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Semimatroids and their Tutte polynomials

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Abstract. We define and study *semimatroids*, a class of objects which abstracts the dependence properties of an affine hyperplane arrangement. We show that geometric semilattices are precisely the posets of flats of semimatroids. We define and investigate the Tutte polynomial of a semimatroid. We prove that it is the universal Tutte-Grothendieck invariant for semimatroids, and we give a combinatorial interpretation for its non-negative integer coefficients.

Keywords. semimatroid, pointed matroid, geometric semilattice, hyperplane arrangement, Tutte polynomial.

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Resumen. En este artículo definimos y estudiamos las *semimatroides*, una clase de objetos que abstraen las propiedades de dependencia de un arreglo de hiperplanos afines. Demostramos que un semiretículo es geométrico si y sólo si es el semiretículo de conjuntos cerrados de una semimatroide. Definimos e investigamos el polinomio de Tutte de una semimatroide. Demostramos que es la invariante universal de Tutte-Grothendieck para la clase de semimatroides, y presentamos una interpretación combinatoria de sus coeficientes, que son enteros no negativos.

1. Introduction.

The goal of this paper is to define and study a class of objects called *semimatroids*. A semimatroid can be thought of as a matroid-theoretic abstraction of the dependence properties of an affine hyperplane arrangement. Many properties of hyperplane arrangements are really facts about their underlying matroidal structure. Therefore, the study of such properties can be carried out much more naturally and elegantly in the setting of semimatroids.

The paper is organized as follows. In Section 2 we define semimatroids, and show how a hyperplane arrangement determines a semimatroid. The following sections provide different ways of thinking about semimatroids. Section

3 shows how a semimatroid "extends" to a matroid, and determines a modular ideal inside it. The semimatroid can be recovered from the matroid and its modular ideal. Section 4 describes the close relationship between semimatroids and strong maps. Semimatroids are described in terms of elementary preimages and single-element coextensions. Section 5 gives a bijection between semimatroids and pointed matroids. Section 6 gives a new characterization of geometric semilattices as posets of flats of semimatroids, extending the classical correspondence between geometric lattices and simple matroids.

The final sections are geared towards the study of the Tutte polynomial of a semimatroid. Section 7 defines the concepts of duality, deletion and contraction. Section 8 defines the Tutte polynomial, and shows that it is the unique Tutte-Grothendieck invariant for the class of semimatroids. Finally, Section 9 gives a combinatorial interpretation for the non-negative coefficients of the Tutte polynomial.

It is worth pointing out that Kawahara discovered semimatroids independently, and described their Orlik-Solomon algebra in [14]. Las Vergnas's work on the Tutte polynomial of a quotient map [17] also overlaps with our work; we say more about this at the end of Section 8.

It is also worth remarking that the semimatroids of [32] are the same ones that we are studying; their name is justified by the correspondence between simple semimatroids and geometric semilattices. The "semimatroids" in [15] and [27] are different objects.

Throughout the paper, we will assume a basic knowledge of matroid theory. For a good introduction to the subject, see [20].

2. **Semimatroids.**

Definition 2.1. *A* semimatroid *is a triple* $(S, \mathcal{C}, r_{\mathcal{C}})$ *consisting of a finite set* S*, a non-empty simplicial complex* $\mathcal{C}$ *on* S*, and a function* $r_{\mathcal{C}} : \mathcal{C} \to \mathbb{N}$*, satisfying the following five conditions.*

(R1) *If* $X \in \mathcal{C}$*, then* $0 \leq r_{\mathcal{C}}(X) \leq |X|$*.*
(R2) *If* $X, Y \in \mathcal{C}$ *and* $X \subseteq Y$*, then* $r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(Y)$*.*
(R3) *If* $X, Y \in \mathcal{C}$ *and* $X \cup Y \in \mathcal{C}$*, then* $r_{\mathcal{C}}(X) + r_{\mathcal{C}}(Y) \geq r_{\mathcal{C}}(X \cup Y) + r_{\mathcal{C}}(X \cap Y)$*.*
(CR1) *If* $X, Y \in \mathcal{C}$ *and* $r_{\mathcal{C}}(X) = r_{\mathcal{C}}(X \cap Y)$*, then* $X \cup Y \in \mathcal{C}$*.*
(CR2) *If* $X, Y \in \mathcal{C}$ *and* $r_{\mathcal{C}}(X) < r_{\mathcal{C}}(Y)$*, then* $X \cup y \in \mathcal{C}$ *for some* $y \in Y - X$*.*

Note that a semimatroid with $\mathcal{C} = 2^S$ is precisely a matroid, since (CR1) and (CR2) hold trivially.

We call S, $\mathcal{C}$ and $r_{\mathcal{C}}$ the *ground set*, *collection of central sets* and *rank function* of the semimatroid $(S, \mathcal{C}, r_{\mathcal{C}})$, respectively. Sometimes we will slightly abuse notation and denote the semimatroid $\mathcal{C}$, when its ground set and rank function are clear. We will denote subsets of S by upper case letters, and elements of S by lower case letters.

Figure 1 shows an example of a semimatroid. The ground set is $[3] = \{1, 2, 3\}$, the collection of central sets is $\mathcal{C} = \{\emptyset, 1, 2, 3, 13, 23\}$, and $r_{\mathcal{C}}(A) = |A|$

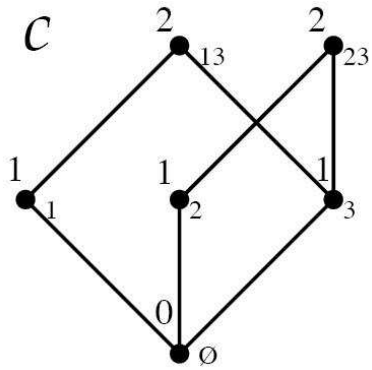

FIGURE 1. A semimatroid.

for $A \in \mathcal{C}$. We draw $\mathcal{C}$ as a poset ordered by containment, and next to each node we indicate its rank.

There is a well-known connection between matroids and central hyperplane arrangements, described below. We describe it below, and show that it extends to a connection between semimatroids and affine hyperplane arrangements.

Given a field $\Bbbk$ and a positive integer n, an *affine hyperplane* in $\Bbbk^n$ is an $(n-1)$-dimensional affine subspace of $\Bbbk^n$. A *hyperplane arrangement* $\mathcal{A}$ in $\Bbbk^n$ is a finite set of affine hyperplanes in $\Bbbk^n$. An arrangement is *central* if all its hyperplanes have a non-empty intersection.

The *rank function* $r_{\mathcal{A}}$ is defined for the central subarrangements $\mathcal{B} \subseteq \mathcal{A}$ by the equation $r_{\mathcal{A}}(\mathcal{B}) = n - \dim \cap \mathcal{B}$. If $\mathcal{A}$ is central, then $r_{\mathcal{A}}$ is defined on all of its subsets, and it is the rank function of a matroid $M_{\mathcal{A}}$ on $\mathcal{A}$. [23, Lect. 3]

Definition 2.2. *For a central arrangement $\mathcal{A}$, let $M_{\mathcal{A}}$ be the matroid of $\mathcal{A}$, with ground set $\mathcal{A}$ and rank function $r_{\mathcal{A}}$.*

Similarly, if $\mathcal{A}$ is a hyperplane arrangement which is not necessarily central, then its rank function determines a semimatroid:

Proposition 2.3. *Let $\mathcal{A}$ be an affine hyperplane arrangement in $\Bbbk^n$. Let $\mathcal{C}_{\mathcal{A}}$ be the collection of central subarrangements of $\mathcal{A}$, and let $r_{\mathcal{A}} : \mathcal{C}_{\mathcal{A}} \to \mathbb{N}$, defined by*

$$r_{\mathcal{A}}(\mathcal{B}) = n - \dim \cap \mathcal{B},$$

be the rank function of $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{C}_{\mathcal{A}}, r_{\mathcal{A}})$ is a semimatroid.

Proof. To each hyperplane $H_i \in \mathcal{A}$ we can associate a vector $v_i \in \Bbbk^n$ (or equivalently a linear functional in the dual vector space) and a constant $c_i \in \Bbbk$, so that H_i is the set of points $x \in \Bbbk^n$ such that $v_i \cdot x = c_i$, with the usual inner product on $\Bbbk^n$. It is easy to see that the rank of a central subset $\{H_{i_1}, \ldots, H_{i_k}\} \in \mathcal{C}_{\mathcal{A}}$ is equal to the rank of the set $\{v_{i_1}, \ldots, v_{i_k}\}$ in $\Bbbk^n$.

From this point of view, axioms (R1), (R2), (R3) are standard facts of linear algebra applied to the vector space $\Bbbk^n$. We now check axioms (CR1) and (CR2).

To check axiom (CR1), assume that $X, Y \in \mathcal{C}$ and $r_{\mathcal{A}}(X) = r_{\mathcal{A}}(X \cap Y)$. Let $A = \cap X$ be the intersection of the hyperplanes in X, and similarly let $B = \cap Y$. Since $X \cap Y \subseteq X$ and $r_{\mathcal{A}}(X \cap Y) = r_{\mathcal{A}}(X)$, we must have $\cap(X \cap Y) = \cap X = A$. Also, $X \cap Y \subseteq Y$ implies $\cap(X \cap Y) \supseteq \cap Y = B$. Therefore $A \supseteq B$, and every hyperplane in $X \cup Y$ contains B. It follows that $X \cup Y \in \mathcal{C}$.

To check axiom (CR2), assume that $X, Y \in \mathcal{C}$ and $r_{\mathcal{A}}(X) < r_{\mathcal{A}}(Y)$. Let $L_X = \{v_i \,|\, H_i \in X\}$ and define similarly L_Y. Since $\text{rank}(L_Y) > \text{rank}(L_X)$, there exists a vector $v \in L_Y$, corresponding to a hyperplane $y \in Y$, which is not in the span of L_X. Thus y has a non-empty intersection with $\cap X$. ☑

Corollary 2.4. *For positive integers $k \leq n$, let S be $[n] = \{1, 2, \ldots, n\}$, let $\mathcal{C}$ consist of the subsets of $[n]$ of size at most k, and let the rank of a set S (with $|S| \leq k$) be its size:*

$$S = [n], \qquad \mathcal{C} = \binom{[n]}{\leq k}, \qquad r_{\mathcal{C}}(S) = |S|.$$

Then $(S, \mathcal{C}, r_{\mathcal{C}})$ is a semimatroid.

Proof. This is clearly the semimatroid of n generic affine hyperplanes in $\mathbb{R}^k$. ☑

We will need the fact that semimatroids satisfy a "local" version of (R1) and (R2) and a stronger version of (CR1) and (CR2), as follows.

Proposition 2.5. *Semimatroids satisfy the following alternative axioms:*
(R2') *If $X \cup x \in \mathcal{C}$ then $r_{\mathcal{C}}(X \cup x) - r_{\mathcal{C}}(X) = 0$ or 1.*
(CR1') *If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) = r_{\mathcal{C}}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup Y) = r_{\mathcal{C}}(Y)$.*
(CR2') *If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) < r_{\mathcal{C}}(Y)$, then $X \cup y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y) = r_{\mathcal{C}}(X) + 1$ for some $y \in Y - X$.*

Proof. **(R2')**: From (R2) we know that $r_{\mathcal{C}}(X \cup x) \geq r_{\mathcal{C}}(X)$. From (R3) we know that $r_{\mathcal{C}}(X \cup x) - r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(x) - r_{\mathcal{C}}(\emptyset)$, and this is 0 or 1 by (R1).

(CR1'): The hypotheses imply that $X \cup Y \in \mathcal{C}$. Then (R2) says that $r_{\mathcal{C}}(Y) \leq r_{\mathcal{C}}(X \cup Y)$, while (R3) says that $r_{\mathcal{C}}(Y) \geq r_{\mathcal{C}}(X \cup Y)$.

