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On the homeotopy group of the non orientable surface of genus three

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Abstract. In this note we prove that, if $N_3 = P\#P\#P$, where $P := \mathbb{R}P^2$, then the canonical homomorphism from $\mathrm{Diff}(N_3)$ onto the homeotopy group $\mathrm{Mod}(N_3)$ has a section. To do this we first prove that $\mathrm{Mod}(N_3) = GL(2, \mathbb{Z})$.

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Resumen. En esta nota probamos que, si $N_3 = P\#P\#P$, donde $P := \mathbb{R}P^2$, entonces el homomorfismo canónico de $\mathrm{Diff}(N_3)$ sobre el grupo de homeotopía $\mathrm{Mod}(N_3)$ tiene una sección. Para hacer esto, primero probamos que $\mathrm{Mod}(N_3) = GL(2, \mathbb{Z})$.

1. Introduction

If M is a closed smooth surface we denote by $\mathrm{Mod}(M)$ the quotient group $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ where $\mathrm{Diff}(M)$ is the group of all diffeomorphisms from M to M and $\mathrm{Diff}_0(M)$ is the normal subgroup of diffeomorphisms isotopic to the identity. We call it the homeotopy group or the extended mapping class group of M.

S. Morita [9], [10] has shown that, if M_g is the closed genus g orientable surface, then the canonical epimorphism

$$\mathrm{Diff}(M_g) \to \mathrm{Mod}(M_g)$$

from the group of diffeomorphisms of M_g onto its extended mapping class group admits no section provided that $g \geq 18$.

When $g \leq 1$ it is easy to show that the homomorphism does have a splitting: If $g = 0$ then $\mathrm{Mod}(M_0) = \mathbb{Z}_2$; a section is defined by sending the non trivial element of $\mathrm{Mod}(M_0)$ to the antipodal map of S^2. Also, for genus one $M_1 =$

$\mathbb{R}^2/\mathbb{Z}^2$ and $\mathrm{Mod}(M_1) = GL(2,\mathbb{Z})$ (cf. [11, p. 26]). The standard linear action of $GL(2,\mathbb{Z})$ on $(\mathbb{R}^2,\mathbb{Z}^2)$ defines a splitting of $\mathrm{Diff}(M_1) \to \mathrm{Mod}(M_1)$.

If N_k is the genus k non-orientable surface (the connected sum of k copies of P) then

$$\mathrm{Diff}(N_k) \to \mathrm{Mod}(N_k),$$

has a section if $k \leq 2$.

For, if $k = 1$ then $\mathrm{Mod}(P) = 1$ (see [4]) and trivially a section exists. If $k = 2$ and we think of N_2 as $S^1 \times S^1$ with identifications $(z,w) \sim (-z,\bar{w})$, then $\mathrm{Mod}(N_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the image of a section is $\{f_{\epsilon_1\epsilon_2} : |\epsilon_1| = |\epsilon_2| = 1\}$ where $f_{\epsilon_1\epsilon_2}(z,w) = (z^{\epsilon_1}, w^{\epsilon_2})$ (see [6], [12]).

Here we will prove that

$$\mathrm{Diff}(N_k) \to \mathrm{Mod}(N_k)\,,$$

also has a section if $k = 3$.

2. **Proofs**

First, we will show that $\mathrm{Mod}(N_3) = GL(2,\mathbb{Z})$.

In [2], using [3], a presentation of $\mathrm{Mod}(N_3)$ is given and one can see that this presentation defines $GL(2,\mathbb{Z})$, (see [7]).

However we feel that this result is not *well* known. In here we will give a proof of the fact that $\mathrm{Mod}(N_3) = \mathrm{Mod}(M_1)(= GL(2,\mathbb{Z}))$ using simple methods in algebraic topology.

We will work in the smooth category.

Let T_0 be a torus minus the interior of a $2-$disk D. An arc α properly embedded in T_0 is trivial if there is a $2-$disk in T_0 whose boundary is the union of α and an arc in ∂T_0. This is equivalent to the condition that α represent the trivial element of $H_1(T_0, \partial T_0; \mathbb{Z}_2)$. In the following lemma $\cup_{i=1}^n \alpha_i/\varphi$ will denote the quotient space of the union of arcs $\cup_{i=1}^n \alpha_i$ obtained by identifying $x \in \partial(\cup_{i=1}^n \alpha_i)$ with $\varphi(x)$.

