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Corcoran's Aristotelian syllogistic as a subsystem of first-order logic

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Abstract. Aristotelian syllogistic has been formalized for some time now by means of a natural deduction system, called D by John Corcoran. In a classical paper, Corcoran proves a completeness theorem for such a system. His proof involves the use of a reduced system, called RD, that is easier to handle and turns out to be equivalent to D. The question remains, however, whether RD is in fact the easiest such system that is equivalent to D. In this paper we answer this question, but raise some more, by embedding system RD in first-order predicate logic.

Keywords. Syllogistic, Aristotle, Corcoran, Completeness, Minimality, First-Order Subsystem.

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Resumen. La silogística aristotélica ha sido formalizada hace ya cierto tiempo por medio de un sistema de deducción natural, llamado D por John Corcoran. En un artículo clásico, Corcoran demuestra un teorema de completitud para dicho sistema. Su demostración involucra el uso de un sistema reducido, llamado RD, que es más fácil de manejar y resulta ser equivalente a D. El problema sigue siendo, sin embargo, si RD es de hecho el sistema más sencillo que es equivalente a D. En este artículo responderemos esta pregunta, pero crearemos otras más, al incrustar el sistema RD en la lógica de predicados de primer orden.

1. Introduction

In the classical paper [3], Corcoran defines two logical systems which he calls D and RD. The former —system D— is intended as a formalization of Aristotle's syllogistic within the framework of modern logic. Aristotelian syllogistic studies the necessity relation between two (general non-empty) terms, S and P, that follows from assuming both a relation between terms P and M, and a relation between terms S and M. Aristotle managed to give a satisfactory account of the aforementioned entailment by using a number of conversion rules and the four 'perfect syllogisms'. Both rules and syllogisms are formalized —in Corcoran's systems— as natural deduction rules (see below). As modern logical systems, the systems D and RD are set up by defining (a) a formal language; (b) a semantic system; and (c) a deduction system. System D formalizes all of the conversion rules and all of the perfect syllogisms, whereas system RD formalizes only some of them. For this reason, system RD is easier to handle, in the sense that we can come up with a completeness theorem for it. Nonetheless, system RD turns out to be equivalent to system D. The outcome of this is a proof of the completeness of system D, and —as a formalization of Aristotle's original 'system'— in turn it implies Aristotelian syllogistic is complete. This is a beautiful result, no doubt. One of its nicest sides comes from the fact that it provides a formal proof for a highly intuitive conviction. By the same token, we believe that there are other intuitive convictions that can be given formal proofs as well. Those intuitive convictions are brought out by, among others, the following questions: (i) RD allowed us to come up with a completeness proof, however, is it the easiest, most interesting system to deal with? That is, is RD a *minimal* system that is equivalent to system D? (ii) If we manage to define a translation function from RD to first-order logic, would it be possible to characterize the fragment of first-order logic this translation defines? (iii) What would happen if we weakened the requirement that the models must consist of non-empty sets? In this paper, we set out to give an answer to the first of these questions.

In the following section, we define both systems and we point out some of their properties. Section 3 defines the translation function from the categorical propositions to the atomic formulas of an appropriate first-order language. We then proceed to prove: (i) that this translation function is a faithful interpretation—that is, that a syllogism is valid in any model for RD iff its translation is true in any model for T_{RD}; (ii) that the axioms of T_{RD} are independent; and (iii) that RD is in fact a minimal system of rules for the Aristotelian syllogistic. We make some comments and remarks in section 4.

2. Corcoran's RD and D systems

The vocabulary consists of a set $R := \{a, e, i, o\}$, and a non-empty set V. The set of *categorical propositions*, $\mathcal{P}_V$, consists of those and only those expressions of the form SaP, SeP, SiP, SoP where $S, P \in V$, and $S \neq P$.

Let $M = \{U_i\}_{i\in I}$ be a family of *non-empty* sets, and let $g : V \to M$. We define the *interpretation function* $[\![\cdot]\!]_{M,g} : \mathcal{P}_V \to \{0, 1\}$ as follows:[1]

[1]It is not hard to see that the present definition provides categorical propositions with their traditional semantics, namely, SaP :'Every S is P', SeP :'No S is P', SiP :'Some S is P', SoP :'Some S is not P'.

Definition 2.1.

1. $[\![SaP]\!]_{M,g} = 1$ *iff* $g(S) \subseteq g(P)$,
2. $[\![SeP]\!]_{M,g} = 1$ *iff* $g(S) \cap g(P) = \emptyset$,
3. $[\![SiP]\!]_{M,g} = 1$ *iff* $g(S) \cap g(P) \neq \emptyset$,
4. $[\![SoP]\!]_{M,g} = 1$ *iff* $g(S) \nsubseteq g(P)$,

Let $d \in \mathcal{P}_V$, $K \subseteq \mathcal{P}_V$, and $[\![\cdot]\!]_{M,g}$ an interpretation function:

1. If $[\![d]\!]_{M,g} = 1$, we say that $[\![\cdot]\!]_{M,g}$ is a *true interpretation* of d.
2. We say that $[\![\cdot]\!]_{M,g}$ is a *true interpretation* of K if $[\![\cdot]\!]_{M,g}$ is a true interpretation of d, for each $d \in K$.
3. $K \models d$ if and only if every true interpretation of K is a true interpretation of d.
4. The (meta-)function called the *contradictory of* function, denoted by $c : \mathcal{P}_V \to \mathcal{P}_V$, is defined as follows:

$$c(d) := \begin{cases} SoP & \text{if } d = SaP, \\ SiP & \text{if } d = SeP, \\ SeP & \text{if } d = SiP, \\ SaP & \text{if } d = SoP, \end{cases}$$

