{n,k}$ for $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ by

$$\varphi(z)^n = \sum_{k=0}^{\infty} a_{n,k}\, z^k\,. \tag{2.1}$$

Now we present the Bürmann-Lagrange formula [PoSz25, p. 125] in a somewhat different form and also for functions ψ with a pole at 0.

The formulas still hold near $w = 0$ if we only assume that φ is analytic near $z = 0$ and $\varphi(0) \neq 0$.

Theorem 2.1. *Let $m \geq 0,\ 0 < \rho \leq 1$ and*

$$\psi(z) = \sum_{k=-m}^{\infty} b_k\, z^k \text{ for } 0 < |z| < \rho\,. \tag{2.2}$$

If $0 < |w| < \rho$ then

$$w\, f'(w)\, \psi(f(w)) = \sum_{n=-m+1}^{\infty} \left(\sum_{k=-m+1}^{n} b_{k-1}\, a_{n,n-k} \right) w^n\,, \tag{2.3}$$

$$\psi(f(w)) = b_0 - \sum_{k=1}^{m} b_{-k}\, a_k^* + \sum_{n=-m}^{\infty} \left(\sum_{k=-m}^{n} \frac{k}{n}\, b_k\, a_{n,n-k} \right) w^n \tag{2.4}$$

where $n = 0$ is omitted in the last outer sum and where $z\,\varphi'(z)/\varphi(z) = \sum a_k^*\, z^k$.

Proof. Since $|f(w)| \leq |w|$ by the Schwarz lemma and since f is univalent in $\mathbb{D}$, we have $0 < |f(w)| < \rho$ for $0 < |w| < \rho$ so that $\psi \circ f$ is analytic in $\{0 < |w| < \rho\}$.

Let $0 < r < \rho$ and $C = \{|w| = r\}$. Let $n \in \mathbb{Z}$. The coefficient of w^n of the function $w\, f'(w)\, \psi(f(w))$ is

$$\frac{1}{2\pi i} \int_C \frac{\psi(f(w))}{w^n}\, f'(w)\, dw = \frac{1}{2\pi i} \int_{f(C)} \frac{\psi(z)\, \varphi(z)^n}{z^n}\, dz\,,$$

where we have substituted $w = z/\varphi(z)$ with $z = f(w)$. This is the coefficient of z^{n-1} of the function

$$\psi(z)\, \varphi(z)^n = \sum_{k=-m+1}^{\infty} b_{k-1}\, z^{k-1} \sum_{j=0}^{\infty} a_{n,j}\, z^j$$

which is equal to the inner sum in (2.3).

Next we apply (2.3) to ψ'. We obtain

$$\frac{d}{dw}\, \psi(f(w)) = \sum_{n=-m}^{\infty} \left(\sum_{k=-m+1}^{n} k\, b_k\, a_{n,n-k} \right) w^{n-1}\,.$$

Integrating we obtain (2.4) except for a constant. The coefficient of w^0 is

$$\frac{1}{2\pi i} \int_C \frac{\psi(f(w))}{w}\, dw = \frac{1}{2\pi i} \int_{f(C)} \frac{\psi(z)}{z} \left(1 - z\, \frac{\varphi'(z)}{\varphi(z)} \right) dz$$

because of (1.3), which gives the value in (2.4). ☑

In particular we obtain

$$w\,f'(w)\,f(w)^{k-1} = \sum_{n=k}^{\infty} a_{n,n-k}\,w^n \qquad \text{for } k \in \mathbb{Z}\,, \tag{2.5}$$

$$f(w)^k = \sum_{n=k}^{\infty} \frac{k}{n}\,a_{n,n-k}\,w^n \qquad \text{for } k \in \mathbb{N}\,. \tag{2.6}$$

3. **Some auxiliary estimates**

A Stolz angle at 1 is an open triangle $\triangle$ symmetric to $\mathbb{R}$ that satisfies $\overline{\triangle} \cap \partial\mathbb{D} = \{1\}$. We say that a function has an angular limit at 1 if this limit exists for $z \to 1$ in every Stolz angle $\triangle$.

