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Conservation laws I: viscosity solutions

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Abstract. In this paper we use the Brouwer-Schauder's fixed point theorem to obtain the existence of local smooth viscosity solutions of the Cauchy problem for the parabolic system

$$\begin{cases} u_t^1 + f_1\left(u^1, u^2, \cdots, u^n\right)_x + g_1\left(u^1, u^2, \cdots, u^n\right) = \varepsilon u_{xx}^1 \\ \vdots \\ u_t^n + f_n\left(u^1, u^2, \cdots, u^n\right)_x + g_n\left(u^1, u^2, \cdots, u^n\right) = \varepsilon u_{xx}^n, \end{cases}$$

with the bounded measurable initial data

$$u^1(x,0) = u_0^1(x), \quad u^2(x,0) = u_0^2(x), \cdots, \quad u^n(x,0) = u_0^n(x).$$

Then based on the local existence and the maximum principle, we get the existence of global smooth solutions for two special systems, one related to the hyperbolic system of quadratic flux and the other related to the LeRoux system.

Keywords. Hyperbolic conservation laws, viscosity solution, Cauchy problem, *a priori* estimate.

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Resumen. En este artículo usamos el teorema de punto fijo de Brouwer-Schauder para obtener la existencia de soluciones locales de viscosidad suave al problema de Cauchy para el sistema parabólico

$$\begin{cases} u_t^1 + f_1\left(u^1, u^2, \cdots, u^n\right)_x + g_1\left(u^1, u^2, \cdots, u^n\right) = \varepsilon u_{xx}^1 \\ \vdots \\ u_t^n + f_n\left(u^1, u^2, \cdots, u^n\right)_x + g_n\left(u^1, u^2, \cdots, u^n\right) = \varepsilon u_{xx}^n, \end{cases}$$

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$$u^1(x,0) = u_0^1(x), \quad u^2(x,0) = u_0^2(x), \cdots, \quad u^n(x,0) = u_0^n(x).$$

Luego, basados en la existencia local y el principio del máximo, obtenemos la existencia de soluciones globales suaves para dos sistemas especiales , uno relacionado con el sistema parabólico de flujo cuadrático y el otro relacionado con el sistema LeRoux.

1. Existence of Viscosity Solutions

Let us consider the following Cauchy problem for the parabolic system:

$$\begin{cases} u_t^1 + f_1\left(u^1, u^2, \cdots, u^n\right)_x + g_1\left(u^1, u^2, \cdots, u^n\right) = \varepsilon u_{xx}^1 \\ \vdots \\ u_t^n + f_n\left(u^1, u^2, \cdots, u^n\right)_x + g_n\left(u^1, u^2, \cdots, u^n\right) = \varepsilon u_{xx}^n \end{cases} \tag{1}$$

with the bounded measurable initial data

$$u^1(x,0) = u_0^1(x), \cdots, u^n(x,0) = u_0^n(x), \tag{2}$$
$$\left|u_0^1(x)\right| \le M, \cdots, \left|u_0^n(x)\right| \le M.$$

Theorem 1.1. **(i)** *Suppose that $f_i \in C^1\left(R^n\right)$ and $g_i\,(i = 1, 2, \cdots, n)$ are locally Lipschitz continuous functions. Then the Cauchy problem (1)-(2) has a unique solution $\left(u^{1\varepsilon}(x,t), \cdots, u^{n\varepsilon}(x,t)\right) \in C^\infty(R \times (0, \tau_0))$ for a small $\tau_0 > 0$ which depends only on the L^∞ norm of the initial data, and*

$$\left|u^{1\varepsilon}(x,t)\right| \le 2M, \cdots, \left|u^{n\varepsilon}(x,t)\right| \le 2M, \qquad \forall (x,t) \in R \times [0, \tau_0).$$

(ii) *Moreover, if the solution $\left(u^{1\varepsilon}(x,t), u^{2\varepsilon}(x,t), \cdots, u^{n\varepsilon}(x,t)\right)$ has an a-priori estimate*

$$\left|u^{1\varepsilon}(x,t)\right| \le M(T), \cdots, \left|u^{n\varepsilon}(x,t)\right| \le M(T), \quad \text{for any } t \in [0,T], \tag{3}$$

then the solution $\left(u^{1\varepsilon}(x,t), \cdots, u^{n\varepsilon}(x,t)\right)$ exists on $R \times [0,T]$.
Particularly, if there exists $N > 0$ so that

$$\left\|u^{1\varepsilon}(x,t)\right\|_{L^\infty(R\times[0,+\infty))} \le N, \cdots, \left\|u^{n\varepsilon}(x,t)\right\|_{L^\infty(R\times[0,+\infty))} \le N,$$

then the solution $\left(u^{1\varepsilon}(x,t), \cdots, u^{n\varepsilon}(x,t)\right) \in C^\infty\left(R \times (0, +\infty)\right)$.