The *deductive system* consists of the following natural deduction rules:[2]

$$\text{(I)}\ \frac{SeP}{PeS} \quad \text{(II)}\ \frac{SaP}{SiP} \quad \text{(III)}\ \frac{\begin{array}{c}MaP\\ SaM\end{array}}{SaP} \quad \text{(IV)}\ \frac{\begin{array}{c}MeP\\ SaM\end{array}}{SeP}$$

$$\text{(V)}\ \frac{SiP}{PiS} \quad \text{(VI)}\ \frac{\begin{array}{c}MaP\\ SiM\end{array}}{SiP} \quad \text{(VII)}\ \frac{\begin{array}{c}MeP\\ SiM\end{array}}{SoP}$$

Definition 2.2. *A sequence $\langle p_1, \ldots, p_n \rangle$ of categorical propositions is a direct deduction of d from K if $d = p_n$ and for each $i \in \{1, \ldots, n\}$ one of the following holds:*

1. *$p_i \in K$, or*
2. *There exists $j < i$ such that p_i is obtained from p_j using rules (I), (II), or (V), or*
3. *There exist $j, k < i$ such that p_i is obtained from p_j and p_k using (III), (IV), (VI), or (VII).*

Definition 2.3. *A sequence $\langle p_1, \ldots, p_n \rangle$ of categorical propositions is an indirect deduction of d from K if there exists $j < n$ such that $p_n = c(p_j)$ and for each $i \in \{1, \ldots, n\}$ one of the following holds:*

[2] Rules (I), (II), and (V) are known, according to the tradition, as the conversion rules among categorical propositions. Rules (III), (IV), (VI), and (VII) are known as the perfect syllogisms.

(1) *$p_i \in K + c(d)$,*[3] *or*
(2) *There exists $j < i$ such that p_i is obtained from p_j by using rules (I), (II), or (V), or*
(3) *There exist $j, k < i$ such that p_i is obtained from p_j and p_k by using (III), (IV), (VI), or (VII).*

We say that $K \vdash_D d$ if and only if there exists either a direct or an indirect deduction of d from K. The logical system so defined is called D.

There is only one difference between systems D and RD, namely, the latter's deductive system consists only of rules (I)-(IV). Whenever a categorical proposition d follows from a set K of categorical propositions according to the latter set of rules, we say that $K \vdash_{RD} d$. Thus having reduced the number of deduction rules allows us to prove the following interesting results. The proofs of these results can be found in [3], or [2].

Lemma 2.1. *If $\langle p_1, \ldots, p_n \rangle$ is an indirect deduction of d from K, then for each $i \in \{1, \ldots, n\}$ there is a direct deduction of p_i from $K + c(d)$. In particular, $K + c(d) \vdash_{RD} p_n$ and $K + c(d) \vdash_{RD} c(p_n)$ in a 'direct way'.*

Theorem 2.4 (Reductio ad absurdum)**.** *If $K + c(d) \vdash_{RD} e$ and $K + c(d) \vdash_{RD} c(e)$, then $K \vdash_{RD} d$.*

Theorem 2.5 ((Strong) Soundness)**.** *If $K \vdash_{RD} d$, then $K \models d$.*[4]

Theorem 2.6 ((Strong) Completeness)**.** *If $K \models d$, then $K \vdash_{RD} d$.*

As usual, completeness is proved via an adequation lemma that is in turn proved via other lemmas and definitions. As we happen to need some of these lemmas, we are going to state them without proof —but see the foregoing references.

Definition 2.7. *Let $\emptyset \neq K \subseteq \mathcal{P}_V$, and define the following sets:*

- *$A(K) := \{F \in \wp(V) : S \in F \& P \notin F, SaP \in K\}$ (Subsets of V containing S but lacking P, for each $SaP \in K$).*
- *$E(K) := \{F \in \wp(V) : S \in F \& P \in F, SeP \in K\}$ (Subsets of V containing both S and P, for each $SeP \in K$).*
- *$U(K) := \wp(V) - (A(K) \cup E(K))$*

[3]If A and B are two sets of categorical propositions, the expression $A + B$ is defined to be $A \cup B$. Moreover, if d and d' are categorical propositions, the expressions $d + d'$ and $A + d$ are defined to be $\{d, d'\}$ and $A \cup \{d\}$ respectively.

[4]It is worth noting that rule (II) is sound with respect to the semantics defined in definition 2.1 only because we required the sets in M to be non-empty. If we weakened this hypothesis, rule (II) would no longer be sound. Now, with regard to the third question in the introduction, viz. what would happen if we weakened the requirement that the models must consists of non-empty sets, it is clear from the previous observation that systems D and RD would not longer be sound with respect to this semantics. Thus, the systems that are sound with respect to it would be different from —not equivalent to— D and RD. It remains to be seen how interesting these models are. [We are indebted to the anonymous referee for this remark.]