(CR2'): By applying (CR2) repeatedly, we see that we can keep on adding elements $y_1, \ldots, y_k$ of Y to the set X, until we reach a set $X \cup y_1 \cup \cdots \cup y_k \in \mathcal{C}$ such that $r_{\mathcal{C}}(X \cup y_1 \cup \cdots \cup y_k) = r_{\mathcal{C}}(Y)$. Now we claim that $r_{\mathcal{C}}(X \cup y_i) = r_{\mathcal{C}}(X) + 1$ for some i. If that was not the case then, since $r_{\mathcal{C}}(X \cup y_1) = r_{\mathcal{C}}(X)$, (CR1') applies to $X \cup y_1$ and $X \cup y_2$. Therefore $X \cup y_1 \cup y_2 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y_1 \cup y_2) = r_{\mathcal{C}}(X \cup y_2) = r_{\mathcal{C}}(X)$. Then (CR1') applies to $X \cup y_1 \cup y_2$ and $X \cup y_3$, so $X \cup y_1 \cup y_2 \cup y_3 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y_1 \cup y_2 \cup y_3) = r_{\mathcal{C}}(X)$. Continuing in this way, we conclude that $X \cup y_1 \cup \cdots \cup y_k \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y_1 \cup \cdots \cup y_k) = r_{\mathcal{C}}(X)$, a contradiction. ☑

Semimatroids, like matroids, have several equivalent definitions. In their context, it is possible to talk about flats, independent sets, spanning sets, bases, circuits, and most other basic matroid concepts. We will say more about this in Section 9. Until then, we will use the rank function approach of Definition 2.1 throughout most of our treatment. We will also need some facts about the closure approach, which we now present.

Definition 2.6. *For a semimatroid* $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ *and a set* $X \in \mathcal{C}$*, the* closure *of* X in $\mathcal{C}$ *is* $\mathrm{cl}_{\mathcal{C}}(X) = \{x \in S \mid X \cup x \in \mathcal{C}, r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X)\}$.

We will sometimes drop the subscript and write $\mathrm{cl}(X)$ instead of $\mathrm{cl}_{\mathcal{C}}(X)$ when it causes no confusion.

Proposition 2.7. *The closure operator of a semimatroid satisfies the following properties, for all* $X, Y \in \mathcal{C}$ *and* $x, y \in S$.
(CLR1) $\mathrm{cl}(X) \in \mathcal{C}$ *and* $r_{\mathcal{C}}(\mathrm{cl}(X)) = r_{\mathcal{C}}(X)$.
(CL1) $X \subseteq \mathrm{cl}(X)$.
(CL2) *If* $X \subseteq Y$ *then* $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.
(CL3) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.
(CL4) *If* $X \cup x \in \mathcal{C}$ *and* $y \in \mathrm{cl}(X \cup x) - \mathrm{cl}(X)$*, then* $X \cup y \in \mathcal{C}$ *and* $x \in \mathrm{cl}(X \cup y)$.

Proof. To check (CLR1), let $\mathrm{cl}(X) = \{x_1, \ldots, x_k\}$. We repeat the argument of the proof of (CR2'). Since $r_{\mathcal{C}}(X \cup x_1) = r_{\mathcal{C}}(X)$, (CR1') applies to $X \cup x_1$ and $X \cup x_2$, so $X \cup x_1 \cup x_2 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x_1 \cup x_2) = r_{\mathcal{C}}(X)$. (CR1') then applies to $X \cup x_1 \cup x_2$ and $X \cup x_3$, so $X \cup x_1 \cup x_2 \cup x_3 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x_1 \cup x_2 \cup x_3) = r_{\mathcal{C}}(X)$. Continuing in this way, we conclude that $X \cup x_1 \cup \cdots \cup x_k \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x_1 \cup \cdots \cup x_k) = r_{\mathcal{C}}(X)$.

(CL1) is trivial.

To check (CL2), let $x \in \mathrm{cl}(X)$. Then $X \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X)$. Applying (CR1') to $X \cup x$ and Y, we conclude that $Y \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(Y \cup x) = r_{\mathcal{C}}(Y)$. Therefore $x \in \mathrm{cl}(Y)$.

We know that $\mathrm{cl}(X) \subseteq \mathrm{cl}(\mathrm{cl}(X))$; so to prove (CL3) it suffices to check the reverse inclusion. Let $x \in \mathrm{cl}(\mathrm{cl}(X))$. Then $\mathrm{cl}(X) \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(\mathrm{cl}(X) \cup x) = r_{\mathcal{C}}(\mathrm{cl}(X)) = r_{\mathcal{C}}(X)$. Therefore, since $\mathrm{cl}(X) \cup x \supseteq X \cup x \supseteq X$ and $\mathcal{C}$ is a simplicial complex, we have $X \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X)$ also; *i.e.*, $x \in \mathrm{cl}(X)$.

Finally, we check (CL4). The assumption that $y \in \mathrm{cl}(X \cup x)$ implies that $X \cup x \cup y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x \cup y) = r_{\mathcal{C}}(X \cup x) \leq r_{\mathcal{C}}(X) + 1$. Since $X \cup y \in \mathcal{C}$, the assumption that $y \notin \mathrm{cl}(X)$ implies that $r_{\mathcal{C}}(X) + 1 = r_{\mathcal{C}}(X \cup y)$. These two results together give $r_{\mathcal{C}}(X \cup x \cup y) = r_{\mathcal{C}}(X \cup y)$; *i.e.*, $x \in \mathrm{cl}(X \cup y)$. ☑

We will later need the following definitions.

Definition 2.8. *A* flat *of a semimatroid* $\mathcal{C}$ *is a set* $A \in \mathcal{C}$ *such that* $\mathrm{cl}(A) = A$. *The* poset of flats $K(\mathcal{C})$ *of a semimatroid* $\mathcal{C}$ *is the set of flats of* $\mathcal{C}$*, ordered by containment.*

3. **Modular ideals.**

In Sections 3, 4 and 5 we will present bijections between the class of semimatroids and other important classes of objects. Figure 2, which we will slowly get to understand by the end of Section 4, is a useful illustration of these bijections.

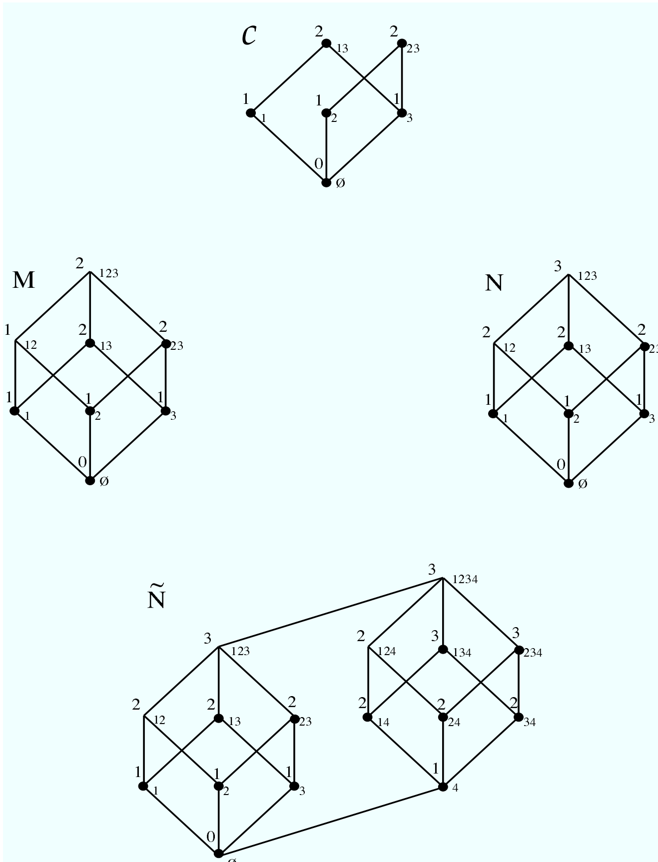

FIGURE 2. The semimatroid $\mathcal{C}$ and its corresponding matroids.

In this section, we show that a semimatroid is equivalent to a pair $(M,\mathcal{I})$ of a matroid M and one of its modular ideals $\mathcal{I}$. We start by showing how we can naturally construct a matroid $M_{\mathcal{C}}$ from a given semimatroid $(S,\mathcal{C},r_{\mathcal{C}})$, by extending the rank function $r_{\mathcal{C}}$ from $\mathcal{C}$ to 2^S. (In Figure 2, this amounts to extending the semimatroid $\mathcal{C}$ of Figure 1 to the matroid M. In Section 4 we will explain what N and $\widetilde{N}$ are.)

Proposition 3.1. *Let* $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ *be a semimatroid. For each subset* $X \subseteq S$*, let* $r(X) = \max\{r_{\mathcal{C}}(Y) \,|\, Y \subseteq X\,,\, Y \in \mathcal{C}\}$*. Then* r *is the rank function of a matroid* $M_{\mathcal{C}} = (S, r)$*.*

Proof. It is clear, but worth remarking explicitly, that $r(X) = r_{\mathcal{C}}(X)$ if $X \in \mathcal{C}$. It will be most convenient to check the three *local* axioms (R1')-(R3') for the rank function of a matroid [7]. Let $X \subseteq S$ and $a, b \in S$ be arbitrary.
(R1') $r(\emptyset) = 0$.
This is trivial.
(R2') $r(X \cup a) - r(X) = 0$ or 1.
This is easy. It is immediate from the definition that $r(X \cup a) \geq r(X)$. Now let $r(X \cup a) = r_{\mathcal{C}}(Y)$ for $Y \subseteq X \cup a$, $Y \in \mathcal{C}$. Then $Y - a \subseteq X$ is also in $\mathcal{C}$, so $r(X) \geq r_{\mathcal{C}}(Y - a) \geq r_{\mathcal{C}}(Y) - 1 = r(X \cup a) - 1$.
(R3') If $r(X \cup a) = r(X \cup b) = r(X)$, then $r(X \cup a \cup b) = r(X)$.
This takes more work. Assume that $r(X \cup a) = r(X \cup b) = r(X) = s$ but $r(X \cup a \cup b) = s + 1$. Let $W \subseteq X \cup a \cup b$, $W \in \mathcal{C}$ be such that $r_{\mathcal{C}}(W) = s + 1$. Notice that W must contain a; otherwise we would have $W \subseteq X \cup b$ and $r_{\mathcal{C}}(W) > r(X \cup b)$. Similarly, W contains b. So let $W = Z \cup a \cup b$; clearly $Z \subseteq X$.

We have $s + 1 = r_{\mathcal{C}}(Z \cup a \cup b) \leq r_{\mathcal{C}}(Z \cup a) + 1 \leq r(X \cup a) + 1 = s + 1$. Therefore $r_{\mathcal{C}}(Z \cup a) = s$. Similarly, $r_{\mathcal{C}}(Z \cup b) = s$. Then, by the submodularity of $r_{\mathcal{C}}$, $r_{\mathcal{C}}(Z) = s - 1$.

Now, since $r(X) = s$, we can find $V \subseteq X$, $V \in \mathcal{C}$ such that $r_{\mathcal{C}}(V) = s$. So we have $V, Z \in \mathcal{C}$ with $s = r_{\mathcal{C}}(V) > r_{\mathcal{C}}(Z) = s - 1$. By (CR2'), we can add an element of V to Z and obtain a set $Y \in \mathcal{C}$ with $X \supseteq Y \supseteq Z$ such that $r_{\mathcal{C}}(Y) = s$. Notice that $Z \cup a \subseteq Y \cup a \subseteq X \cup a$ and $r(Z \cup a) = r(X \cup a) = s$. Thus $r(Y \cup a) = s$. Similarly, $r(Y \cup b) = s$ and $r(Y \cup a \cup b) = s + 1$.