Lemma 2.1. *Let T_0 be the torus minus the interior of a $2-$disk. Let $\varphi : \partial T_0 \to \partial T_0$ be a fixed point free involution. Let $\alpha_1, \ldots, \alpha_n$, with n odd, be disjoint arcs properly embedded in T_0 such that $\varphi\partial\,(\cup_{i=1}^n \alpha_i) = \partial\,(\cup_{i=1}^n \alpha_i)$, $\cup_{i=1}^n \alpha_i/\varphi$ is connected and $\sum_{i=1}^n [\alpha_i] = 0 \in H_1(T_0, \partial T_0; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then at least one α_i is trivial.*

Proof. Let a, b, c be the nontrivial elements of $H_1(T_0, \partial T_0; \mathbb{Z}_2)$. Let $a_1, \ldots, a_p$ be the arcs of $\{\alpha_1, \ldots, \alpha_n\}$ which represent a. Let $b_1, \ldots, b_q$ those which represent b and $c_1, \ldots, c_r$ those which represent c.

Assume no α_i is trivial, that is $[\alpha_i] \neq 0$ for all i. Then $0 = \sum_{i=0}^n [\alpha_i] = pa + bq + rc$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $p + q + r = n$, an odd number. If one of the numbers p, q, r is even then the other two must also be even; but this contradicts the fact that n is odd. Therefore p, q, r are all odd.

Notice that for any i and any j the $0-$spheres $\partial\alpha_i$ and ∂b_j are linked in ∂T_0 (meaning that both components of $\partial T_0 - \partial\alpha_i$ contain one point of ∂b_j).

Similarly ∂b_j and ∂c_k are linked, and ∂c_k and ∂a_i are linked, for any values of i, j, k. Also ∂a_i and ∂a_j are not linked ∂b_i and ∂b_j are not linked, ∂c_i and ∂c_k are not linked if $i \neq j$.

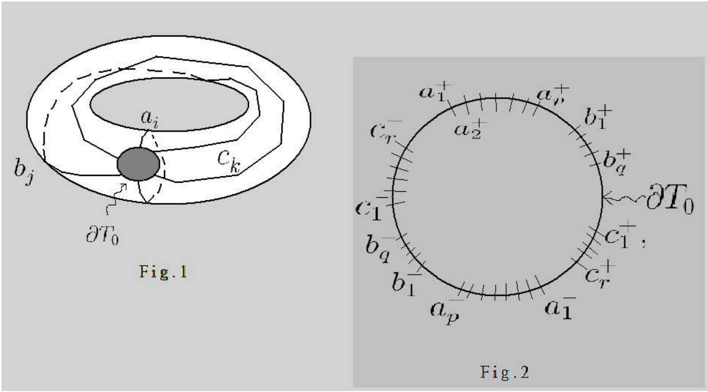

FIGURE 1

This implies that after renumbering the a's, b's and c's the arrangement of the points of $\cup_{i=1}^{n} \partial\alpha_i$ in ∂T_0 is: $a_1^+, a_2^+, \ldots, a_p^+, b_1^+, \ldots, b_q^+, c_1^+, \ldots, c_r^+, a_p^-, \ldots,$ $a_2^-, a_1^-, b_q^-, \ldots, b_1^-, c_r^-, \ldots, c_1^-$ as shown in figure 1; here $\partial a_i = \{a_i^+, a_i^-\}$, $\partial b_j = \{b_j^+, b_j^-\}$ and $\partial c_k = \{c_k^+, c_k^-\}$.

But then the number of components of $\cup\alpha_i/\varphi$ is $\frac{p+1}{2} + \frac{q+1}{2} + \frac{r+1}{2} > 1$ (think of φ as the antipodal involution), contradicting that $\cup\alpha_i/\varphi$ is connected. Hence at least one α_i is trivial. ☑

We write $N = N_3$ henceforth.

Proposition 2.1. *Let μ and α be simple closed curves in N representing the element of order 2 in $H_1(N;\mathbb{Z})$. Then α is isotopic to μ.*

Proof. Write $N = T_0 \cup P_0$, the union of a punctured torus T_0 and a Möbius band P_0, with $T_0 \cap P_0 = \partial T_0 = \partial P_0$. We think of P_0 as an $I-$bundle over the circle and denote by $\varphi\colon \partial T_0 \to \partial T_0$ the fixedpoint free involution that interchanges the boundary points of each fiber.

We may assume that μ is the image of a section of this bundle. We may also assume that α intersects ∂T_0 minimally, that is, $|\alpha' \cap \partial T_0| \geq |\alpha \cap \partial T_0|$ for any curve α' ambient isotopic to α. We claim that $|\alpha \cap \partial T_0| = 0$.