Lemma 2.2. *If $K \subset \mathcal{P}_V$ is maximal consistent, $M := \wp(U(K))$, and $i : V \to M$ is such that $i(S) := \{F \in U(K) : S \in F\}$, then the following claims hold:*

(i) *If $S \in V$, then $i(S) \neq \emptyset$.*
(ii) *$SaP \in K$ iff $U(K)$ contains no set containing S but lacking P.*
(iii) *$SeP \in K$ iff $U(K)$ contains no set containing both S and P.*
(iv) *$SiP \in K$ iff $U(K)$ contains a set containing both S and P.*
(v) *$SoP \in K$ iff $U(K)$ contains a set containing S but lacking P.*

Lemma 2.3. *If $K \subseteq \mathcal{P}_V$ is maximal consistent, $M := \wp(U(K))$, and $i : V \to M$ is such that $i(S) := \{F \in U(K) : S \in F\}$, then $[\![\cdot]\!]_{\wp(U(K)),i}$ is a true interpretation of K.*

3. System RD as a subsystem of first-order logic

In the first part of this section we define a translation function from the categorical propositions to the atomic formulas of an appropriate first-order language. The aim of this translation function is to find a correspondence between the interpretation functions in Corcoran's RD system and the models of a first-order theory, that we shall call T_{RD}. Thus, we prove that every syllogism valid in RD is valid in T_{RD}.

The first-order language that we use is defined as follows. We use a similarity type $\tau = \{A, E\}$ without equality symbol,[5] with only two binary relations. We call T_{RD} the theory defined by the following axioms:

A_1 $\forall x, y\, (xEy \leftrightarrow yEx)$
A_2 $\forall x, y\, (xAy \to \neg(xEy))$
A_3 $\forall x, y, z\, (yAz \wedge xAy \to xAz)$
A_4 $\forall x, y, z\, (yEz \wedge xAy \to xEz)$

T_{RD} is a consistent set, thanks to the following model: Let S be a non-empty set and define $\mathcal{M} = \langle M, A^{\mathcal{M}}, E^{\mathcal{M}} \rangle$, where

$$M = \wp(S) - \{\emptyset\}$$
$$xA^{\mathcal{M}}y \Longleftrightarrow x \subseteq y$$
$$xE^{\mathcal{M}}y \Longleftrightarrow x \cap y = \emptyset$$

Definition 3.1. *A formula μ is an aristotelian syllogism (or syllogism, for short) iff μ is of the form $\forall x, y, z(\phi_1 \wedge \phi_2 \to \phi_3)$ where:*

(1) *ϕ_i is an atomic formula or the negation of an atomic formula ($i = 1, 2, 3$).*
(2) *Each variable x, y, z appears exactly in two formulas ϕ_i, ϕ_j, $i \neq j$.*

[5]The consequence of leaving out the equality symbol is that the atomic sentences are only of the form xAy or xEy.

3.1. **Translation.** From now on, we suppose that V is equal to the set of variables of the first-order language $\mathcal{L}_\tau$ we are working with.

We define the function $TR : \mathcal{P}_V \to \mathcal{L}_\tau$ in the following manner:

(1) $TR(xay) = xAy$
(2) $TR(xey) = xEy$
(3) $TR(xiy) = \neg(xEy)$
(4) $TR(xoy) = \neg(xAy)$

Note that TR is injective, so we can define its inverse function restricted to the image of TR —that is, the atomic formulas and their negations.

Now we will show how one can construct an interpretation function of RD from a model of T_{RD}. This is achieved as follows. Firstly, we build up the set of atomic formulas —in the first-order language— that are true *and false*[6] in the model of T_{RD} under a particular interpretation g. Secondly, we translate this set back into RD by means of TR^{-1}. Next, we prove that this latter set is maximal consistent, and therefore, by lemma 2.3, we can find a true interpretation of it.

Definition 3.2. *Let $\mathcal{M} = \langle M, A^{\mathcal{M}}, E^{\mathcal{M}} \rangle$ and $g : V \to M$ be such that $\mathcal{M} \models T_{RD}$. [Recall that V, the set of terms of the language of RD, is equal to the set of variables of the language of T_{RD}.] We define:*[7]

$$K_{\mathcal{M},g} := \left\{TR^{-1}(xAy) : \mathcal{M} \models xAy[g]\right\} \cup \left\{TR^{-1}(xEy) : \mathcal{M} \models xEy[g]\right\}$$
$$\cup \left\{TR^{-1}(\neg(xAy)) : \mathcal{M} \not\models xAy[g]\right\} \cup \left\{TR^{-1}(\neg(xEy)) : \mathcal{M} \not\models xEy[g]\right\}$$

Note that $K_{\mathcal{M},g} \subseteq \mathcal{P}_V$, so we can define

$$TR(K_{\mathcal{M},g}) = \{TR(d) : d \in K_{\mathcal{M},g}\}$$

From the last definitions it is not hard to see that $\mathcal{M} \models TR(K_{\mathcal{M},g})[g]$.

Lemma 3.1. *$TR(c(d)) \vdash \neg TR(d)$ and $\neg TR(c(d)) \vdash TR(d)$.*

Proof. We leave the proof to the reader. ☑

Lemma 3.2. *If $K \vdash_{RD} d$, then $T_{RD} + TR(K) \vdash TR(d)$.*

Proof. We shall prove, by induction on the length of the deduction, that if there is a direct deduction of d from K of the form $\langle p_1, \ldots, p_n \rangle$, then $T_{RD} + TR(K) \vdash p_i$ for all $i \leq n$.