Proposition 3.1. *Let g be analytic in $\mathbb{D}$ and*

$$g(z) \sim b(1-z)^{\beta} \ \textit{as } z \to 1 \ \textit{angularly.} \tag{3.1}$$

where $b \neq 0$ and $\beta \neq 0$. Then

$$g'(z) \sim -\beta\,b(1-z)^{\beta-1} \ \textit{as } z \to 1 \ \textit{angularly.} \tag{3.2}$$

Proof. By (3.1) the function $(1-z)^{-\beta}\,g(z)$ has the angular limit $b \neq \infty$ at 1. It follows [Po92, Prop. 4.8] that

$$(1-z)^{-\beta+1}\,g'(z) + \beta(1-z)^{-\beta}\,g(z) = (1-z)\,\frac{d}{dz}\,\left[(1-z)^{-\beta}\,g(z)\right]$$

has the angular limit 0 at 1. Hence (3.2) follows from (3.1). ☑

Proposition 3.2. *Let g be analytic in $\mathbb{D}$ and*

$$(1-x)^{\alpha}\,g(x) \to 0 \ \textit{as } x \to 1-\,, \tag{3.3}$$

$$|1-z|^{\alpha}\,|g(z)| \le c < \infty \ \textit{for } z \in \mathbb{D} \tag{3.4}$$

where $1 < \alpha < \infty$. Then

$$\int_{-\pi}^{\pi} |g(re^{it})|\,dt = o\left((1-r)^{1-\alpha}\right) \ \textit{as } r \to 1- \ . \tag{3.5}$$

Proof. We establish (3.5) for $0 \le t \le \pi$. The analytic function $(1-z)^{\alpha}\,g(z)$ is bounded because of (3.4) and therefore has the angular limit 0 at 1 because of (3.3), see [Po92, Th. 4.3].

Given $\varepsilon \in (0,1)$ there exists $r_0 \in \left(\frac{1}{2}\,,1\right)$ such that

$$|1-re^{it}|^{\alpha}\,|g(re^{it})| < \varepsilon \text{ for } r_0 < r < 1,\ |t| \le \delta = (1-r)/\varepsilon\,.$$

For $r_0 < r < 1$ we therefore have

$$\int_0^{\delta} |g(re^{it})|\,dt < \varepsilon \int_0^{\delta} \frac{|1-re^{it}|^{2-\alpha}}{|1-re^{it}|^2}\,dt\,. \tag{3.6}$$

If $1 < \alpha \le 2$ this is

$$\le \varepsilon(1+\varepsilon^{-2})^{(2-\alpha)/2}\,(1-r)^{2-\alpha}\int_0^\delta \frac{dt}{|1-re^{it}|^2} < 4\pi\varepsilon^{\alpha-1}(1-r)^{1-\alpha}\,.$$

If $2 \le \alpha < \infty$ the last expression in (3.6) is

$$\le \varepsilon(1-r)^{(2-\alpha)}\int_0^\delta \frac{dt}{|1-re^{it}|^2} \le 2\pi\varepsilon(1-r)^{1-\alpha}\,.$$

Since $|1-re^{it}| \ge 2rt/\pi$ we obtain from (3.4) that

$$\int_\delta^\pi |g(re^{it})|\,dt \le \int_\delta^\infty \frac{c\,\pi^\alpha}{t^\alpha}\,dt = \frac{c\,\pi^\alpha\varepsilon^{\alpha-1}}{\alpha-1}\,(1-r)^{1-\alpha}$$

because $\delta = (1-r)/\varepsilon$. These estimates prove (3.5). ☑

It is well known that, for $\alpha > 0$,

$$(-1)^n\binom{-\alpha}{n} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \qquad (n\to\infty)\,. \tag{3.7}$$

The following theorem is the key to the later results.

Theorem 3.3. *Let* $1 < \alpha < \infty$ *and let*

$$h(z) = \sum_{n=0}^\infty c_n\,z^n \tag{3.8}$$

be analytic in $\mathbb{D}$. *We suppose that*

$$(1-x)^{\alpha-1}\,h(x) \to a \in \mathbb{C} \text{ as } x \to 1\,, \tag{3.9}$$

$$\sup_{z\in\mathbb{D}} |1-z|^\alpha\,|h'(z)| < \infty\,. \tag{3.10}$$

Then

$$c_n \sim \frac{a}{\Gamma(\alpha-1)}\,n^{\alpha-2} \text{ as } n\to\infty\,. \tag{3.11}$$

Proof. It follows from (3.9) and (3.10) that

$$|h(x)| \le \frac{c_0}{(1-x)^{\alpha-1}}\ (0\le x\le 1),\ |h'(\zeta)| \le \frac{c_1}{|1-\zeta|^\alpha}\ (\zeta\in\mathbb{D})\,.$$