Now we have $Y, Z \cup a \cup b \in \mathcal{C}$ with $s + 1 = r_{\mathcal{C}}(Z \cup a \cup b) > r_{\mathcal{C}}(Y) = s$. Once again, (CR2') guarantees that we can add an element of $Z \cup a \cup b$ to Y to obtain an element of rank $s + 1$ in $\mathcal{C}$. But $Z \subseteq Y$, so the only elements of $Z \cup a \cup b$ which may not be in Y are a and b. Also, we saw that $r(Y \cup a) = r(Y \cup b) = s$. This is a contradiction. ☑

The following definitions will be important to us.

Definition 3.2. *A pair* $\{X, Y\}$ *of subsets of* S *is a* modular pair *of the matroid* $M = (S, r)$ *if* $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$*.*

Definition 3.3. [13] *A* modular ideal $\mathcal{I}$ *of a matroid* $M = (S, r)$ *is a non-empty collection of subsets of* S *satisfying the following three conditions.*
(MI1) $\mathcal{I}$ *is a simplicial complex.*
(MI2) $\{a\} \in \mathcal{I}$ *for every non-loop* a *of* M*.*
(MI3) *If* $\{X, Y\}$ *is a modular pair in* M *and* $X, Y \in \mathcal{I}$*, then* $X \cup Y \in \mathcal{I}$*.*

Proposition 3.4. *For any semimatroid* $(S, \mathcal{C}, r_{\mathcal{C}})$*, the collection* $\mathcal{C}$ *is a modular ideal of* $M_{\mathcal{C}}$*.*

Proof. We denote the rank function of $M_{\mathcal{C}}$ by r and the rank function of $\mathcal{C}$ by $r_{\mathcal{C}}$. Of course, $r_{\mathcal{C}}$ is just the restriction of r to $\mathcal{C}$.

Axioms (MI1) and (MI2) of a modular ideal are satisfied trivially. We reformulate (MI3) as follows:

(MI3) If $A, B, C \subseteq S$ are pairwise disjoint, $A \cup B, A \cup C \in \mathcal{C}$ and $r(A \cup B \cup C) - r(A \cup B) = r(A \cup C) - r(A)$, then $A \cup B \cup C \in \mathcal{C}$.

We can assume that B and C are non-empty; if one of them is empty, the claim is trivial. We prove (MI3) by induction on $|B| + |C|$.

The first case is $|B| + |C| = 2$; let $B = \{b\}$ and $C = \{c\}$. First assume that $r(A \cup b)$ and $r(A \cup c)$ are different; say, $r_{\mathcal{C}}(A \cup b) < r_{\mathcal{C}}(A \cup c)$. By (CR2), we can add some element of $A \cup c$ to $A \cup b$ and obtain a set in $\mathcal{C}$. This element can only be c, so $A \cup b \cup c \in \mathcal{C}$.

Assume then that $r_{\mathcal{C}}(A \cup b) = r_{\mathcal{C}}(A \cup c) = s$. If $r_{\mathcal{C}}(A) = s$, (CR1) implies that $A \cup b \cup c \in \mathcal{C}$. Assume then that $r_{\mathcal{C}}(A) = s - 1$, and therefore $r(A \cup b \cup c) = s + 1$ by hypothesis. There is a subset of $A \cup b \cup c$ in $\mathcal{C}$ of rank $s + 1$; since it cannot be contained in $A \cup b$ or $A \cup c$, it must be of the form $B \cup b \cup c$ for some $B \subseteq A$. But then we have $r_{\mathcal{C}}(A \cup b) < r_{\mathcal{C}}(B \cup b \cup c)$. By (CR2), we can add some element of $B \cup b \cup c$ to $A \cup b$ and obtain a set in $\mathcal{C}$. This element can only be c, so $A \cup b \cup c \in \mathcal{C}$.

Having established the base case of the induction, we proceed with the inductive step. Assume that $|B| + |C| \geq 3$ and, without loss of generality, $|B| \geq 2$. Let $b \in B$. Applying the submodularity of r twice, we get that $d = r(A \cup B \cup C) - r(A \cup B) \leq r(A \cup b \cup C) - r(A \cup b) \leq r(A \cup C) - r(A) = d$. It follows that $r(A \cup b \cup C) - r(A \cup b) = d$ also.

We can apply the induction hypothesis to the sets $A, \{b\}, C$, since $A \cup b, A \cup C \in \mathcal{C}$ and $|\{b\}| + |C| < |B| + |C|$. We conclude that $A \cup b \cup C \in \mathcal{C}$. We can then apply the induction hypothesis to the sets $A \cup b, B - b, C$, since $A \cup B, A \cup b \cup C \in \mathcal{C}$ and $|B - b| + |C| < |B| + |C|$. We conclude that $A \cup B \cup C \in \mathcal{C}$, as desired. ☑

Propositions 3.1 and 3.4 show us how to obtain a pair $(M, \mathcal{I})$ of a matroid M and one of its modular ideals $\mathcal{I}$, given a semimatroid $\mathcal{C}$. Now we show that it is possible to recover $\mathcal{C}$ from the pair $(M, \mathcal{I})$.

Proposition 3.5. *Let $M = (S, r)$ be a matroid, and let $\mathcal{I}$ be a modular ideal of M. Let $r_{\mathcal{I}}$ be the restriction of the rank function of M to $\mathcal{I}$. Then $(S, \mathcal{I}, r_{\mathcal{I}})$ is a semimatroid.*

Proof. The rank function $r_{\mathcal{I}}$ inherits axioms (R1)-(R3) from r_M. (CR1) is easy. If $X, Y \in \mathcal{I}$ and $r_{\mathcal{I}}(X) = r_{\mathcal{I}}(X \cap Y)$, then $r(Y) = r(X \cup Y)$ by submodularity. Thus $\{X, Y\}$ is a modular pair in M, and $X \cup Y \in \mathcal{I}$.

Now we check (CR2). We start by showing that $\mathcal{I}$ must contain every independent set of M. In fact, assume that I is a minimal independent set which is not in $\mathcal{I}$. Since $\mathcal{I}$ contains all non-loop elements, I has at least two elements a and b. Then $\mathcal{I}$ contains the modular pair $\{I - a, I - b\}$, so it contains their union I, a contradiction.

Now take $X, Y \in \mathcal{I}$ with $r(X) < r(Y)$, and pick $y \in Y$ such that $r(X \cup y) = r(X) + 1$. Let X' be an independent subset of X of rank $r(X)$; then $X' \cup y$ is an independent set of rank $r(X) + 1$. Therefore $\mathcal{I}$ contains the modular pair $\{X' \cup y, X\}$, so it contains their union $X \cup y$. ☑

Theorem 3.6. *Let S be a finite set. Let* $\mathrm{Semimat}(S)$ *be the set of semimatroids on S. Let* $\mathrm{MatId}(S)$ *be the set of pairs $(M, \mathcal{I})$ of a matroid M on S and a modular ideal $\mathcal{I}$ of M.*

(1) *The assignment $(S, \mathcal{C}, r_{\mathcal{C}}) \mapsto (M_{\mathcal{C}}, \mathcal{C})$ is a map* $\mathrm{Semimat}(S) \to \mathrm{MatId}(S)$.
(2) *The assignment $(M, \mathcal{I}) \mapsto (S, \mathcal{I}, r_{\mathcal{I}})$ is a map* $\mathrm{MatId}(S) \to \mathrm{Semimat}(S)$.
(3) *The two maps above are inverses of each other, and give a one-to-one correspondence between* $\mathrm{Semimat}(S)$ *and* $\mathrm{MatId}(S)$.

Proof. The first and second parts are restatements of Propositions 3.1 and 3.4 and Proposition 3.5, respectively.

Denote the maps $\mathrm{Semimat}(S) \to \mathrm{MatId}(S)$ and $\mathrm{MatId}(S) \to \mathrm{Semimat}(S)$ above by f and g respectively. It is immediate that $g \circ f$ is the identity map in $\mathrm{Semimat}(S)$. To check that $f \circ g$ is the identity map in $\mathrm{MatId}(S)$, we need to show the following. Given a matroid $M = (S, r)$ and a modular ideal $\mathcal{I}$ of M, $r(X) = \max\{r(Y) \mid Y \subseteq X, Y \in \mathcal{I}\}$ for all $X \subseteq S$. But this is easy: it is clear that $r(X) \geq \max\{r(Y) \mid Y \subseteq X, Y \in \mathcal{I}\}$. Equality is attained because X has an independent subset X' of rank $r(X)$; since X' is independent, it is in $\mathcal{I}$. ☑

Before we continue our analysis, we state explicitly a simple property of semimatroids and modular ideals which is implicit in the proofs above.

In a semimatroid $(S, \mathcal{C}, r_{\mathcal{C}})$, all the maximal sets in $\mathcal{C}$ have the same rank, which we denote $r_{\mathcal{C}}$. In a modular ideal $\mathcal{I}$ of a matroid $M = (S, r)$, all the maximal sets have maximum rank $r = r(S)$.

4. Elementary preimages and single-element coextensions.

Now we show that a semimatroid is also equivalent to a pair $(M, \widetilde{N})$ of a matroid M and one of its rank-increasing single-element coextensions $\widetilde{N}$. To do it, we outline the correspondence between the modular ideals, the elementary preimages and the rank-increasing single-element coextensions of a matroid.

This correspondence is just the dual of the well understood correspondence between the modular filters, the elementary quotients, and the rank-preserving single-element extensions of a matroid [10], [13], [16]. Therefore we omit all the proofs of these results, and refer the reader to the relevant literature.

The reader may want to to keep in mind Figure 2. Recall $\mathcal{C}$ is a semimatroid, and M is its extension to a matroid. From $\mathcal{C}$ and M we will obtain an elementary preimage of M, which is the matroid N, and a rank-increasing single-element coextension of M, which is the matroid $\widetilde{N}$.

Definition 4.1. *A* quotient map $N \to M$ *is a pair of matroids M, N on the same ground set such that every flat of M is a flat of N.*

There are several other equivalent definitions of quotient maps, including the following.

Proposition 4.2. [16, Proposition 8.1.6] *Let M and N be two matroids on the set S. The following are equivalent:*
(i) $N \to M$ is a quotient map.
(ii) For any $A \subseteq S$, $\mathrm{cl}_N(A) \subseteq \mathrm{cl}_M(A)$.
(iii) For any $A \subseteq B \subseteq S$, $r_N(B) - r_N(A) \geq r_M(B) - r_M(A)$.

Definition 4.3. *An* elementary quotient map *is a quotient map $N \to M$ such that $r(N) - r(M) = 0$ or 1.*

We will focus our attention on elementary quotient maps. Their importance is the following. Perhaps the most useful notion of a morphism in the category of matroids is that of a *strong map*. Every strong map between matroids can be regarded essentially as a quotient map, followed by an embedding of a submatroid into a matroid. Also, every quotient map can be factored into a sequence of elementary quotient maps. Therefore, elementary quotient maps are essentially the building blocks of strong maps. For more information on this topic, we refer the reader to [16].

Definition 4.4. *An* elementary preimage *of a matroid M is a matroid N on the same ground set such that $N \to M$ is an elementary quotient map.*

Elementary preimages are relevant in our investigation because they are equivalent to modular ideals:

Theorem 4.5. [13, Proposition 6.5] *Let $M = (S, r_M)$ be a matroid. Let $\mathrm{ModId}(M)$ be the set of modular ideals of M and let $\mathrm{Preim}(M)$ be the set of elementary preimages of M.*

(1) *Given $\mathcal{I} \in \mathrm{ModId}(M)$, define the rank function $r_N : 2^S \to \mathbb{N}$ by:*
$$r_N(A) = \begin{cases} r_M(A) & \text{if } A \in \mathcal{I} \\ r_M(A) + 1 & \text{if } A \notin \mathcal{I} \end{cases}$$
Then $N = (S, r_N)$ is a matroid, and $N \in \mathrm{Preim}(M)$.