(i) If $p_i \in K$, then $TR(p_i) \in TR(K)$ and therefore $T_{RD} + TR(K) \vdash TR(p_i)$

(ii) Let's suppose that p_i is obtained from p_k by using rule I ($k < i$). By the induction hypothesis we have that $T_{RD} + TR(K) \vdash TR(p_k)$, thus by axiom A_1, substitution and modus ponens, $T_{RD} + TR(K) \vdash TR(p_i)$. Similarly with rule II.

(iii) Let's suppose that p_i is obtained from p_k, p_m by using rule III ($k, m < i$). By induction hypothesis we have that $T_{RD} + TR(K) \vdash TR(p_k)$ and $T_{RD} +$

[6]We are indebted to the anonymous referee for this remark.

[7]We are using the notion of satisfaction as defined in [4, p. 81].

$TR(K) \vdash TR(p_m)$, therefore $T_{RD} + TR(K) \vdash TR(p_k) \wedge TR(p_m)$. Thus, by axiom A_3, substitution and modus ponens, $T_{RD} + TR(K) \vdash TR(p_i)$. Similarly with rule IV.

On the other hand, if $\langle p_1, \ldots, p_n \rangle$ is an indirect deduction, then by lemma 2.1 we can find direct deductions of p_n and $c(p_n)$ from $K + c(d)$. By the previous part of this proof, and lemma 3.1, we have that $T_{RD} + TR(K) + TR(c(d))$ is inconsistent, and by the *reductio ad absurdum* theorem of first-order logic (Cf. [4]), it follows that $T_{RD} + TR(K) \vdash \neg TR(c(d))$. Finally, again by lemma 3.1, we have $T_{RD} + TR(K) \vdash TR(d)$, and we are done. ☑

Corollary 3.1. *If $K_{\mathcal{M},g} \vdash_{RD} d$, then $T_{RD} + TR(K_{\mathcal{M},g}) \vdash TR(d)$.*

Lemma 3.3. *$K_{\mathcal{M},g}$ is maximal consistent.*

Proof. Suppose towards a contradiction that $K_{\mathcal{M},g}$ is inconsistent. Then there is a sentence d such that $K_{\mathcal{M},g} \vdash_{RD} d$ and $K_{\mathcal{M},g} \vdash_{RD} c(d)$. By corollary 3.1, $T_{RD} + TR(K_{\mathcal{M},g}) \vdash TR(d)$ and $T_{RD} + TR(K_{\mathcal{M},g}) \vdash TR(c(d))$, and by lemma 3.1 it follows that $T_{RD} + TR(K_{\mathcal{M},g})$ is inconsistent. This contradicts that $\mathcal{M} \models T_{RD} + TR(K_{\mathcal{M},g})[g]$.

In order to prove $K_{\mathcal{M},g}$ is maximal consistent, we shall prove that for any $d \in \mathcal{P}_V$, we have that either $d \in K_{\mathcal{M},g}$ or $c(d) \in K_{\mathcal{M},g}$. Let's suppose that $d \notin K_{\mathcal{M},g}$, and that d is of the form xay. From the definition of $K_{\mathcal{M},g}$ it follows that $\mathcal{M} \not\models xAy[g]$, which in turn implies that $TR^{-1}(\neg(xAy)) = c(xay) \in K_{\mathcal{M},g}$. The case d of the form xey is similar. As for the case xiy, we have that xiy does not belong to $K_{\mathcal{M},g}$ only if $\mathcal{M} \models xAy[g]$ and therefore, $xay = c(xiy) \in K_{\mathcal{M},g}$. The case xoy is similar. ☑

The foregoing lemmas, including those in the previous section, allow us to show that, starting from a first-order model of T_{RD}, $\mathcal{M}$, and a valuation function $g : V \to M$, we can build up an interpretation function of the RD system. This latter interpretation function is defined by using $M^{\mathcal{M}} := \wp(U(K_{\mathcal{M},g}))$ and $i : V \to M^{\mathcal{M}}$ such that $i(x) := \{F \in U(K_{\mathcal{M},g}) : x \in F\}$. From lemma 2.3 it is not hard to see that $[\![\cdot]\!]_{M^{\mathcal{M}},i}$ is a true interpretation of $K_{\mathcal{M},g}$. The following theorem shows that we have a nice relationship between these two models.

Theorem 3.3. *The following claims hold:*

(1) $\mathcal{M} \models xAy[g]$ *iff* $[\![xay]\!]_{M^{\mathcal{M}},i} = 1$
(2) $\mathcal{M} \models xEy[g]$ *iff* $[\![xey]\!]_{M^{\mathcal{M}},i} = 1$

Proof. We will only prove claim 1. Claim 2 is proved in a similar way.

($\Rightarrow$) If $\mathcal{M} \models xAy[g]$ then $TR^{-1}(xAy) \in K_{\mathcal{M},g}$. Since $[\![\cdot]\!]_{M^{\mathcal{M}},i}$ is a true interpretation of $K_{\mathcal{M},g}$, then $[\![xay]\!]_{M^{\mathcal{M}},i} = 1$.