Let $z\in\mathbb{D}$ and $|1-z|<1$; the case $|1-z|\ge 1$ is simpler. Let C be the circular arc $\{\zeta\in\mathbb{D} : |1-\zeta| = |1-z|\}$ and let $x\in(0,1)$ be the point where C intersects $\mathbb{R}$. Integrating over C we obtain

$$|h(z)-h(x)| \le \int_x^z |h'(\zeta)||d\zeta| \le \frac{\pi}{2}\,|1-z|\,\frac{c_1}{|1-z|^\alpha}\,,$$

and since $1-x=|1-z|$ we conclude that

$$|1-z|^{\alpha-1}|h(z)| \le c_2 \text{ for } z \in \mathbb{D}\,. \tag{3.12}$$

It follows by (3.9) that $(1-z)^{\alpha-1}h(z)$ has the angular limit a; see e.g. [Po92, Th. 4.3]. Therefore we conclude from Proposition 3.1 that $(1-x)^{\alpha}\,h'(x) \to (\alpha-1)a$ as $x \to 1-$. Hence we can apply Proposition 3.2 to the function

$$g(z) = z\,h'(z) - \frac{(\alpha-1)a}{(1-z)^{\alpha}} = \sum_{n=0}^{\infty}\left(n\,c_n - (\alpha-1)\,a\binom{-\alpha}{n}(-1)^n\right) z^n\,; \tag{3.13}$$

the condition (3.4) is satisfied due to (3.10). We conclude from (3.5) with $r = 1-n^{-1}$ that the coefficients of g are $o(n^{\alpha-1})$ so that (3.11) follows from (3.13) and (3.7). ☑

4. **A fractional derivative condition**

In this section we consider the following condition and its consequences.

(A) The function $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and satisfies $\varphi(0) \neq 0$ and

$$\varphi(x) - x \;\sim\; b(1-x)^{\beta} \text{ as } x \to 1- \tag{4.1}$$

where $0 < b < \infty$ and $1 < \beta \le 2$. Note that we only require radial and not unrestricted approach to $z = 1$.

Proposition 4.1. *If condition* (A) *holds then* $\varphi(1) = \varphi'(1) = 1$ *as angular limits and, as* $z \to 1$ *angularly,*

$$\varphi(z) - z \;\sim\; b(1-z)^{\beta}, \tag{4.2}$$

$$1-\varphi'(z) \;\sim\; \beta\, b(1-z)^{\beta-1}, \tag{4.3}$$

$$\varphi''(z) \;\sim\; \beta(\beta-1)\, b(1-z)^{\beta-2}. \tag{4.4}$$

Proof. We see from (4.1) that $(1-\varphi(x))/(1-x) \to 1$. Hence φ has the angular derivative 1 at 1 so that $\varphi'(1) = 1$ [Po92, Prop. 4.7] and it follows from the Julia-Wolff lemma [Po92, Th. 4.13] that

$$\frac{1+\varphi(z)}{1-\varphi(z)} = \frac{1+z}{1-z} + p(z) \qquad (z \in \mathbb{D})$$

where $\operatorname{Re} p(z) > 0$ and thus $|\arg p(z)| < \frac{\pi}{2}$. Hence

$$h(z) = \log\frac{\varphi(z)-z}{(1-z)^{\beta}} = \log\frac{p(z)(1-\varphi(z))}{2(1-z)^{\beta-1}}$$

satisfies $|\operatorname{Im} h(z)| < (\beta+2)\,\pi/2$ and is therefore a Bloch function [Po92, Sect. 4.2]. Since $h(x) \to \log b$ as $x \to 1$ by (4.1), it follows that h has the angular limit at 1. This is the assertion (4.2), and we obtain (4.3) and (4.4) by applying Proposition 3.1 twice. ☑

Let f be again the fixed point function of φ, see (1.1).