(2) *Given $N \in \mathrm{Preim}(M)$, let $\mathcal{I} = \{A \in S : r_M(A) = r_N(A)\}$. Then $\mathcal{I} \in \mathrm{ModId}(M)$.*

(3) *The two maps $\mathrm{ModId}(M) \to \mathrm{Preim}(M)$ and $\mathrm{Preim}(M) \to \mathrm{ModId}(M)$ defined above are inverses of each other. They establish a one-to-one correspondence between $\mathrm{ModId}(M)$ and $\mathrm{Preim}(M)$.*

Corollary 4.6. *Given a finite set S, let $\mathrm{MatPreim}(S)$ be the set of pairs (M, N) of a matroid M on S and one of its elementary preimages N. Then there are one-to-one correspondences between $\mathrm{Semimat}(S)$, $\mathrm{MatId}(S)$ and $\mathrm{MatPreim}(S)$.*

Proof. Combine Theorems 3.6 and 4.5. ☑

Definition 4.7. *Let M be a matroid on the ground set S and let p be an element not in S. A* single-element coextension *of M by p is a matroid $\widetilde{N}$ on the set $S \cup p$ such that $M = \widetilde{N}/p$. $\widetilde{N}$ is* rank-increasing *if $r(\widetilde{N}) > r(M)$.*

It is worth remarking that most single-element coextensions of M by p are rank-increasing. The only one which is not rank-increasing is the matroid $\widetilde{N}$ on $S \cup p$ such that $r_{\widetilde{N}}(A \cup p) = r_{\widetilde{N}}(A) = r_M(A)$ for all $A \subseteq S$; *i.e.*, the one where p is a loop.

Theorem 4.8. [16, dual of Theorem 8.3.2] *Let M be a matroid and p be an element not in its ground set. Let $\mathrm{Coext}(M,p)$ be the set of rank-increasing single-element coextensions of M by p.*

(1) *Given $N \in \mathrm{Preim}(M)$, define $r_{\widetilde{N}} : 2^{S \cup p} \to \mathbb{N}$ by*

$$\begin{aligned} r_{\widetilde{N}}(A) &= r_N(A) \\ r_{\widetilde{N}}(A \cup p) &= r_M(A) + 1 \end{aligned}$$

for $A \subseteq S$. Then $\widetilde{N} = (S \cup p, r_{\widetilde{N}})$ is a matroid, and $\widetilde{N} \in \mathrm{Coext}(M,p)$.

(2) *If $\widetilde{N} \in \mathrm{Coext}(M,p)$, then the matroid $N = \widetilde{N} - p$ is in $\mathrm{Preim}(M)$.*

(3) *The two maps $\mathrm{Preim}(M) \to \mathrm{Coext}(M,p)$ and $\mathrm{Coext}(M,p) \to \mathrm{Preim}(M)$ defined above are inverses of each other. They establish a one-to-one correspondence between $\mathrm{Preim}(M)$ and $\mathrm{Coext}(M,p)$.*

Corollary 4.9. *Given a finite set S and an element $p \notin S$, let $\mathrm{MatCoext}(S,p)$ be the set of pairs $(M, \widetilde{N})$, where M is a matroid on S and $\widetilde{N}$ is one of its rank-increasing single-element coextensions by p. Then there are one-to-one correspondences between $\mathrm{Semimat}(S)$, $\mathrm{MatId}(S)$, $\mathrm{MatPreim}(S)$ and $\mathrm{MatCoext}(S,p)$.*

Proof. Combine Theorems 3.6, 4.5 and 4.8. ☑

We briefly mention that given a matroid M, there are other objects in correspondence with the modular ideals of M. Two such examples are the *modular cocuts* of M and the *colinear subclasses* of M. They are the duals of modular cuts and linear subclasses, respectively.

A modular cocut $\mathcal{U}$ of M is a collection of circuit unions of M satisfying two conditions. First, if $U_1 \subseteq U_2$ are circuit unions and $U_2 \in \mathcal{U}$, then $U_1 \in \mathcal{U}$. Second, if $U_1, U_2 \in \mathcal{U}$ and $\{U_1, U_2\}$ is a modular pair in M, then $U_1 \cup U_2 \in \mathcal{U}$.

A colinear subclass $\mathcal{C}$ of M is a set of circuits of M such that if $C_1, C_2 \in \mathcal{C}$ and $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$, and $C_3 \subseteq C_1 \cup C_2$ is a circuit, then $C_3 \in \mathcal{C}$.

The details and proofs of the (dual) correspondences appear in [20, Theorem 7.2.2] and [10], respectively.

We end this section by summarizing the correspondences and objects of Sections 3 and 4 in Figure 2, which we can now understand completely.

To the semimatroid $\mathcal{C}$, we have assigned a pair $(M, \mathcal{I}) \in \mathrm{MatId}(S)$, a pair $(M, N) \in \mathrm{MatPreim}(S)$ and a pair $(M, \widetilde{N}) \in \mathrm{MatCoext}(S)$. To obtain the matroid M, we add the subsets of [3] not in $\mathcal{C}$ to get the Boolean algebra $2^{[3]}$.

In Figure 2, we have placed big nodes on the sets of this poset which are in the diagram of $\mathcal{C}$, and small nodes on the new sets. To obtain the rank function of M, we copy the rank function of $\mathcal{C}$ on the big nodes. On each small node, we put the largest number that we can find on a big node below it. The big nodes form the modular ideal $\mathcal{I}$ of M.

To obtain the matroid N, we simply leave the rank function of M fixed on the big nodes, and increase it by 1 on the little nodes.

Finally, to obtain the matroid $\widetilde{N}$, we glue two Boolean algebras $2^{[3]}$, to obtain a Boolean algebra $2^{[4]}$ on 4 elements. (We have omitted most of the "diagonal" edges of this poset for clarity.) On the lower copy of the Boolean algebra, we put the rank function of N. On the upper copy, we put the rank function of M, increased by 1.

5. **Pointed matroids.**

We now establish a correspondence between semimatroids and pointed matroids.

Definition 5.1. [5] *A* pointed matroid *is a pair* (M, p) *of a matroid* M *and a distinguished element* p *of its ground set.*

Pointed matroids are a combinatorial tool often used in the study of affine hyperplane arrangements. The connection between them is the following. Consider an affine arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ in $\mathbb{R}^n$, where H_i is defined by the equation $v_i \cdot x = c_i$.

Definition 5.2. *The* cone over $\mathcal{A}$ *is the arrangement* $c\mathcal{A} = \{H'_1, \ldots, H'_k, H\}$ *in* $\mathbb{R}^{n+1}$*, where* H'_i *is defined* [1] *by the equation* $v_i \cdot x = c_i x_{n+1}$ *for* $1 \leq i \leq k$*, and* H *is the* additional hyperplane $x_{n+1} = 0$.

Being a central arrangement, $c\mathcal{A}$ has a matroid $M_{c\mathcal{A}}$ on the ground set $c\mathcal{A}$ associated to it. To the arrangement $\mathcal{A}$, we associate the pointed matroid $(M_{c\mathcal{A}}, H)$.

Theorem 5.3. *Let* S *be a set and let* $p \notin S$*. Let* Pointedmat(S, p) *be the set of pointed matroids* (M, p) *on* $S \cup p$ *such that* p *is not a loop of* M*. There are one-to-one correspondences between* Semimat(S), MatId(S), MatPreim(S), MatCoext(S, p) *and* Pointedmat(S, p).

Proof. It suffices to show a correspondence between MatCoext(S, p) and Pointedmat(S, p). The elements of MatCoext(S, p) are the pairs $(\widetilde{N}/p, \widetilde{N})$ for all matroids $\widetilde{N}$ on $S \cup p$ such that $r(\widetilde{N}) > r(\widetilde{N}/p)$; *i.e.*, such that p is not a loop. The map $(\widetilde{N}/p, \widetilde{N}) \mapsto (\widetilde{N}, p)$ establishes the desired bijection. ☑

At this point, given a set S and an element $p \notin S$, we have bijections between Semimat(S), MatId(S), MatPreim(S), MatCoext(S, p) and Pointedmat(S, p),

[1] with a slight abuse of notation

provided by Theorems 3.6, 4.6, 4.9, and 5.3. The bijection $\text{Pointedmat}(S,p) \to \text{Semimat}(S)$ is an important one. We have obtained it as the composition of four bijections, and now we wish to describe it explicitly.

Theorem 5.4. *Let S be a set and let $p \notin S$.*

(1) *For $(\widetilde{N},p) \in \text{Pointedmat}(S,p)$, let $\mathcal{C} = \{A \subseteq S \,|\, p \notin \text{cl}_{\widetilde{N}}(A)\}$ and let $r_{\mathcal{C}}$ be the restriction of $r_{\widetilde{N}}$ to $\mathcal{C}$. Then $(S,\mathcal{C},r_{\mathcal{C}})$ is a semimatroid.*

(2) *For $(S,\mathcal{C},r_{\mathcal{C}}) \in \text{Semimat}(S)$, define $r_{\widetilde{N}} : 2^{S\cup p} \to \mathbb{N}$ by*

$$r_{\widetilde{N}}(A) = \begin{cases} r_{\mathcal{C}}(A) & \text{if } A \in \mathcal{C} \\ \max\{r_{\mathcal{C}}(B) \,|\, B \subseteq A, B \in \mathcal{C}\} + 1 & \text{if } A \notin \mathcal{C} \end{cases}$$

$$r_{\widetilde{N}}(A \cup p) = \begin{cases} r_{\widetilde{N}}(A) + 1 & \text{if } A \in \mathcal{C} \\ r_{\widetilde{N}}(A) & \text{if } A \notin \mathcal{C} \end{cases}$$

for $A \subseteq S$. Then $r_{\widetilde{N}}$ is a rank function on $S \cup p$, and $(\widetilde{N},p) \in \text{Pointedmat}(S,p)$.

(3) *The two maps $\text{Pointedmat}(S,p) \to \text{Semimat}(S)$ and $\text{Semimat}(S) \to \text{Pointedmat}(S,p)$ defined above are inverses. They establish a one-to-one correspondence between $\text{Pointedmat}(S,p)$ and $\text{Semimat}(S)$.*

Proof. We will show that, if we start with $(\widetilde{N},p) \in \text{Pointedmat}(S,p)$ and trace the bijections of Theorems 5.3, 4.8, 4.5 and 3.6, we obtain the semimatroid $\mathcal{C}(\widetilde{N},p)$. Under the bijection of Theorem 5.3, $(\widetilde{N},p) \in \text{Pointedmat}(S,p)$ corresponds to $(\widetilde{N}/p,\widetilde{N}) \in \text{MatCoext}(S,p)$. Under the bijection of Theorem 4.8, $\widetilde{N} \in \text{Coext}(\widetilde{N}/p)$ corresponds to $\widetilde{N} - p \in \text{Preim}(\widetilde{N}/p)$. $\widetilde{N} - p \in \text{Preim}(\widetilde{N}/p)$, under the bijection of Theorem 4.5, corresponds to the modular ideal $\mathcal{C} = \{A \subseteq S \,|\, r_{\widetilde{N}/p}(A) = r_{\widetilde{N}-p}(A)\} \in \text{ModId}(\widetilde{N}/p)$. Since p is not a loop of $\widetilde{N}$, $r_{\widetilde{N}/p}(A) = r_{\widetilde{N}}(A \cup p) - 1$ and $r_{\widetilde{N}-p}(A) = r_{\widetilde{N}}(A)$. Therefore $\mathcal{C} = \{A \subseteq S \,|\, p \notin \text{cl}_{\widetilde{N}}(A)\}$. Finally, under the bijection of Theorem 3.6, $(\widetilde{N}/p,\mathcal{C}) \in \text{MatId}(S)$ corresponds to $(S,\mathcal{C},r_{\mathcal{C}}) \in \text{Semimat}(S)$. Similarly, if we start with a semimatroid $(S,\mathcal{C},r_{\mathcal{C}})$ and keep track of its successive images under the bijections of Theorems 3.6, 4.5, 4.8 and 5.3, we get the pointed matroid $(\widetilde{N},p)$ described. This theorem then becomes a consequence of the previous ones. ☑

It is not difficult to see that, under the coning construction, the central subsets of a hyperplane arrangement $\mathcal{A}$ correspond to the subsets of $c\mathcal{A}$ whose closure in $M_{c\mathcal{A}}$ does not contain the additional hyperplane H. Theorem 5.4 shows that, for semimatroids, the natural analog of the cone of a semimatroid $\mathcal{C}$ is the matroid $\widetilde{N}$ of the pointed matroid $(\widetilde{N},p)$ corresponding to it.