($\Leftarrow$) If $[\![xay]\!]_{M^{\mathcal{M}},i} = 1$, then $i(x) \subseteq i(y)$. So, for every $F \in U(K_{\mathcal{M},g})$, if $x \in F$, then $y \in F$ [check out the definition of i!]. This implies that $U(K_{\mathcal{M},g})$ contains no set containing x but lacking y. As $K_{\mathcal{M},g}$ is maximal consistent, by lemma 2.2(ii) we have that $xay \in K_{\mathcal{M},g}$ and thus $xAy \in TR(K_{\mathcal{M},g})$. Therefore $\mathcal{M} \models xAy[g]$. ☑

We have constructed an interpretation function of RD from a model of T_{RD} that has very interesting properties (Cf. Theorem 3.3). Conversely, we can construct a model of T_{RD} from an interpretation function of RD that has similar interesting properties (Cf. Theorem 3.5). This is what we will set out to do in what follows.

Definition 3.4. *Let $M = \{U_i\}_{i \in I}$ be a family of non-empty sets and $g : V \to M$. We define $\mathcal{M}^M := \langle M, A^{\mathcal{M}}, E^{\mathcal{M}} \rangle$ where*

$$M = \{U_i\}_{i \in I}$$
$$g(x) A^{\mathcal{M}} g(y) \Longleftrightarrow g(x) \subseteq g(y)$$
$$g(x) E^{\mathcal{M}} g(y) \Longleftrightarrow g(x) \cap g(y) = \emptyset$$

Theorem 3.5. *The following claims are true:*

(1) $\mathcal{M}^M \models T_{RD}$.
(2) $[\![xay]\!]_{M,g} = 1$ *iff* $\mathcal{M}^M \models xAy[g]$.
(3) $[\![xey]\!]_{M,g} = 1$ *iff* $\mathcal{M}^M \models xEy[g]$.
(4) *For every* $d \in \mathcal{P}_V$, $[\![d]\!]_{M,g} = 1$ *iff* $\mathcal{M}^M \models TR(d)[g]$.

Proof. By construction, it is straightforward to see that $\mathcal{M}^M \models T_{RD}$. We will only prove claim 2. Claim 3 is proved in a similar way. Claim 4 is a straightforward consequence of the two previous claims.

We have that $[\![xay]\!]_{M,g} = 1$ iff $g(x) \subseteq g(y)$ iff $\mathcal{M}^M \models xAy[g]$. ☑

The two previous results give us a nice way to go from an interpretation function in one logical system to a model in the other one, which leads us to the following proof that the translation function we have defined is a *faithful interpretation*. That is, that a syllogism is valid in RD iff its translation is a theorem of T_{RD}. This is stated in the following theorem:

Theorem 3.6. *Let μ be the syllogism $\forall x, y, z(\phi_1 \wedge \phi_2 \to \phi_3)$, then:*

$$T_{RD} \vdash \mu \quad \textit{iff} \quad TR^{-1}(\phi_1) + TR^{-1}(\phi_2) \vdash_{RD} TR^{-1}(\phi_3).$$

The proof of this theorem runs as follows. Since we have a correspondence between interpretation functions and models, if some syllogism is such that there exists a true interpretation of its premises that is not a true interpretation of its conclusion, then there is a model of T_{RD} for which the translation of this syllogism is false (Cf. Lemma 3.4), and viceversa (Cf. Lemma 3.5). This allows us to prove the contrapositive of each of the implications of Theorem 3.6.

Lemma 3.4. *Let μ be a syllogism. If there exist M, g such that $[\![TR^{-1}(\phi_1)]\!]_{M,g} = 1$, $[\![TR^{-1}(\phi_2)]\!]_{M,g} = 1$ and $[\![TR^{-1}(\phi_3)]\!]_{M,g} = 0$, then there is $\mathcal{M} \models T_{RD}$ such that $\mathcal{M} \not\models \mu$.*

Proof. By theorem 3.5, from M, g we can build up a model $\mathcal{M}$ such that $\mathcal{M} \models T_{RD}$, and such that $[\![d]\!]_{M,g} = 1$ iff $\mathcal{M}^M \models TR(d)[g]$ for every $d \in \mathcal{P}_V$. If we assume that $[\![TR^{-1}(\phi_1)]\!]_{M,g} = 1$, $[\![TR^{-1}(\phi_2)]\!]_{M,g} = 1$ and $[\![TR^{-1}(\phi_3)]\!]_{M,g} = 0$, then $\mathcal{M}^M \models \phi_1[g]$, $\mathcal{M}^M \models \phi_2[g]$ and $\mathcal{M}^M \not\models \phi_3[g]$. By definition of satisfaction we have that $\mathcal{M}^M \not\models (\phi_1 \wedge \phi_2 \rightarrow \phi_3)[g]$, then $\mathcal{M}^M \not\models \forall x, y, z(\phi_1 \wedge \phi_2 \rightarrow \phi_3)$. ☑

Lemma 3.5. *Let μ be the syllogism $\forall x, y, z(\phi_1 \wedge \phi_2 \longrightarrow \phi_3)$. If there is $\mathcal{M} \models T_{RD}$ such that $\mathcal{M} \not\models \mu$, then there are M, g such that $[\![TR^{-1}(\phi_1)]\!]_{M,g} = 1$, $[\![TR^{-1}(\phi_2)]\!]_{M,g} = 1$ and $[\![TR^{-1}(\phi_3)]\!]_{M,g} = 0$.*