Theorem 4.2. *Under the assumption* (A) *the domain* $F = f(\mathbb{D})$ *has tangents of angles* $\pm\frac{\pi}{2\beta}$ *at* 1 *and*

$$1 - f(w) \sim b^{-1/\beta}(1-w)^{1/\beta}, \tag{4.5}$$

$$f'(w) \sim (\beta b)^{-1}(1-f(w))^{1-\beta} \sim \beta^{-1} b^{-\frac{1}{\beta}}(1-w)^{\frac{1}{\beta}-1}, \tag{4.6}$$

$$f''(w) \sim (\beta-1)\,\beta^{-2}\, b^{-\frac{1}{\beta}}\,(1-w)^{\frac{1}{\beta}-2} \tag{4.7}$$

as $w \to 1$, $w \in \mathbb{D}$, *thus for unrestricted approach.*

Proof. (a) Let $\triangle$ be a Stolz angle in 1 of opening $\alpha > \pi/\beta$ and let $\varepsilon > 0$. If $z = 1 - \rho\, e^{i\vartheta}$ with $|\vartheta| < \frac{\pi}{2}$ then, by (4.2),

$$\begin{aligned}|\varphi(z)|^2 &= |z + b(1-z)^\beta + o(\rho^\beta)|^2 \\ &= |z|^2 + 2b\,\mathrm{Re}\,[(1-z)^\beta] + o(\rho^\beta)\end{aligned}$$

as $\rho \to 0$ and thus

$$|\varphi(z)|^2 - |z|^2 = \rho^\beta(2b\cos(\beta\vartheta) + o(1))\,.$$

This is positive for $\beta|\vartheta| < \frac{\pi}{2} - \varepsilon$ and negative for $\beta|\vartheta| > \frac{\pi}{2} + \varepsilon$ for small ρ. Hence the domain $F = \{z \in \mathbb{D} : |\varphi(z)| > |z|\}$ has tangents of angles $\pm\pi/(2\beta)$ at 1. In particular, F lies within some Stolz angle near 1.

(b) We obtain from Proposition 4.1 that $1 - z\,\varphi'(z)/\varphi(z) \sim \beta b(1-z)^{\beta-1}$ as $z \to 1$ angularly. Since $f(\mathbb{D})$ lies in a Stolz angle by part (a), we conclude from (1.3) with $z = f(w)$ that

$$f'(w) = \frac{1+o(1)}{\beta b}\,(1-f(w))^{1-\beta} \tag{4.8}$$

as $w \to 1$, $w \in \mathbb{D}$ and therefore

$$\begin{aligned}(1-f(w))^\beta &= \beta\int_w^1 (1-f(\omega))^{\beta-1}\,\frac{1+o(1)}{\beta b}\,(1-f(\omega))^{1-\beta}d\omega \\ &= (b^{-1}+o(1))\,(1-w)\,.\end{aligned}$$

Hence (4.5) holds, and (4.6) follows from (4.8).

By a short calculation we obtain from (1.3) that

$$f''(w) = \frac{w^2\, f'(w)^3}{f(w)}\,\varphi''(f(w)) + 2\,\frac{f'(w)^2}{f(w)} - 2\,\frac{f'(w)}{w}\,. \tag{4.9}$$

Hence we see from (4.4) and (4.6) that

$$f''(w) \sim (\beta b)^{-3}(1-f(w))^{3-3\beta}\beta(\beta-1)\,b(1-f(w))^{\beta-2}$$

which implies (4.7) in view of (4.5). ☑

Let $a_{n,k}$ be the coefficients of $\varphi(z)^n$, see (2.1). We come to our main theorem.

Theorem 4.3. *Suppose that condition* (A) *holds and that* $\overline{f(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$. *Let*

$$\psi(z) = \frac{\chi(z)}{(1-z)^{\gamma}} = \sum_{k=0}^{\infty} b_k\, z^k\,,\ \gamma \ge 0 \tag{4.10}$$

where χ *is analytic in* $\mathbb{D}$ *and has a finite angular limit* $\chi(1) \neq 0$. *Then*

$$\sum_{k=1}^{n} b_{k-1}\, a_{n,n-k} \sim \frac{\chi(1)\, b^{\frac{\gamma-1}{\beta}}}{\beta\, \Gamma(1+(\gamma-1)/\beta)}\, n^{\frac{\gamma-1}{\beta}} \quad \text{as } n \to \infty\,. \tag{4.11}$$