The triple of matroids $(\widetilde{N},\widetilde{N}-p,\widetilde{N}/p) = (\widetilde{N},N,M)$ is sometimes called the *triple of the pointed matroid* $(\widetilde{N},p)$. We will also call it the *triple of the semimatroid* $\mathcal{C}$.

6. **Geometric semilattices.**

We now discuss geometric semilattices and their relationship to semimatroids. We start by recalling some poset terminology. For more background information, see for example [22, Chapter 3].

A *meet semilattice* is a poset K such that any subset $S \subseteq K$ has a *greatest lower bound* or *meet* $\wedge S$: an element such that $\wedge S \leq s$ for all $s \in S$, and $\wedge S \leq t$ for any $t \in K$ such that $t \leq s$ for all $s \in S$. Such posets have a minimum element $\hat{0}$.

Notice that if a set S of elements of a meet semilattice has an upper bound, then it has a least upper bound, or *join* $\vee S$. It is the meet of the upper bounds of S.

A *lattice* is a poset L such that any subset $S \subseteq L$ has a greatest lower bound and a least upper bound. Clearly, if a meet semilattice has a maximum element, then it is a lattice.

A meet semilattice K is *ranked* with *rank function* $r : K \to \mathbb{N}$ if, for all $x \in K$, every maximal chain from $\hat{0}$ to x has the same length $r(x)$. An *atom* is an element of rank 1. A set of atoms A is *independent* if it has an upper bound and $r(\vee A) = |A|$.

Definition 6.1. *A* geometric semilattice *is a ranked meet semilattice satisfying the following two conditions.*
(G1) *Every element is a join of atoms.*
(G2) *The independent sets of atoms are the independent sets of a matroid.*
A geometric lattice *is a ranked lattice satisfying (G1) and (G2).*

Geometric lattices arise very naturally in matroid theory from the following result. Recall that a matroid $M = (S, r)$ is *simple* if $r(x) = 1$ for all $x \in S$ and $r(\{x, y\}) = 2$ for all $x, y \in S, x \neq y$.

Theorem 6.2. [3], [12] *A poset is a geometric lattice if and only if it is isomorphic to the poset of flats of a matroid. Furthermore, each geometric lattice is the poset of flats of a unique simple matroid, up to isomorphism.*

From this point of view, semimatroids are the "right" generalization of matroids, as the following theorem shows.

Definition 6.3. *A semimatroid* $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ *is* simple *if* $\{x\} \in \mathcal{C}$ *and* $r_{\mathcal{C}}(x) = 1$ *for all* $x \in S$*, and* $r_{\mathcal{C}}(\{x, y\}) = 2$ *for all* $\{x, y\} \in \mathcal{C}$ *with* $x \neq y$.

Theorem 6.4. *A poset is a geometric semilattice if and only if it is isomorphic to the poset of flats of a semimatroid. Furthermore, each geometric semilattice is the poset of flats of a unique simple semimatroid, up to isomorphism.*

To prove Theorem 6.4 we use the following two propositions.

Proposition 6.5. [28] *A poset K is a geometric semilattice if and only if there is a geometric lattice L with an atom p such that $K = L - [p, \hat{1}]$.* [2] *Furthermore, L and p are uniquely determined by K.*

Proposition 6.6. *Let $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ be a semimatroid, and let $(\widetilde{N}, p)$ be the pointed matroid on $S \cup p$ corresponding to it under the bijection of Theorem 5.4. Let $K(\mathcal{C})$ and $L(\widetilde{N})$ be the posets of flats of $\mathcal{C}$ and $\widetilde{N}$. Then $K(\mathcal{C}) = L(\widetilde{N}) - [p, \hat{1}]$.*

Proof. Since both posets are ordered by containment, we only need to show the equality of the sets $K(\mathcal{C})$ and $L(\widetilde{N}) - [p, \hat{1}]$.

First we show that $K(\mathcal{C}) \subseteq L(\widetilde{N}) - [p, \hat{1}]$. Let $X \in K(\mathcal{C})$. Then for all $x \notin X$ such that $X \cup x \in \mathcal{C}$, $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X) + 1$, and therefore $r_{\widetilde{N}}(X \cup x) = r_{\widetilde{N}}(X) + 1$. To check that X is a flat in $\widetilde{N}$, we need to show that this equality still holds if $X \cup x \notin \mathcal{C}$. This is not difficult: if that is the case and $x \neq p$, then $r_{\widetilde{N}}(X \cup x) = \max\{r_{\mathcal{C}}(Y) \,|\, Y \subseteq X \cup x, Y \in \mathcal{C}\} + 1 \geq r_{\mathcal{C}}(X) + 1 = r_{\widetilde{N}}(X) + 1$. Clearly then equality must hold. The case $x = p$ is easier, but needs to be checked separately. Hence $K(\mathcal{C}) \subseteq L(\widetilde{N})$, and since no element of $\mathcal{C}$ contains p, $K(\mathcal{C}) \subseteq L(\widetilde{N}) - [p, \hat{1}]$.

The inverse inclusion is easier. If X is a flat in $\widetilde{N}$ not containing p, then $r_{\widetilde{N}}(X \cup x) = r_{\widetilde{N}}(X) + 1$ for all $x \notin X$. When $X \cup x \in \mathcal{C}$, this equality says that $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X) + 1$. Therefore X is a flat in $\mathcal{C}$ also. ☑

Proof of Theorem 6.4. It is not difficult to check that the bijection of Theorem 5.4 restricts to a bijection between *simple pointed matroids* (pointed matroids $(\widetilde{N}, p) \in \text{Pointedmat}(S, p)$ such that $\widetilde{N}$ is simple) and simple semimatroids. The result then follows combining this fact with Theorem 6.2 and Propositions 6.5 and 6.6. ☑

7. Duality, deletion and contraction.

Like matroids, semimatroids have natural notions of duality, deletion and contraction, which we now define.

Definition 7.1. *Let $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ be a semimatroid. Extend the function $r_{\mathcal{C}}$ to a matroid rank function $r : 2^S \to \mathbb{N}$ as in Proposition 3.1. Define the simplicial complex $\mathcal{C}^* = \{X \subseteq S \,|\, S - X \notin \mathcal{C}\}$, and the rank function $r^* : \mathcal{C}^* \to \mathbb{N}$ by $r^*(X) = |X| - r + r(S - X)$. The* dual of $\mathcal{C}$ *is the triple $\mathcal{C}^* = (S, \mathcal{C}^*, r^*)$.*

Proposition 7.2. *$\mathcal{C}^*$ is a semimatroid.*

Proof. It is possible to simply check that $\mathcal{C}^*$ satisfies the axioms of a semimatroid. It is shorter to proceed as follows.

Consider the pair $(M, N) \in \text{MatPreim}(S)$ associated to $\mathcal{C}$ under Corollary 4.6. It is known [16, Proposition 8.1.6(f)] that if N is an elementary preimage

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[2] Here $[p, \hat{1}]$ denotes the interval of elements greater than or equal to p in the poset L.

of M, then M^* is an elementary preimage of N^*. From the pair $(N^*, M^*) \in \mathrm{MatPreim}(S)$, we then get a semimatroid using Corollary 4.6 again. It is straightforward to check that this semimatroid is precisely $\mathcal{C}^*$. ☑

Proposition 7.3. *For any semimatroid $\mathcal{C}$, we have that $(\mathcal{C}^*)^* = \mathcal{C}$.*

Proof. This is easy to check directly from the definition. ☑

Definition 7.4. *Let $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ be a semimatroid and let $e \in S$ be such that $\{e\} \in \mathcal{C}$. Let $\mathcal{C}/e = \{A \subseteq S - e \,|\, A \cup e \in \mathcal{C}\}$ and, for $A \in \mathcal{C}/e$, let $r_{\mathcal{C}/e}(A) = r_{\mathcal{C}}(A \cup e) - r_{\mathcal{C}}(e)$. The* contraction of e from $\mathcal{C}$ *is the triple $\mathcal{C}/e = (S - e, \mathcal{C}/e, r_{\mathcal{C}/e})$.*

Proposition 7.5. *$\mathcal{C}/e$ is a semimatroid.*

Proof. Checking the axioms of a semimatroid is straightforward. ☑

Definition 7.6. *Let $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ be a semimatroid and let $e \in S$ be such that $\{e\} \in \mathcal{C}$. Let $\mathcal{C} - e = \{A \in \mathcal{C} \,|\, e \notin A\}$ and, for $A \in \mathcal{C} - e$, let $r_{\mathcal{C}-e}(A) = r_{\mathcal{C}}(A)$. The* deletion of e from $\mathcal{C}$ *is the triple $\mathcal{C} - e = (S - e, \mathcal{C} - e, r_{\mathcal{C}-e})$.*

Proposition 7.7. *$\mathcal{C} - e$ is a semimatroid.*

Proof. Checking the axioms of a semimatroid is straightforward. ☑

Again, as with matroids, there are two special kinds of elements that we need to pay special attention to when we perform deletion and contraction.

Definition 7.8. *A* loop *of a semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ is an element $e \in S$ such that $\{e\} \in \mathcal{C}$ and $r_{\mathcal{C}}(e) = 0$.*

Definition 7.9. *An* isthmus *of a semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ is an element $e \in S$ such that, for all $A \in \mathcal{C}$, $A \cup e \in \mathcal{C}$ and $r_{\mathcal{C}}(A \cup e) = r_{\mathcal{C}}(A) + 1$.*

Lemma 7.10. *If $e \in S$ is a loop of the semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$, then $r_{\mathcal{C}/e} = r_{\mathcal{C}}$. Otherwise, $r_{\mathcal{C}/e} = r_{\mathcal{C}} - 1$.*

Proof. Clearly $r_{\mathcal{C}/e} \leq r_{\mathcal{C}}$. If e is a loop, consider any $A \in \mathcal{C}$. (CR1') applies to $\{e\}$ and A, so $A \cup e \in \mathcal{C}$ and $r_{\mathcal{C}}(A \cup e) = r_{\mathcal{C}}(A)$. Therefore the maximum rank $r_{\mathcal{C}}$ in $\mathcal{C}$ is achieved for some $A \in \mathcal{C}/e$. But then we have $r_{\mathcal{C}/e}(A) = r_{\mathcal{C}}(A \cup e) - 0 = r_{\mathcal{C}}$, so $r_{\mathcal{C}/e} = r_{\mathcal{C}}$.

If e is not a loop, then for all $A \in \mathcal{C}/e$ we have $r_{\mathcal{C}/e}(A) = r_{\mathcal{C}}(A \cup e) - 1$, so $r_{C/e} \leq r_C - 1$. Equality holds: if we start with $\{e\} \in \mathcal{C}$ and repeatedly apply (CR2') with an element of $\mathcal{C}$ of rank $r_{\mathcal{C}}$, we can obtain a set $A \cup e$ of rank $r_{\mathcal{C}}$. Then $r_{C/e}(A) = r_C - 1$. ☑

Lemma 7.11. *If $e \in S$ is an isthmus of the semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$, then $r_{\mathcal{C}-e} = r_{\mathcal{C}} - 1$. Otherwise, $r_{\mathcal{C}-e} = r_{\mathcal{C}}$.*

Proof. Clearly $r_{\mathcal{C}-e} \leq r_{\mathcal{C}}$. If e is an isthmus then it is clear from the definition that $r_{\mathcal{C}-e} = r_{\mathcal{C}} - 1$.