Proof. In an analogous manner to the proof above, the proof is simple. Consider the interpretation function $[\![\cdot]\!]_{M^{\mathcal{M}},i}$ we built up right before Theorem 3.3. If $\mathcal{M} \not\models \mu$, then in particular $\mathcal{M} \not\models (\phi_1 \wedge \phi_2 \rightarrow \phi_3)[g]$ for some g. By the definition of satisfaction, it follows that $\mathcal{M} \models \phi_1[g]$, $\mathcal{M} \models \phi_2[g]$ and $\mathcal{M} \not\models \phi_3[g]$. By Theorem 3.3 we have $[\![TR^{-1}(\phi_1)]\!]_{M^{\mathcal{M}},i} = 1$, $[\![TR^{-1}(\phi_2)]\!]_{M^{\mathcal{M}},i} = 1$ and $[\![TR^{-1}(\phi_3)]\!]_{M^{\mathcal{M}},i} = 0$. ☑

Proof of Theorem 3.6. We will prove the contrapositive of both implications:

$\Rightarrow$) Suppose that $TR^{-1}(\phi_1) + TR^{-1}(\phi_2) \nvdash_{RD} TR^{-1}(\phi_3)$. We have that $TR^{-1}(\phi_1) + TR^{-1}(\phi_2) \not\models TR^{-1}(\phi_3)$ (Theorem 2.6). Hence, there is a true interpretation of $TR^{-1}(\phi_1)$ and $TR^{-1}(\phi_2)$ which is not true for $TR^{-1}(\phi_3)$. By lemma 3.4, there is $\mathcal{M} \models T_{RD}$ such that $\mathcal{M} \not\models \mu$. Thus $T_{RD} \not\models \mu$, so $T_{RD} \nvdash \mu$ (strong soundness of first-order logic).

$\Leftarrow$) Similar to the proof above, but using lemma 3.5 instead. ☑

Now, it just so happens that Theorem 3.6 can be generalized in a more fruitful way[8] as Theorem 3.7:

Theorem 3.7. *Let K be a set of categorical propositions and d a single categorical proposition in the RD system, then $K \vdash_{RD} d$ iff $T_{RD} + TR(K) \vdash TR(d)$*

Proof. We shall prove each direction in the theorem:

$\Rightarrow$) This direction is Lemma 3.2.

$\Leftarrow$) Assume that $K \nvdash_{RD} d$. Therefore, there is an interpretation function $[\![\cdot]\!]_{M,g}$ such that it is a true interpretation of K but $[\![d]\!]_{M,g} = 0$. By theorem 3.5, we can build up a model $\mathcal{M}$ such that $\mathcal{M} \models T_{RD}$, and such that $[\![d']\!]_{M,g} = 1$ iff $\mathcal{M}^M \models TR(d')[g]$ for every $d' \in \mathcal{P}_V$. Since we have assumed that $[\![d']\!]_{M,g} = 1$ for every $d' \in K$, and that $[\![d]\!]_{M,g} = 0$, it follows again from theorem 3.5 that $\mathcal{M} \models T_{RD} + TR(K)$, but $\mathcal{M} \not\models TR(d)$. By soundness of first-order logic it follows that $T_{RD} + TR(K) \nvdash TR(d)$. ☑

[8] We say that the generalization in question is more fruitful because we can draw from it other interesting results concerning the systems D and RD, and the latter's translation in first-order logic. For these results, see [1].

3.2. **Independence of $\mathbf{T_{RD}}$'s axioms.** Let T^i_{RD} be the set T_{RD} except for axiom A_i ($i = 1, 2, 3, 4$). In this subsection, we shall build up a model $\mathcal{M}_i$ such that $\mathcal{M}_i \models T^i_{RD}$ and $\mathcal{M}_i$ is not a model of axiom A_i, for $i = 1, 2, 3, 4$. Using these models $\mathcal{M}_i$'s we will prove the independence of the axioms in T_{RD}. The following models meet the desired conditions:

(1) Let be $\mathcal{M}_1 = \langle M, A^{\mathcal{M}}, E^{\mathcal{M}} \rangle$, where:

$$M = \{a, b, c\}, \qquad A^{\mathcal{M}} = \{(b, c)\}, \qquad E^{\mathcal{M}} = \{(a, c), (c, a)(b, a)\}.$$

(2) Let be $\mathcal{M}_2 = \langle M, A^{\mathcal{M}}, E^{\mathcal{M}} \rangle$, where:

$$M = \{a, b, c\}, \qquad A^{\mathcal{M}} = \{(a, b), (a, c)\}$$

$$E^{\mathcal{M}} = \{(a, a), (a, b), (b, a), (b, c), (c, b), (a, c), (c, a)\}.$$

(3) Let be $\mathcal{M}_3 = \langle M, A^{\mathcal{M}}, E^{\mathcal{M}} \rangle$, where:

$$M = \{a, b, c\}, \qquad A^{\mathcal{M}} = \{(a, b), (b, c)\}, \qquad E^{\mathcal{M}} = \emptyset.$$

(4) Let be $\mathcal{M}_4 = \langle M, A^{\mathcal{M}}, E^{\mathcal{M}} \rangle$, where:

$$M = \{a, b, c\}, \qquad A^{\mathcal{M}} = \{(a, b)\}, \qquad E^{\mathcal{M}} = \{(b, c), (c, b)\}.$$

3.3. **Minimality of RD.** We say that RD is *minimal* if there is no proper subset of its set of rules (see p. 70), let's say Min, such that for every $d \in \mathcal{P}_V$ and every $K \subseteq \mathcal{P}_V$, $K \vdash_{RD} d$ iff $K \vdash_{Min} d$.