Proof. (a) We apply Theorem 3.3 with $\alpha = 2 + \frac{\gamma-1}{\beta} > 1$ and

$$h(w) = w\psi(f(w))\, f'(w) = \sum_{n=0}^{\infty} c_n\, w^n\,. \tag{4.12}$$

We have $\chi(f(w)) \to \chi(1)$ because $f(1) = 1$ and $F = f(\mathbb{D})$ lies in a Stolz angle by Theorem 4.2. Hence we obtain from (4.5), (4.6) and (4.10) that

$$(1-w)^{\alpha-1} h(w) \sim \chi(1)\,(1-w)^{1+\frac{\gamma-1}{\beta}}\, b^{\frac{\gamma}{\beta}}\,(1-w)^{-\frac{\gamma}{\beta}}\, \beta^{-1}\, b^{-\frac{1}{\beta}}\,(1-w)^{\frac{1}{\beta}-1}$$

which converges to $\chi(1)\,\beta^{-1}\, b^{(\gamma-1)/\beta}$ as $w \to 1$. We shall verify (3.10) in part (b). Then it follows from (3.11) that

$$c_n \sim c\, n^{(\gamma-1)/\beta} \text{ as } n \to \infty \tag{4.13}$$

where c is the factor in (4.11), and (4.11) now is a consequence of (2.3) (with $m = 0$) in the Bürmann-Lagrange formula.

(b) Since the angular limit $\chi(1)$ exists, we have $(1-z)\,\chi'(z) \to 0$ [Po92, Prop. 4.8] and thus, by (4.10),

$$\psi'(z) = \frac{\gamma\,\chi(z) + (1-z)\,\chi'(z)}{(1-z)^{\gamma+1}} = O\left(\frac{1}{|1-z|^{\gamma+1}}\right) \tag{4.14}$$

as $z \to 1$, $z \in F = f(\mathbb{D})$. Hence we obtain from Theorem 4.2 that

$$\begin{aligned} h'(w) &= \psi(f(w)) f'(w) + w\psi'(f(w))\, f'(w)^2 + w\psi(f(w))\, f''(w) \\ &= O\left(|1-f(w)|^{-\gamma-1+2-2\beta}\right) + O\left(|1-f(w)|^{-\gamma} |1-w|^{\frac{1}{\beta}-2}\right) \\ &= O\left(|1-w|^{(1-\gamma)/\beta-2}\right) = O\left(|1-w|^{-\alpha}\right) \end{aligned} \tag{4.15}$$

as $w \to 1$, $w \in \mathbb{D}$. It follows that $|1-w|^{\alpha}|h'(w)|$ is bounded for $w \in \mathbb{D}$, $|w-1| \le \delta$ for some $\delta > 0$.

Furthermore f is continuous and injective in $\overline{\mathbb{D}}$. Since $f(1) = 1$ it follows that $|1-f(w)|$ is bounded away from 0 in $U = \{w \in \mathbb{D} : |w-1| > \delta\}$. By assumption we have $\overline{f(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$ and it follows from [MePo05, Th. 2.2] that f is analytic in $\overline{U}$. Moreover $\psi'(f(w))$ is bounded in U. Hence we see from (4.15) that $|1-w|^{\alpha}\, |h'(w)|$ is bounded also in U. ☑

5. **Applications to probability theory**

Now we assume that φ has the form

$$\varphi(z) = \sum_{k=0}^{\infty} a_k\, z^k, \quad a_k \geq 0 \quad (k = 0, 1, \ldots) \tag{5.1}$$

and satisfies $\varphi(0) \neq 0$, $\varphi(1) = 1$ and $\varphi'(1) = 1$. Thus φ is the generating function of a random variable X with values in $\mathbb{N}_0$ and expectation $\mathbb{E}(X) = \varphi'(1) = 1$. Let

$$S_n = X_1 + \ldots + X_n \qquad (n = 0, 1, \ldots)$$

where the X_ν are independent random variables with $\mathbb{P}(X_\nu = k) = a_k$ for all ν and k. Since the X_ν are independent, the power $\varphi(z)^n$ has the coefficients $\mathbb{P}(S_n = k)$ and thus, by (2.1)

$$a_{n,k} = \mathbb{P}(S_n = k) \text{ for } n, k \in \mathbb{N}_0\,. \tag{5.2}$$