If e is not an isthmus, there are two cases. If there is an $A \in \mathcal{C}$ such that $A \cup e \notin \mathcal{C}$, take a maximal one. It is also a maximal set in $\mathcal{C}$, so it has maximum rank $r_{\mathcal{C}}$; and $A \in \mathcal{C} - e$, so $r_{\mathcal{C}-e} = r_{\mathcal{C}}$. The other possibility is that for all $A \in \mathcal{C}$, we have $A \cup e \in \mathcal{C}$ and $r(A \cup e) = r(A)$. In this case it is also clear that $r_{\mathcal{C}-e} = r_{\mathcal{C}}$. ☑

Lemma 7.12. *If $e \in S$ is a loop or an isthmus of the semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$, then $\mathcal{C} - e = \mathcal{C}/e$.*

Proof. This is clear from Lemmas 7.10 and 7.11 and their proofs. ☑

8. **The Tutte polynomial.**

With the background results that we have established, we are now able to define and study the Tutte polynomial of a semimatroid. We follow the treatment of Tutte polynomials of matroids given in [8].

Definition 8.1. *The* Tutte polynomial *of a semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ is defined by*

$$T_{\mathcal{C}}(x,y) = \sum_{X \in \mathcal{C}} (x-1)^{r_{\mathcal{C}} - r_{\mathcal{C}}(X)} (y-1)^{|X| - r_{\mathcal{C}}(X)}. \tag{8.1}$$

If $\mathcal{C} = 2^S$, then $\mathcal{C}$ is a matroid and $T_{\mathcal{C}}$ is its usual Tutte polynomial. If $\mathcal{A}$ is a hyperplane arrangement and $\mathcal{C}_{\mathcal{A}}$ is the semimatroid determined by it, then the Tutte polynomial of the semimatroid $\mathcal{C}_{\mathcal{A}}$ is precisely the Tutte polynomial of the arrangement $\mathcal{A}$, as defined and studied in [2]. That paper focuses on enumerative aspects arising from the computation of these polynomials; here we will concentrate our attention on matroid-theoretical considerations.

Example. Figure 3 shows a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^3$, consisting of the five planes $x_1 + x_2 + x_3 = 0, x_1 = x_2, x_2 = x_3, x_3 = x_1$ and $x_1 + x_2 + x_3 = 1$.

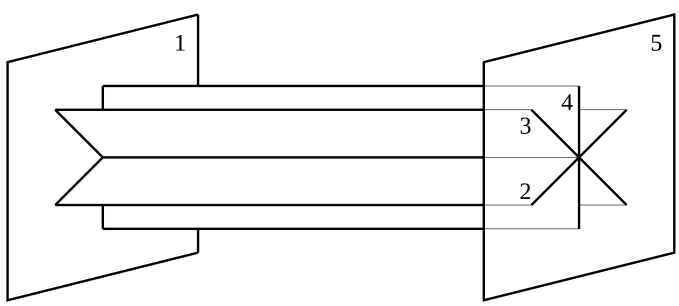

FIGURE 3. The arrangement $\mathcal{A}$.

Table 1 shows all the central subsets of $\mathcal{A}$, and their contributions to the Tutte polynomial of $\mathcal{A}$.

TABLE 1. Computing the Tutte polynomial $T_{\mathcal{A}}(x,y)$.

central subset of $\mathcal{A}$	contribution to $T_{\mathcal{A}}(x,y)$
$\emptyset$	$(x-1)^3(y-1)^0$
$1,2,3,4,5$	$(x-1)^2(y-1)^0$
$12,13,14,23,24,25,34,35,45$	$(x-1)^1(y-1)^0$
$123,124,134,235,245,345$	$(x-1)^0(y-1)^0$
234	$(x-1)^1(y-1)^1$
$1234,2345$	$(x-1)^0(y-1)^1$

We find that

$$\begin{aligned} T_{\mathcal{A}}(x,y) &= (x-1)^3+5(x-1)^2+9(x-1)+6+(x-1)(y-1)+2(y-1) \\ &= x^3+2x^2+xy+x+y. \end{aligned}$$

As in the matroid setting, the Tutte polynomial of a semimatroid satisfies the following simple recursive formula, known as the *deletion-contraction* relation.

Proposition 8.2. *Let $\mathcal{C}=(S,\mathcal{C},r_{\mathcal{C}})$ be a semimatroid, and let $e\in S$. If $\{e\}\notin\mathcal{C}$ then $T_{(S,\mathcal{C},r_{\mathcal{C}})}(x,y)=T_{(S-e,\mathcal{C},r_{\mathcal{C}})}(x,y)$. If $\{e\}\in\mathcal{C}$, then*

(i) $T_{\mathcal{C}}(x,y)=T_{\mathcal{C}-e}(x,y)+T_{\mathcal{C}/e}(x,y)$ *if e is neither an isthmus nor a loop,*
(ii) $T_{\mathcal{C}}(x,y)=x\,T_{\mathcal{C}-e}(x,y)$ *if e is an isthmus, and*
(iii) $T_{\mathcal{C}}(x,y)=y\,T_{\mathcal{C}/e}(x,y)$ *if e is a loop.*

Proof. The first statement is clear from the definitions. Now, when $\{e\}\in\mathcal{C}$, we have

$$\begin{aligned} T_{\mathcal{C}}(x,y) &= \sum_{\substack{X\in\mathcal{C}\\ e\notin X}}(x-1)^{r_{\mathcal{C}}-r_{\mathcal{C}}(X)}(y-1)^{|X|-r_{\mathcal{C}}(X)}+ \\ &\qquad\qquad \sum_{X\cup e\in\mathcal{C}}(x-1)^{r_{\mathcal{C}}-r_{\mathcal{C}}(X\cup e)}(y-1)^{|X\cup e|-r_{\mathcal{C}}(X\cup e)}. \end{aligned}$$

Notice that, if $r_{\mathcal{C}}=r_{\mathcal{C}-e}$, the first sum in the right hand side is exactly the Tutte polynomial of $\mathcal{C}-e$. If, on the other hand, $r_{\mathcal{C}}=r_{\mathcal{C}-e}+1$, the only difference is that we get an extra factor of $(x-1)$. More precisely, in view of Lemma 7.11, the first sum of the right hand side is $T_{\mathcal{C}-e}(x,y)$ if e is not an isthmus, and $(x-1)T_{\mathcal{C}-e}(x,y)$ if it is an isthmus. Similarly, from Lemma 7.10, the second sum is $T_{\mathcal{C}/e}(x,y)$ if e is not a loop, and $(y-1)T_{\mathcal{C}/e}(x,y)$ if it is a loop. These two observations, together with Lemma 7.12, complete the proof of *(i)-(iii)*. ☑

Definition 8.3. *Two matroids $(S_1,\mathcal{C}_1,r_{\mathcal{C}_1})$ and $(S_2,\mathcal{C}_2,r_{\mathcal{C}_2})$ are* isomorphic *if there is a bijection $f:S_1\to S_2$ which induces an isomorphism of simplicial complexes $f:\mathcal{C}_1\to\mathcal{C}_2$ such that $r_{\mathcal{C}_1}(c)=r_{\mathcal{C}_2}(f(c))$ for all $c\in\mathcal{C}_1$.*

A function f on the class $\mathbb{S}$ of semimatroids is called a *semimatroid invariant* if $f(\mathcal{C}_1) = f(\mathcal{C}_2)$ for all $\mathcal{C}_1 \cong \mathcal{C}_2$. An invariant is called a *Tutte-Grothendieck invariant* (or *T-G invariant*) if it satisfies the conditions of Proposition 8.2. The following theorem shows that the Tutte polynomial is not only a T-G invariant; in fact it is the universal T-G invariant on the class of semimatroids. Any other *generalized T-G invariant*, that is, an invariant satisfying the conditions of Theorem 8.5, is an evaluation of the Tutte polynomial. This result is well-known for matroids [6], [19].

Definition 8.4. *For a semimatroid $\mathcal{C} = (S, \mathcal{C}, r)$, let $\#\mathcal{C}$ be the number of elements $x \in S$ such that $\{x\} \in \mathcal{C}$. A semimatroid is* non-trivial *if $\#\mathcal{C} \neq 0$.*

Theorem 8.5. *Let $\mathbb{S}$ be the class of non-trivial semimatroids. Let $\mathbb{k}$ be a field and $a, b \in \mathbb{k}$; and let R be a commutative ring containing $\mathbb{k}$. Let $f : \mathbb{S} \to R$ be a* generalized T-G invariant*; that is, suppose:*

(i) *If $\mathcal{C}_1 \cong \mathcal{C}_2$ then $f(\mathcal{C}_1) = f(\mathcal{C}_2)$.*
(ii) *If $e \in S$ is neither an isthmus nor a loop in $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ and $\{e\} \in \mathcal{C}$, then $f(\mathcal{C}) = af(\mathcal{C} - e) + bf(\mathcal{C}/e)$.*
(iii) *If e is an isthmus in $\mathcal{C}$, then $f(\mathcal{C}) = f(I)f(\mathcal{C} - e)$.*
(iv) *If e is a loop in $\mathcal{C}$, then $f(\mathcal{C}) = f(L)f(\mathcal{C}/e)$.*
(v) *If $e \in S$ and $\{e\} \notin \mathcal{C}$ then $f(S, \mathcal{C}, r_{\mathcal{C}}) = f(S - e, \mathcal{C}, r_{\mathcal{C}})$.*

Then the function f is given by $f(\mathcal{C}) = a^{\#\mathcal{C} - r_{\mathcal{C}}}\, b^{r_{\mathcal{C}}}\, T_{\mathcal{C}}(f(I)/b, f(L)/a)$ for $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$.

Here $I = (\{i\}, \{\emptyset, \{i\}\}, r)$ denotes the semimatroid consisting of a single isthmus i, and $L = (\{l\}, \{\emptyset, \{l\}\}, r)$ denotes the semimatroid consisting of a single loop l.

Proof. We can proceed by induction. The only non-trivial semimatroids which cannot be decomposed using $(ii), (iii), (iv)$ and (v) are I and L, in which case the formula for $f(\mathcal{C})$ holds trivially. It simply remains to show that $a^{\#\mathcal{C} - r_{\mathcal{C}}}\, b^{r_{\mathcal{C}}} T_{\mathcal{C}}(f(I)/b, f(L)/a)$ satisfies the relations $(ii), (iii), (iv)$ and (v). This is straightforward from Proposition 8.2. ☑

We conclude this section with some remarks about the relationship between the Tutte polynomial of a semimatroid $\mathcal{C}$, the Tutte polynomials of its associated triple $(\widetilde{N}, N, M)$, and the Tutte polynomial of the dual semimatroid $\mathcal{C}^*$.

Recall that the characteristic polynomial $\chi_{\mathcal{A}}(q)$ of an affine hyperplane arrangement $\mathcal{A}$ in an n-dimensional vector space is given by

$$\chi_{\mathcal{A}}(q) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{|\mathcal{B}|} q^{n - r(\mathcal{B})}.$$

Characteristic polynomials behave nicely with respect to the coning construction of Definition 5.2.

Proposition 8.6. ([18, Proposition 2.51]) *For any arrangement $\mathcal{A}$,*

$$\chi_{c\mathcal{A}}(q) = (q-1)\chi_{\mathcal{A}}(q).$$

This proposition tells us that, to study characteristic polynomials of arrangements, we can essentially focus our attention on central arrangements.