Suppose $|\alpha \cap \partial T_0| > 0$. Then we can assume that $\alpha \cap P_0$ consists of n $I-$fibers $f_1, \ldots, f_n$ and $\alpha \cap T_0$ is the union of n disjoint arcs $\alpha_1, \ldots, \alpha_n$ properly embedded in T_0. As $H_1(N) = H_1(P_0, \partial P_0) \oplus H_1(T_0, \partial T_0) = \mathbb{Z}_2 \oplus \mathbb{Z}^2$ and as α represents the element of order two, then we must have that $\sum [f_i] \neq 0$ in

$H_1(P_0, \partial P_0; \mathbb{Z}_2)$ (that is, n must be odd) and $\sum[\alpha_i] = 0$ in $H_1(T_0, \partial T_0; \mathbb{Z}_2)$. By Lemma 2.1, at least one α_i must be trivial and so we can isotope α_i to reduce the number of components of its intersection with ∂T_0. This contradicts our minimality assumption. Hence $|\alpha \cap \partial T_0| = 0$ and, since α is not trivial, it is isotopic to μ. ☑

Proposition 2.2. *Let* $N = T_0 \cup P_0$ *with* $T_0 \cap P_0 = \partial T_0 = \partial P_0$. *Then any diffeomorphism* h *of* N *is isotopic to one leaving* T_0 *and* P_0 *invariant.*

Proof. Let μ be the image of a section of P_0. By Proposition 2.1, $h\mu$ is ambient isotopic to μ so we may assume that h leaves μ invariant. But then we can also assume that it leaves its tubular neighborhood P_0 invariant. ☑

Theorem 2.3. *The natural homomorphism*

$$\psi\colon \mathrm{Mod}(N) \to \mathrm{Aut}(H_1(N)/\mathrm{Torsion}(H_1(N))(\cong GL(2, \mathbb{Z}))\,,$$

is an isomorphism.

Proof. Again write $N = T_0 \cup P_0$ and $T = T_0 \cup D$. Any automorphism of $H_1(T)$ is induced by a diffeomorphism of T which can be isotoped so that the 2−disk D is invariant. Hence any automorphism of $H_1(T_0, \partial T_0)$ is induced by a diffeomorphism of T_0. Since any diffeomorphism of ∂P_0 can be extended to a diffeomorphism of P_0 (a nice exercise), it follows that any automorphism of $H_1(N)/\mathrm{Torsion}(H_1(N)$ is induced by a diffeomorphism of N. Thus ψ is an epimorphism.

Suppose now that $\psi(h)$ is the identity. By Proposition 2.2, h is isotopic to a diffeomorphism leaving T_0 invariant. Now, $h|_{T_0}$ induces the identity on $H_1(T_0, \partial T_0)$ and is therefore isotopic to id_{T_0} and a diffeomorphism of P_0 which is the identity on ∂P_0 is isotopic rel ∂ to id_{P_0}. Hence h is isotopic to id_N. This proves that ψ is a monomorphism. ☑

Theorem 2.4. *The natural homomorphism* $\mathrm{Diff}(N) \to \mathrm{Mod}(N)$ *has a section.*

Proof. Let $T = \mathbb{R}^2/\mathbb{Z}^2$. Consider the blow up $B(T)$ of T at the identity element e of T. Recall $B(T) = (T - \{e\}) \cup P^1$ where P^1 is the space of one-dimensional vector subspaces of $\mathbb{R}^2$. The blow up $B(T)$ is diffeomorphic to N.

If f is a linear automorphism of $\mathbb{R}^2$ with $f(\mathbb{Z}^2) = \mathbb{Z}^2$, it induces a diffeomorphism of $T - \{e\}$, a diffeomorphism of P^1 and a diffeomorphism of $B(T)$ (cf.[1, Lemma 2.1]). Thus the standard linear action of $GL(2, \mathbb{Z})$ on T induces an action of $GL(2, \mathbb{Z})$ on $B(T)$.

Hence we have a homomorphism

$$GL(2, \mathbb{Z}) \to \mathrm{Diff}(B(T))\,,$$

which composed with

$$\mathrm{Diff}(B(T)) \to \mathrm{Mod}B(T) \xrightarrow{\approx} \mathrm{Aut}(H_1(N)/\mathrm{Torsion}(H_1(N))$$

is an isomorphism. ☑

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