The minimality of RD would be obtained by solving the following questions:

(a) Are the axioms in T_{RD} independent from each other?

(b) Let μ be the syllogism $\forall x, y, z(\phi_1 \wedge \phi_2 \longrightarrow \phi_3)$: is it true that

$$T_{RD} \vdash \mu \quad \Longrightarrow \quad TR^{-1}(\phi_1) + TR^{-1}(\phi_2) \vdash_{RD} TR^{-1}(\phi_3)?$$

(c) Let μ be the syllogism $\forall x, y, z(\phi_1 \wedge \phi_2 \longrightarrow \phi_3)$ and Min_i the set of RD's rules except for rule i (with i= I, II, III, IV): could it be true that

$$T^i_{RD} \vdash \mu \quad \Longleftarrow \quad TR^{-1}(\phi_1) + TR^{-1}(\phi_2) \vdash_{Min_i} TR^{-1}(\phi_3)?$$

In fact, if (a), (b) and (c) are true, we can conclude that RD is a minimal system for the Aristotelian syllogistic:

Theorem 3.8. *If (a), (b) and (c) are true, then for i= I, II, III, IV there is a syllogism* $\forall x, y, z \left(\phi^i_1 \wedge \phi^i_2 \rightarrow \phi^i_3\right)$ *such that:*

(1) $TR^{-1}\left(\phi^i_1\right) + TR^{-1}\left(\phi^i_2\right) \vdash_{RD} TR^{-1}\left(\phi^i_3\right)$ *and*

(2) $TR^{-1}\left(\phi^i_1\right) + TR^{-1}\left(\phi^i_2\right) \nvdash_{Min_i} TR^{-1}\left(\phi^i_3\right)$.

Proof. We will prove each case in turn. We start out from the last two cases, because they are easier to prove:

Case i=III: If we take axiom A_3 we have:

$$T_{RD} \vdash \forall x, y, z(yAz \wedge xAy \rightarrow xAz) \qquad \& \qquad T^3_{RD} \nvdash \forall x, y, z(yAz \wedge xAy \rightarrow xAz)$$

From this, (b) and (c) we obtain:

$$yaz + xay \vdash_{RD} xaz \qquad \& \qquad yaz + xay \nvdash_{Min_{III}} xaz$$

Case i=IV: If we take the axiom A_4 we have:

$$T_{RD} \vdash \forall x, y, z(yEz \wedge xAy \to xEz) \qquad \& \qquad T^4_{RD} \nvdash \forall x, y, z(yEz \wedge xAy \to xEz)$$

Thus by (b) and (c) we get:

$$yez + xay \vdash_{RD} xez \qquad \& \qquad yez + xay \nvdash_{Min_{IV}} xez$$

Case i=I: It's enough to show a syllogism μ for which $T_{RD} \vdash \mu$ and $T^1_{RD} \not\models \mu$. Let μ be the syllogism $\forall x, y, z\, (yAz \wedge \neg\, (xEy) \to \neg\, (xEz))$. The following is a deduction of xEy from $T_{RD} + yAz + xEz$:

1.	xEz	Premise
2.	$xEz \to zEx$	It is derived from axiom A_1 in T_{RD}
3.	zEx	MP between 1. and 2.
4.	yAz	Premise
5.	$zEx \wedge yAz$	Tautological deduction from 3. and 4.
6.	$(zEx \wedge yAz) \to yEx$	It is derived from axiom A_4 in T_{RD}
7.	yEx	MP between 5. and 6.
8.	$yEx \to xEy$	It is derived from axiom A_1 in T_{RD}
9.	xEy	MP between 7. and 8.

From the deduction above it follows that $T_{RD} + yAz \wedge \neg(xEy) + xEz$ is inconsistent and therefore $T_{RD} \vdash yAz \wedge \neg(xEy) \to \neg(xEz)$. By the generalization theorem we get $T_{RD} \vdash \mu$. Finally, it is not hard to see that $\mathcal{M}_1 \models \neg\mu$.

Case i=II: It is enough to show a syllogism μ such that $T_{RD} \vdash \mu$ and $T^2_{RD} \not\models \mu$. Let μ be the syllogism $\forall x, y, z(yEz \wedge xAy \to \neg(xAz))$. The following is a deduction of $\neg(xAz)$ from $T_{RD} + yEz \wedge xAy$:

1.	$yEz \wedge xAy$	Premise
2.	$yEz \wedge xAy \to xEz$	It is derived from axiom A_4 in T_{RD}
3.	xEz	MP between 1. and 2.
4.	$xAy \to \neg(xEz)$	It is derived from axiom A_2 in T_{RD}
5.	$\neg(xAz)$	Tautological deduction of 3. and 4.

From the deduction theorem it follows that $T_{RD} \vdash yEz \wedge xAy \to \neg(xAz)$. By the generalization theorem we get $T_{RD} \vdash \mu$. It is not hard to see that $\mathcal{M}_2 \models \neg\mu$. ☑

Item (a) follows from §3.2 and (b) follows from Theorem 3.6. Item (c) follows from Corollary 3.2 below. This completes the proof of the minimality of RD as a system for the Aristotelian syllogistic.