Proposition 8.6 generalizes immediately to semimatroids. As we saw in Theorem 5.4, the analog of the cone $c\mathcal{A}$ of an arrangement $\mathcal{A}$ is the matroid $\widetilde{N}$ of the semimatroid $\mathcal{C}$. If, in analogy with the definition for arrangements, we define the *characteristic polynomial of the semimatroid* $\mathcal{C}$ to be

$$\chi_{\mathcal{C}}(q) = (-1)^r T_{\mathcal{C}}(1-q, 0) = \sum_{X \in \mathcal{C}} (-1)^{|X|} q^{r-r(X)}.$$

We have the following proposition.

Proposition 8.7. *For any semimatroid $\mathcal{C}$,*

$$\chi_{\widetilde{N}}(q) = (q-1)\chi_{\mathcal{C}}(q).$$

We might wonder if this result generalizes to the Tutte polynomial. It turns out that this situation is not so simple. Let

$$U_{\mathcal{C}}(x,y) = \sum_{X \notin \mathcal{C}} (x-1)^{r_M - r_M(X)} (y-1)^{|X| - r_M(X)}. \tag{8.2}$$

Then, by looking at the defining sums of T_M, T_N and $T_{\widetilde{N}}$, it is easy to see that $T_M = T_{\mathcal{C}} + U_{\mathcal{C}}$, $T_N = (x-1)T_{\mathcal{C}} + U_{\mathcal{C}}/(y-1)$, and $T_{\widetilde{N}} = xT_{\mathcal{C}} + y/(y-1)U_{\mathcal{C}}$. (The third of these equations proves Proposition 8.7.) This means that we *can* express the Tutte polynomial of $\mathcal{C}$ in terms of the Tutte polynomials of these three matroids M, N and $\widetilde{N}$, by solving for $T_{\mathcal{C}}$ in any two of these three equations. However, $T_{\mathcal{C}}$ does not only depend on $T_{\widetilde{N}}$. The relation between the characteristic polynomials is very simple because, when we substitute $x = 1-q$ and $y = 0$, the second term in the expression of $T_{\widetilde{N}}$ vanishes

We conclude that the Tutte polynomial of a semimatroid is closely related to the Tutte polynomials of its associated triple $(\widetilde{N}, N, M)$. However, the relationship is not simple enough that we can derive our results on Tutte polynomials of semimatroids as simple consequences of the analogous results for matroids.

Now let us discuss duality and the Tutte polynomial. For matroids M, we know that $T_{M^*}(x,y) = T_M(y,x)$. This is not the case for a semimatroid $\mathcal{C}$. In fact, it is not difficult to see that $T_{\mathcal{C}^*}(x,y) = U_{\mathcal{C}}(y,x)/(x-1)$.

It is possible to define a three-variable Tutte-like polynomial of a semimatroid which is more compatible with duality. In a slightly different language, this was done by Las Vergnas [17], who defined the concept of the *Tutte polynomial of a quotient map*. In fact, if the semimatroid $\mathcal{C}$ corresponds to the quotient map $N \to M$ under Corollary 4.6, then our definition of the Tutte polynomial of $\mathcal{C}$ coincides with the coefficient of z in Las Vergnas's definition of the Tutte polynomial of the quotient map $N \to M$. In particular, the upcoming Theorem 9.5 can be derived from his analogous theorem for quotient maps. His argument uses the deletion-contraction relation; our approach will give us additional information about the structure of a semimatroid.

9. **Basis activity.**

We now show that the Tutte polynomial of a semimatroid has nonnegative coefficients, by giving a combinatorial interpretation of them. Crapo showed that the coefficients of the Tutte polynomial of a matroid count the bases with a given internal and external activity [11]. Our interpretation in the case of semimatroids is analogous, and our proof is similar to his. There are some subtleties involved in extending this result to semimatroids, so we will need to give slightly different definitions of internal and external activity.

In this section we will work with a fixed semimatroid $\mathcal{C} = (S, \mathcal{C}, r)$. As mentioned after Definition 2.1, we will sometimes call the sets in $\mathcal{C}$ central sets. Proposition 3.1 shows that the rank function r extends to a matroid rank function on 2^S, which we will also call r. No confusion arises from this notation because the semimatroid and matroid rank functions have the same value where they are both defined.

A *basis* of $\mathcal{C} = (S, \mathcal{C}, r)$ is a set $B \in \mathcal{C}$ such that $|B| = r(B) = r$. A set $X \in \mathcal{C}$ is *dependent* if $r(X) < |X|$ and *independent* otherwise. A *circuit* C of $\mathcal{C}$ is a minimal dependent set in $\mathcal{C}$. Clearly such a set satisfies $r(C) = |C| - 1$. A *cocircuit* D is a minimal subset of S whose deletion from $\mathcal{C}$ makes the rank of $\mathcal{C}$ decrease; *i.e.*, one such that $r(S - D) < r$, where $r = r(S)$ is the rank of $\mathcal{C}$. Clearly a cocircuit satisfies $r(S - D) = r - 1$.

Lemma 9.1. *Let B be a basis of $\mathcal{C}$, and let $e \notin B$ be such that $B \cup e \in \mathcal{C}$. Then $B \cup e$ contains a unique circuit.*

Proof. Since $B \cup e \in \mathcal{C}$ is dependent, it contains a circuit. Now assume that it contains two different circuits C_1 and C_2. By (R3) we know that

$$\begin{aligned} r(C_1 \cap C_2) + r(C_1 \cup C_2) &\leq r(C_1) + r(C_2) \\ &= |C_1| - 1 + |C_2| - 1 \\ &= |C_1 \cap C_2| - 1 + |C_1 \cup C_2| - 1. \end{aligned}$$

But $r(B \cup e) = |B \cup e| - 1$ so, by (R2'), $r(X) \geq |X| - 1$ for all $X \subseteq B \cup e$. Therefore $r(C_1 \cap C_2) = |C_1 \cap C_2| - 1$ and $r(C_1 \cup C_2) = |C_1 \cup C_2| - 1$. Thus $C_1 \cap C_2$ is a dependent set in $\mathcal{C}$, and it is a proper subset of the circuit C_1. This is a contradiction. ☑

Lemma 9.2. *Let B be a basis of $\mathcal{C}$, and let $i \in B$. Then $S - B \cup i$ contains a unique cocircuit.*

Proof. The deletion of $S - B \cup i$ from $\mathcal{C}$ makes the rank of $\mathcal{C}$ decrease, so this set contains a cocircuit. Assume that it contains two different cocircuits B_1

and B_2. Then

$$\begin{aligned} r(S-(B_1\cap B_2)) &= r((S-B_1)\cup(S-B_2)) \\ &\leq r(S-B_1)+r(S-B_2)-r((S-B_1)\cap(S-B_2)) \\ &= (r-1)+(r-1)-r(S-(B_1\cup B_2)). \end{aligned}$$

But $S-(B_1\cup B_2)\supseteq B-i$ and $r(B-i)=r-1$, so $r(S-(B_1\cup B_2))\geq r-1$. It follows that $r(S-(B_1\cap B_2))\leq r-1$. Hence the removal of $B_1\cap B_2$ makes the rank of the semimatroid decrease, and $B_1\cap B_2$ is a proper subset of the cocircuit B_1. This is a contradiction. ☑

From now on, we will fix a linear order on S. Now each k-subset of S corresponds to a strictly increasing sequence of k numbers between 1 and $|S|$. For each $0\leq k\leq |S|$, order the k-subsets of S using the lexicographic order on these sequences.

Definition 9.3. *Let B be a basis of $\mathcal{C}$. An element $e\notin B$ is an* externally active element *for B if $B\cup e\in\mathcal{C}$ and e is the smallest element*[3] *of the unique circuit in $B\cup e$. Let $E(B)$ be the set of externally active elements for B, and let $e(B)=|E(B)|$. We call $e(B)$ the* external activity *of B.*

Definition 9.4. *Let B be a basis of $\mathcal{C}$. An element $i\in B$ is an* internally active element *in B if i is the smallest element of the unique cocircuit in $S-B\cup i$. Let $I(B)$ be the set of internally active elements for B, and let $i(B)=|I(B)|$. We call $i(B)$ the* internal activity *of B.*

Now we are in a position to state the main theorem of this section.

Theorem 9.5. *For any semimatroid $\mathcal{C}$,*

$$T_{\mathcal{C}}(x,y)=\sum_{B \text{ basis of } \mathcal{C}} x^{i(B)}y^{e(B)}.$$

Theorem 9.5 shows that the coefficients of the Tutte polynomial are non-negative integers. The coefficient of q^it^e is equal to the number of bases of $\mathcal{C}$ with internal activity i and external activity e.

We still have some work to do before we can prove Theorem 9.5. The next step will be to give a very useful characterization of internally and externally active elements. From now on, when proving results about internally and externally active elements, we will always use Lemmas 9.6 and 9.7 instead of the original definitions.

Given $X\subseteq S$ and an element e, let $X_{>e}=\{x\in X\,|\,x>e\}$. Define $X_{<e}$ analogously.

Lemma 9.6. *Let B be a basis of $\mathcal{C}$ and let $e\notin B$ be such that $B\cup e\in\mathcal{C}$. Then e is externally active for B if and only if $r(B_{>e}\cup e)=r(B_{>e})$.*

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[3] according to the fixed linear order

Proof. First assume that $r(B_{>e} \cup e) = r(B_{>e})$. Then $B_{>e} \cup e \in \mathcal{C}$ is dependent, so it contains a circuit C; e is clearly the smallest element in this circuit. But C must also be the unique circuit contained in $B \cup e$. Therefore e is an externally active element for B.

Now assume that e is externally active for B. The unique circuit in $B \cup e$ obviously contains e; call it $C \cup e$. Then $C \subseteq B_{>e}$. By submodularity, we have $r(B_{>e}) + r(C \cup e) \geq r(B_{>e} \cup e) + r(C)$. But $r(C \cup e) = r(C)$, so $r(B_{>e}) \geq r(B_{>e} \cup e)$ and the desired result follows. ☑

Lemma 9.7. *Let B be a basis and $i \in B$. Then i is internally active in B if and only if $r(B - i \cup S_{<i}) < r$.*[4]

Proof. First assume that $r(B - i \cup S_{<i}) < r$. Then the removal of $(S - B)_{>i} \cup i$ makes the rank of the semimatroid drop, so $(S - B)_{>i} \cup i$ contains a cocircuit. This cocircuit must contain i; call it $D \cup i$, where $D \subseteq (S - B)_{>i}$. The smallest element of this cocircuit is i, and this cocircuit must also be the unique cocircuit contained in $S - B \cup i$. Therefore i is an internally active element of B.

Now assume that i is internally active in B. Let $S - D \cup i$ be the unique cocircuit in $S - B \cup i$, where $D \supseteq B$. Since i is the smallest element in this cocircuit, $D \supseteq S_{<i}$. Therefore $B \cup S_{<i} \subseteq D$ and, since $S - D \cup i$ is a cocircuit, $r(B - i \cup S_{<i}) < r(D - i) < r$. ☑

Now we wish to present a different description of sets in $\mathcal{C}$. To do it, we need two definitions. For each $X \subseteq S$, let dX be the lexicographically largest basis of X. For each independent set X, which is necessarily in $\mathcal{C}$, let uX be the lexicographically smallest basis of $\mathcal{C}$ which contains X.[5] Notice that, for any $X \subseteq S$, udX is a basis of $\mathcal{C}$.

Definition 9.8. *Let $\mathcal{T}$ be the set of triples (B, I, E) such that B is a basis of $\mathcal{C}$, $I \subseteq I(B)$ is a set of internally active elements for B, and $E \subseteq E(B)$ is a set of internally active elements of B.*

We will establish a bijection between $\mathcal{T}$ and $\mathcal{C}$. Define two maps ϕ_1 and ϕ_2 as follows. Given $(B, I, E) \in \mathcal{T}$, let $\phi_1(B, I, E) = B - I \cup E$. Given $X \in \mathcal{C}$, let $\phi_2(X) = (udX, udX - dX, X - dX)$. We will show that the maps ϕ_1 and ϕ_2 give the desired bijection: every set $X \in \mathcal{C}$ can be written uniquely in the form $X = B - I \cup E$ where B is a basis of $\mathcal{C}$, $I \subseteq I(B)$ and $E \subseteq E(B)$.