Proposition 5.1. *Let φ be given by (5.1) with $\varphi(1) = \varphi'(1) = 1$ and suppose that*

$$\sum_{k=1}^{m} k^2\, a_k \,\sim\, c\, m^{2-\beta} \qquad (m \to \infty) \tag{5.3}$$

where $1 < \beta \leq 2$ and $0 < c < \infty$. Then condition (A) *of Section 4 is satisfied with*

$$b = \frac{c\,\Gamma(3-\beta)}{\beta(\beta-1)}\,. \tag{5.4}$$

An explicit example is given [MePo05, Ex. 6.2] by

$$\varphi(z) = z + (2\beta)^{-1}\,(1-z)^{\beta} + \tfrac{1}{4}\,(1-z)^2\,.$$

Proof. The case $\beta = 2$ is easy. Therefore we assume that $1 < \beta < 2$. It follows from (5.3) and (3.7) that

$$\sum_{k=1}^{m} k(k-1)a_k \,\sim\, c\,\Gamma(3-\beta)\,(-1)^m \binom{\beta-3}{m}$$

and therefore, as $x \to 1$,

$$\frac{\varphi''(x)}{1-x} \;=\; \sum_{m=2}^{\infty} \left(\sum_{k=1}^{m} (k-1)k\, a_k \right) x^{m-2} \,\sim\, \frac{c\,\Gamma(3-\beta)}{(1-x)^{3-\beta}}\,.$$

Now we multiply by $1 - x$ and integrate twice using $\varphi'(1) = 1$ and $\varphi(1) = 1$. We obtain (4.1) with b given by (5.4). ☑

The generating function φ is called *aperiodic* if there does not exist $q > 1$ such that $a_k = 0$ for $k \not\equiv 0 \mod q$. If φ is aperiodic then $|\varphi(z)| < 1$ for $z \in \overline{\mathbb{D}}$, $z \neq 1$, see e.g. [MePo05, Sect. 7]. Thus the condition $\overline{f(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$ of Theorem 4.3 is satisfied. Hence we obtain from Theorem 4.3:

Theorem 5.2. *Let the generating function φ be aperiodic and let condition* (A) *of Section 4 be satisfied. Let $\gamma \geq 0$ and*

$$\psi(z) = \frac{\chi(z)}{(1-z)^\gamma} = \sum_{k=0}^{\infty} b_k z^k \tag{5.5}$$

where χ is analytic in $\mathbb{D}$ and $\chi(1) \neq 0$. Then

$$\sum_{k=1}^{n} b_{k-1}\,\mathbb{P}(S_n = n-k) \sim \frac{\chi(1)\, b^{\frac{\gamma-1}{\beta}}}{\beta\,\Gamma(1+(\gamma-1)/\beta)}\, n^{\frac{\gamma-1}{\beta}} \quad \text{as } n \to \infty\,. \tag{5.6}$$

If the variance σ^2 of X is finite then we see from (4.4) that $\beta = 2$ and $b = \sigma^2/2$. Hence (5.6) becomes

$$\sum_{k=1}^{n} b_{k-1}\,\mathbb{P}(S_n = n-k) \sim \frac{\chi(1)\,\sigma^{\gamma-1}}{2^{(\gamma+1)/2}\,\Gamma((1+\gamma)/2)}\, n^{\frac{\gamma-1}{2}}\,. \tag{5.7}$$

Now we give some specific applications where we always assume that condition (A) holds and that φ is aperiodic.

5.1. *The limit behaviour of S_n.* Let Z be any random variable with values in $\mathbb{N}_0$ that is independent of the sums S_n. We apply Theorem 5.2 with $\gamma = 0$ and χ the generating function of Z. Then $\chi(1) = 1$ and the sum in (4.11) becomes

$$\sum_{k=1}^{n} \mathbb{P}(Z = k-1)\,\mathbb{P}(S_n = n-k)\,.$$

Since the Z and S_n are independent we obtain

$$\mathbb{P}(S_n + Z = n) \sim \frac{b^{-1/\beta}}{\beta\,\Gamma(1-1/\beta)}\, n^{-\frac{1}{\beta}} \quad \text{as } n \to \infty\,. \tag{5.8}$$

5.2. *Asymmetry near the equilibrium.* Since $\mathbb{E}(S_n) = n$ the equilibrium is reached if $S_n = n$. Now we apply Theorem 5.2 with $\gamma = 1$ and $\chi(z) = 1$. We obtain

$$\mathbb{P}(S_n < n) = \sum_{k=1}^{n} \mathbb{P}(S_n = n-k) \to \frac{1}{\beta} \quad \text{as } n \to \infty\,.$$

This value is $> \frac{1}{2}$ if $\beta < 2$. This does not contradict the law of large numbers because this law only says that $S_n/n \to 1$ but does not say anything about $\mathbb{P}(S_n/n < 1)$.