Lemma 3.6. *If* $TR^{-1}(\phi_1) + TR^{-1}(\phi_2) \vdash_{RD} TR^{-1}(\phi_3)$, *then* $T_{RD} + \phi_1 + \phi_2 \vdash \phi_3$

Proof. Suppose $\langle p_1, \ldots, p_n \rangle$ is a direct deduction of $TR^{-1}(\phi_3)$ from $TR^{-1}(\phi_1)+TR^{-1}(\phi_2)$. We shall prove that we can translate $\langle p_1, \ldots, p_n \rangle$ into a proof $\langle \alpha_1, \ldots, \alpha_r \rangle$ in first-order logic with premisses $T_{RD} + \phi_1 + \phi_2$. To begin with, we have that $p_1 \in TR^{-1}(\phi_1) + TR^{-1}(\phi_2)$. Thus, define α_1 as ϕ_1 or ϕ_2 as appropriate. Now suppose we have already translated $p_1, \ldots, p_m$ and defined $\alpha_1, \ldots, \alpha_k$. If $p_{m+1} \in TR^{-1}(\phi_1) + TR^{-1}(\phi_2)$, define again α_{k+1} as ϕ_1 or ϕ_2 as appropriate. If there is a p_j with $j \leq m$ such that p_{m+1} comes from p_j by applying rule (I), define α_{k+1} as axiom A_1, α_{k+2} as the appropriate instantiation of axiom A_1 —which happens to be of the form $TR(p_j) \to TR(p_{m+1})$—, and α_{k+3} as $TR(p_{m+1})$. Note that the induction hypothesis is that there is a $l \leq k$ such that α_l is $TR(p_j)$. The case when rule (II) is applied is similar. If there are p_j, p_l with $j, l \leq m$ such that p_{m+1} comes from p_j and p_l by applying rule (III), define

α_{k+1} as axiom A_3

α_{k+2} as the appropriate instantiation of axiom A_3
—which happens to be of the form $TR(p_j) \wedge TR(p_l) \to TR(p_{m+1})$

α_{k+3} as $TR(p_j) \to \big(TR(p_l) \to \big(TR(p_j) \wedge TR(p_l)\big)\big)$

α_{k+4} as $TR(p_l) \to \big(TR(p_j) \wedge TR(p_l)\big)$

α_{k+5} as $TR(p_j) \wedge TR(p_l)$

α_{k+6} as $TR(p_{m+1})$

The induction hypothesis is that there are $r, s \leq k$ such that α_r is $TR(p_j)$ and α_s is $TR(p_l)$. The case when rule (IV) is applied is similar.

On the other hand, suppose $\langle p_1, \ldots, p_n \rangle$ is an indirect deduction. By Lemma 2.1 and the previous part we can prove that $T_{RD} + \phi_1 + \phi_2 + \neg\phi_3$ is inconsistent, which implies that $T_{RD} + \phi_1 + \phi_2 \vdash \phi_3$. ☑

Corollary 3.2. *If* $TR^{-1}(\phi_1) + TR^{-1}(\phi_2) \vdash_{Min_i} TR^{-1}(\phi_3)$*, then* $T^i_{RD} \vdash \mu$

Proof. It is not hard to see in the last proof that if a deduction of $TR^{-1}(\phi_3)$ from $TR^{-1}(\phi_1) + TR^{-1}(\phi_2)$ does not use rule (i), then we do not use axiom A_i in a deduction of ϕ_3 from $T_{RD} + \phi_1 + \phi_2$. Therefore, we can deduce ϕ_3 from $T^i_{RD} + \phi_1 + \phi_2$. By the deduction and generalization theorems we obtain $T^i_{RD} \vdash \mu$. ☑

4. **Remarks**

Two remarks with respect to Theorem 3.6 are in order. First of all, Theorem 3.6 is stronger than we actually need for the proof of the minimality of RD, since we only need one direction. Second of all, we have two different proofs of the right-to-left direction of the claim made in Theorem 3.6, namely, the one in the proof of the Theorem and another in Lemma 3.6.

As of yet, we have been unable to prove the converse of Corollary 3.2, even though we strongly believe it is true.

We need to say some words about the role item (a) plays in the proof of Theorem 3.8. To begin with, if (a) were false, that is, if $T^i_{RD} \vdash A_i$ for some i, then it would follow that $T^i_{RD} \vdash \mu$ iff $\phi_1 + \phi_2 \vdash \phi_3$. Since $T^i_{RD} \vdash \mu$ entails[9] $\phi_1 + \phi_2 \vdash_{Min_i} \phi_3$, then we would get $\phi_1 + \phi_2 \vdash_{Min_i} \phi_3$ iff $\phi_1 + \phi_2 \vdash_{RD} \phi_3$ (we always have $\phi_1 + \phi_2 \vdash_{Min_i} \phi_3$ implies $\phi_1 + \phi_2 \vdash_{RD} \phi_3$). In others words, if (a) were false, RD would not be the minimal system. On the other hand, the minimality of RD does not follow only from (a). Item (a) says something about T_{RD} in first order logic. A connection between T_{RD} and RD needs to be made so we can say something about RD on the grounds of a conclusion stated about T_{RD}. Here is where (b) and (c) come in. In any case, the role played by item (a) also comes in a contextual form, that is, it encourages us to strive for the desired conclusion. Last but not least, item (a) provides us with the appropriate syllogisms for the cases III and IV in the proof of Theorem 3.8.

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[9] As we said before, we do not know this for sure. If this could not be proved then the role of (a) would not be altogether clear.

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