Example. Recall the arrangement $\mathcal{A}$ introduced at the beginning of Section 8. Table 2 illustrates the bijection between $\mathcal{T}$ and $\mathcal{C}$ in that case. Theorem 9.5 and Table 2 imply that $T_{\mathcal{A}}(x, y) = x^3 + 2x^2 + xy + x + y$, confirming our computation at the beginning of Section 8.

Lemma 9.9. *The map ϕ_1 maps $\mathcal{T}$ to $\mathcal{C}$.*

[4] In fact, this is true if and only if $r(B - i \cup S_{<i}) = r - 1$.

[5] We will extend the definition of uX to all $X \subseteq S$ after the proof of Lemma 9.14. For simplicity, we postpone the full definition until then.

TABLE 2. The bijection between $\mathcal{T}$ and $\mathcal{C}$.

B	$I(B)$	$E(B)$	possible $B - I \cup E$
123	123	-	$\emptyset, 1, 2, 3, 12, 13, 23, 123$
124	12	-	$4, 14, 24, 124$
134	1	2	$34, 134, 234, 1234$
235	23	-	$5, 25, 35, 235$
245	2	-	$45, 245$
345	-	2	$345, 2345$

Proof. Let $(B, I, E) \in \mathcal{T}$. For all $e \in E$, $B \cup e$ is central and $r(B \cup e) = r(B)$, so $e \in \mathrm{cl}(B)$. Therefore $E \subseteq \mathrm{cl}(B)$ and $B \cup E \subseteq \mathrm{cl}(B)$. Since $\mathrm{cl}(B) \in \mathcal{C}$, this implies that $B \cup E \in \mathcal{C}$, and $B - I \cup E \in \mathcal{C}$ as well. ☑

Lemma 9.10. *The map ϕ_2 maps $\mathcal{C}$ to $\mathcal{T}$.*

Proof. Let $X \in \mathcal{C}$. Let $D = dX$ and $U = udX$, so that $\phi_2(X) = (U, U - D, X - D)$. We need to show three things.

First, we need U to be a basis for X. This is immediate.

Next, we need the elements of $U - D$ to be internally active in U. Let $x \in U - D$. Since U is the smallest basis for $\mathcal{C}$ containing D, for any element $x' < x$ not in U we have $r(U - x \cup x') = r - 1 = r(U - x)$. By submodularity, we can conclude that $r(U - x \cup S_{<x}) = r - 1$, which is exactly what we wanted.

Finally, we need to show that the elements of $X - D$ are externally active in U. Let $x \in X - D$. First notice that $x \notin U$, because $D \cup x$ is dependent: $r(D \cup x) \leq r(X) = r(D)$. Also notice that $U \cup x$ is central, applying (CR1) to $D \cup x$ and U. Now observe the following. We know that D is the largest basis for X. Therefore $r(D - x' \cup x) = r(D) - 1$ for all $x' \in D_{<x}$. By submodularity, it follows that $r(D - D_{<x} \cup x) = r(D) - |D_{<x}|$. We can rewrite this as $r(D_{>x} \cup x) = r(D_{>x})$ since D is independent. Since $D_{>x} \subseteq U_{>x}$, submodularity implies that $r(U_{>x} \cup x) = r(U_{>x})$. This shows that x is an externally active element in U. ☑

Proposition 9.11. *The map ϕ_1 is a bijection from $\mathcal{T}$ to $\mathcal{C}$, and the map ϕ_2 is its inverse.*

Proposition 9.11 is the main ingredient of our proof of Theorem 9.5. Before proving it, we need some lemmas.

Lemma 9.12. *For all $(B, I, E) \in \mathcal{T}$, we have $r(B - I \cup E) = r - |I|$.*

Proof. We start by showing that $r(B - i \cup e) = r - 1$ for all $i \in I(B), e \in E(B)$. If $e < i$, do the following. Since i is internally active, $r(B - i \cup S_{<i}) = r - 1 = r(B - i)$, and therefore $r(B - i \cup e) = r - 1$. Otherwise, if $i < e$, then

$B_{>e} \subseteq B - i$. Since e is externally active, $r(B_{>e} \cup e) = r(B_{>e})$. Submodularity then implies that $r(B - i \cup e) = r(B - i) = r - 1$.

Now that we know this, submodularity implies that $r(B - i \cup E) = r - 1$ for all $i \in I(B), E \subseteq E(B)$. Applying submodularity again, we get $r(B - I \cup E) = r - |I|$ for all $I \subseteq I(B), E \subseteq E(B)$. ☑

Lemma 9.13. *For all $(B, I, E) \in \mathcal{T}$, we have $d(B - I \cup E) = B - I$.*

Proof. Lemma 9.12 tells us that $B - I$ is a basis for $B - I \cup E$; we need to show that it is the largest one. Consider an arbitrary $(r - |I|)-$ subset X of $B - I \cup E$ with $X > B - I$. We will show that X is not a basis for $B - I \cup E$.

Let $X = (B - I) - (b_1 \cup \cdots \cup b_k) \cup (e_1 \cup \cdots \cup e_k)$, where the b_i's are in $B - I$ and the e_i's are in E. Since $X > B - I$ we can assume, without loss of generality, that $b_1 < e_1, \ldots, e_k$.

From Lemma 9.12 we know that $r(B - I \cup e_i) = r - |I|$ for all $1 \leq i \leq k$. Also, as we saw in the proof of Lemma 9.12, having $b_1 \in B$, $e_i \in E(B)$ and $b_1 < e_i$ implies that $r(B - b_1 \cup e_i) = r - 1$. Combining these two inequalities and using submodularity, we get that $r(B - I - b_1 \cup e_i) = r - |I| - 1$ for all $1 \leq i \leq k$. Invoking submodularity once again, we get that $r((B - I) - b_1 \cup (e_1 \cup \cdots \cup e_k)) = r - |I| - 1$. Therefore $r(X) = r((B - I) - (b_1 \cup \cdots \cup b_k) \cup (e_1 \cup \cdots \cup e_k)) \leq r - |I| - 1 < r(B - I \cup E)$. It follows that X is not a basis for $B - I \cup E$. ☑

Lemma 9.14. *For all $(B, I, E) \in \mathcal{T}$, we have $ud(B - I \cup E) = B$.*

Proof. In view of Lemma 9.13, we need to show that $u(B - I) = B$. Clearly B is a basis of $\mathcal{C}$ containing $B - I$; now we show that it is the smallest one.

Let $X = B - (b_1 \cup \cdots \cup b_k) \cup (c_1 \cup \cdots \cup c_k)$ be an r-tuple smaller than B, where the b_i's are in I (since X must contain $B - I$) and the c_i's are in S. We will show that X is not a basis for $\mathcal{C}$. Once again we can assume, without loss of generality, that $c_1 < b_1, \ldots, b_k$.

Since each b_i is internally active, $r(B - b_i \cup S_{<b_i}) = r - 1$, and hence $r(B - b_i \cup c_1) = r - 1$. Submodularity gives $r(B - (b_1 \cup \cdots \cup b_k) \cup c_1) = r - k$, which in turn gives $r(X) = r(B - (b_1 \cup \cdots \cup b_k) \cup (c_1 \cup \cdots \cup c_k)) \leq (r - k) + (k - 1) < r$. ☑

So far we have only defined uX for independent sets X of $\mathcal{C}$. We can extend the definition to arbitrary subsets $X \subseteq S$ as follows. If X is dependent, then there is no basis of $\mathcal{C}$ containing it. Instead, we consider all the minimal sets of rank r which contain X. Let uX be the lexicographically smallest of those sets. Then we can say even more.

Lemma 9.15. *For all $(B, I, E) \in \mathcal{T}$, we have $u(B - I \cup E) = B \cup E$ and $du(B - I \cup E) = B$.*

We will not need Lemma 9.15 to prove Proposition 9.11 and Theorem 9.5. We state it for completeness, but we omit its proof, which is very similar to the proofs of Lemmas 9.13 and 9.14.

Proof of Proposition 9.11. Checking that $\phi_1 \circ \phi_2$ is the identity map in $\mathcal{C}$ is immediate, and Lemmas 9.13 and 9.14 imply that $\phi_2 \circ \phi_1$ is the identity map in $\mathcal{T}$. $\square$

Proof of Theorem 9.5. Using the bijection of Proposition 9.11, the sets in $\mathcal{C}$ are precisely the sets of the form $B - I \cup E$, where B is a basis, $I \subseteq I(B)$ and $E \subseteq E(B)$. Also, from Lemma 9.12, $r(B - I \cup E) = r - |I|$.

Therefore we have

$$\begin{aligned}
T_{\mathcal{C}}(x,y) &= \sum_{X \in \mathcal{C}} (x-1)^{r-r(X)}(y-1)^{|X|-r(X)} \\
&= \sum_{B \text{ basis}} \sum_{I \subseteq I(B)} \sum_{E \subseteq E(B)} (x-1)^{r-r(B-I\cup E)}(y-1)^{|B-I\cup E|-r(B-I\cup E)} \\
&= \sum_{B \text{ basis}} \sum_{I \subseteq I(B)} \sum_{E \subseteq E(B)} (x-1)^{|I|}(y-1)^{|E|} \\
&= \sum_{B \text{ basis}} (1+(x-1))^{|I(B)|}(1+(y-1))^{|E(B)|} \\
&= \sum_{B \text{ basis}} x^{i(B)} y^{e(B)}.
\end{aligned}$$

as desired. ☑

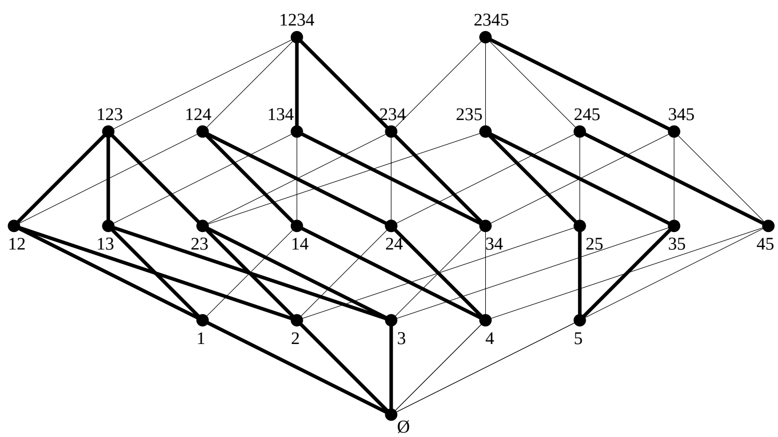

FIGURE 4. The decomposition of $\mathcal{C}$ into intervals.

Regard the simplicial complex $\mathcal{C}$ as a poset, ordering its faces by inclusion. There is a nice way to understand Theorem 9.5 in terms of this poset. Proposition 9.11 gives us a way of classifying the faces of $\mathcal{C}$ according to the basis of $\mathcal{C}$ that they correspond to under the map ud (or du). This classification decomposes the poset into disjoint intervals, where each interval is a Boolean algebra of the form $[B - I(B), B \cup E(B)]$ for a basis B. This is illustrated in Figure 4 for the arrangement $\mathcal{A}$ considered at the beginning of Section 8; recall Table 2. If we look at the interval corresponding to basis B, and add the contributions of its elements to the right-hand side of (8.1), we simply get the monomial $q^{i(B)}t^{e(B)}$.

Note that this decomposition is well-known for matroids. [4, Prop. 7.3.6]

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