We introduce random variables T_n with values in $\{1, \ldots, n\}$ by $T_n = n - S_n$ for $S_n < n$ and $\mathbb{P}(T_n = k) = \mathbb{P}(S_n = n-k \mid S_n < n)$. It follows from Theorem 5.2 with $\psi(z) = (1-z)^{-2}$ and $\psi(z) = 2(1-z)^{-3}$ that

$$\mathbb{E}(T_n) \sim \frac{b^{1/\beta}}{\Gamma(1+1/\beta)}\, n^{\frac{1}{\beta}}\,,\ \mathbb{V}(T_n) \sim \left(\frac{2b^{2/\beta}}{\Gamma(1+2/\beta)} - \frac{b^{2/\beta}}{\Gamma(1+1/\beta)^2}\right) n^{\frac{2}{\beta}}\,.$$

5.3. *The first return to equilibrium.* Now we introduce a random variable N with values in $\mathbb{N}$ by

$$N = n \Leftrightarrow S_n = n\,,\ S_\nu \neq \nu \quad (1 \leq \nu < n)\,.$$

Thus N is when S_n reaches its equilibrium for the first time.

Now $(S_n = n)$ is a recurrent event [Fe68, p. 311] and we see from (1.3) and from (2.5) with $k = 0$ that

$$\sum_{n=1}^{\infty} \mathbb{P}(S_n = n)\, w^n = \frac{1}{1 - w\,\varphi'(f(w))}\,.$$

Hence it follows [Fe68, p. 311] that

$$w\,\varphi'(f(w)) = \sum_{n=1}^{\infty} \mathbb{P}(N = n)\, w^n\,. \tag{5.9}$$

Since $\varphi'(1) = 1$ and thus $f(1) = 1$, we see that the random variable N is not defective.

Now we argue as in the proof of Theorem 4.3 with $\psi = \varphi''$. It follows from (5.9) that

$$h(w) = w\,\varphi''(f(w))\, f'(w) = \sum_{n=0}^{\infty} n\,\mathbb{P}(N = n + 1)\, w^n\,; \tag{5.10}$$

this notation agrees with (4.12). We consider $\alpha = 1 + \frac{1}{\beta} > 1$. It follows from Proposition 4.1 and Theorem 4.2 that, as $w \to 1-$,

$$\begin{aligned}(1 - w)^{\alpha-1} h(w) \quad &\sim \quad (1 - w)^{\frac{1}{\beta}} \beta(\beta - 1)\, b(1 - f(w))^{\beta-2} f'(w) \\ &\to \quad (\beta - 1)\, b^{\frac{1}{\beta}}\,.\end{aligned}$$

With some effort the condition (3.10) is verified as in part (b) of the proof of Theorem 4.3. Hence we obtain from (5.10) and Theorem 3.3 that

$$n\,\mathbb{P}(N = n + 1) \sim (\beta - 1)\, b^{\frac{1}{\beta}} \Gamma(1/\beta)^{-1}\, n^{\frac{1}{\beta - 1}}$$

and therefore

$$\mathbb{P}(N = n) \sim \frac{(1 - \beta)\, b^{1/\beta}}{\Gamma(1/\beta)}\, n^{\frac{1}{\beta} - 2} \quad \text{as } n \to \infty\,.$$

If σ is finite then $\beta = 2$ and we obtain $\mathbb{P}(N = n) \sim \frac{\sigma}{\sqrt{2\pi}}\, n^{-3/2}$.

5.4. *The total progeny in a branching process.* We consider a Galton-Watson branching process [Fe68] [AtNe72]. Let Z_k denote the number of individuals in the k-th generation where $Z_0 = q$ is given; in general it is assumed that $Z_0 = 1$. These individuals reproduce independently and the number of children of each individual is distributed like X. Then

$$Y = \sum_{k=0}^{\infty} Z_k \leq \infty$$

is the total progeny, that is the total number of all individuals over all generations; see e.g. [Fe68, p. 298] [KaNa94]. It was shown in [MePo05, Sect. 6] that the fixed point function f of φ is the generating function of Y if $Z_0 = 1$. Since we now start with q individuals reproducing independently, we have

$$f(w)^q = \sum_{n=q}^{\infty} \mathbb{P}(Y = n)\, w^n \,.$$

Hence we obtain from (2.6) and (5.8) that

$$\mathbb{P}(Y = n) = \frac{q}{n}\, \mathbb{P}(S_n = n - q) \;\sim\; \frac{q\, b^{-1/\beta}}{\beta\, \Gamma(1 - 1/\beta)}\, n^{-\frac{1}{\beta}-1} \,. \tag{5.11}$$

If $\sigma < \infty$ we have $\mathbb{P}(Y = n) \;\sim\; \frac{q}{\sqrt{2\pi}\,\sigma}\, n^{-3/2}$.

6. The case $\varphi'(1) < 1$

Let φ again be analytic in $\mathbb{D}$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) \neq 0$ and $\varphi(1) = 1$. Now we assume that

$$\varphi'(z) \to \mu < 1 \text{ as } z \to 1,\; z \in \mathbb{D} \,. \tag{6.1}$$

The fixed point function f satisfies $f(1) = 1$ and now

$$f'(w) \to \frac{1}{1-\mu}\,,\; \frac{1 - f(w)}{1 - w} \;\to\; \frac{1}{1-\mu} \text{ as } w \to 1,\, w \in \mathbb{D} \tag{6.2}$$

by (1.3). Hence $f(\mathbb{D})$ is tangential to $\partial\mathbb{D}$ at 1 [Po92, p. 80]. This is the reason why we have to allow unrestricted approach in (6.1). The situation is more complicated than for $\varphi'(1) = 1$ and we only prove one result.

Theorem 6.1. *Suppose that* $1 < \beta < 2$, $c \in \mathbb{C}$, $c \neq 0$ *and*

$$\varphi''(z) \;\sim\; c(1 - z)^{\beta-2},\, \varphi'''(z) = O\left(|1 - z|^{\beta-3}\right) \tag{6.3}$$

as $z \to 1$, $z \in \mathbb{D}$. *If* $\overline{f(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$ *then, for every* $k \in \mathbb{Z}$,

$$a_{n,n-k} \;\sim\; \frac{c(1-\mu)^{-\beta-1}}{\Gamma(2-\beta)}\, n^{-\beta} \;\; as\; n \to \infty \,. \tag{6.4}$$

Proof. We have $\alpha = 3 - \beta > 1$ because $\beta < 2$. We apply Theorem 3.3 to

$$h(w) = \frac{d}{dw}\left(f'(w)\, f(w)^{k-1}\right) = \sum_{n=k}^{\infty} (n-1)\, a_{n,n-k}\, w^{n-2}; \tag{6.5}$$

see (2.5). We restrict ourselves to the case $k = 1$ to simplify some technical details.

It follows from (4.9), (6.2) and (6.3) that

$$h(w) = f''(w) \;\sim\; c(1-\mu)^{-\beta-1}\, (1 - w)^{\beta-2} \text{ as } w \to 1,\, w \in \mathbb{D} \,. \tag{6.6}$$

Now we differentiate (4.9) and obtain from (6.3) that

$$\begin{aligned} h'(w) &= f'''(w) = O(f''(w)\,\varphi''(f(w))) + O(\varphi'''(f(w)) \\ &= O\left(|1-w|^{2\beta-4}\right) + O\left(|1-w|^{\beta-3}\right) = O\left(|1-w|^{-\alpha}\right) \end{aligned}$$

because $\beta > 1$. As in part (b) of the proof of Theorem 4.3, we see that (3.10) holds. Hence it follows from (3.11), (6.5) and (6.6) that

$$(n-1)\,a_{n,n-1} \sim \frac{c(1-\mu)^{-\beta-1}}{\Gamma(2-\beta)}\,n^{1-\beta} \text{ as } n \to \infty$$

which implies (6.4) for $k = 1$. ☑

Now let φ be an aperiodic probability generating function with $\mathbb{E}(X) < 1$ that satisfies (6.3) with $1 < \beta < 2$. Then it follows from (6.4) that

$$\mathbb{P}(S_n = n - q) \sim c_1\, n^{-\beta} \quad (n \to \infty),\ c_1 \neq 0 \tag{6.7}$$

and we obtain from (5.11) that the total progeny Y in a branching process satisfies $\mathbb{P}(Y = n) \sim c_2\, n^{-\beta-1},\ c_2 \neq 0$.

The relation is not always (or never ?) true if $\beta = 2$, that is for finite variance. Consider for instance the case that φ is analytic in $\{|z| < R\}$ with $R > 1$. Then large deviation theory shows that

$$\mathbb{P}(S_n = n-1) = O(\rho^n) \quad (n \to \infty) \text{ for some } \rho < 1$$

which is very much smaller than (6.7). See e.g. [Gä77] and see [MePo05, Sect. 7] for details.

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