Proof. **(i)** The Cauchy problem (1)-(2) is equivalent to the following integral equations:

$$\begin{aligned} u^i(x,t) = &\int_{-\infty}^{+\infty} u_0^i(\xi) G^\varepsilon\left(x-\xi, t\right) d\xi \\ &+ \int_0^t \int_{-\infty}^{+\infty} f_i\left(u^1(\xi,\tau), \cdots, u^n(\xi,\tau)\right) G_\xi^\varepsilon(x-\xi, t-\tau) \\ &- g_i\left(u^1(\xi,\tau), \cdots, u^n(\xi,\tau)\right) G^\varepsilon(x-\xi, t-\tau) d\xi d\tau \ , \end{aligned}$$

where $G^\varepsilon(x,t) = \dfrac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{x^2}{4\varepsilon t}}$, $i = 1, \cdots, n$.

For $\forall \tau > 0$, set

$$B_\tau = \left\{ \left(u^1(x,t), \cdots, u^n(x,t)\right) \mid u^i(x,t) \in C^\infty(R \times (0,\tau)), \right.$$
$$\left. \left\|u^i(x,t)\right\|_{L^\infty(R\times[0,\tau])} \le 2M,\ i = 1, \cdots, n \right\},$$

$$B = \left\{ \left(u^1(x,t), \cdots, u^n(x,t)\right) \mid u^i(x,t) \in C^\infty(R \times (0,\tau)) \right.$$
$$\left. \cap\, L^\infty\left(R \times [0,\tau]\right),\ i = 1, \cdots, n \right\}.$$

It is easy to see that B is a Banach space under the norm

$$\left\|(u^1, \cdots, u^n)\right\|_B = \left\|u^1\right\|_{L^\infty(R\times[0,\tau])} + \cdots + \left\|u^n\right\|_{L^\infty(R\times[0,\tau])}$$

and B_τ is a bounded closed convex subset of B.

Define an operator $\mathbf{T}$ on B_τ,

$$\mathbf{T}\left(u^1, \cdots, u^n\right) = \left(T_1(u^1, \cdots, u^n), \cdots, T_n(u^1, \cdots, u^n)\right)$$

$\forall \left(u^1, \cdots, u^n\right) \in B_\tau$, where

$$\begin{aligned} T_i\left(u^1, \cdots, u^n\right) = & \int_{-\infty}^{+\infty} u_0^i(\xi) G^\varepsilon(x-\xi, t) d\xi \\ & + \int_0^t \int_{-\infty}^{+\infty} f_i\left(u^1(\xi,\tau), \cdots, u^n(\xi,\tau)\right) G_\xi^\varepsilon(x-\xi, t-\tau) \\ & - g_i\left(u^1(\xi,\tau), \cdots, u^n(\xi,\tau)\right) G^\varepsilon(x-\xi, t-\tau) d\xi d\tau \end{aligned}$$

with $i = 1, \cdots, n$.

We assert that there exists a $\tau_0 > 0$ such that for any $\left(u^1, \cdots, u^n\right) \in B_{\tau_0}$, $\mathbf{T}(u^1, \cdots, u^n) \in B_{\tau_0}$, and also $\mathbf{T}$ is a contractive mapping.

In fact, if $\left(u^1, \cdots, u^n\right)$, $\left(u_1^1, \cdots, u_1^n\right)$, $\left(u_2^1, \cdots, u_2^n\right) \in B_\tau$, then there exist positive constants K, L so that

$$\left|f_i\left(u^1 \cdots, u^n\right)\right| \le K, \qquad \left|g_i\left(u^1, \cdots, u^n\right)\right| \le K$$

and

$$\left|f_i\left(u_2^1, \cdots, u_2^n\right) - f_i\left(u_1^1, \cdots, u_1^n\right)\right| \le L\left(\left|u_2^1 - u_1^1\right| + \cdots + \left|u_2^n - u_1^n\right|\right), \qquad (4)$$
$$\left|g_i\left(u_2^1, \cdots, u_2^n\right) - g_i\left(u_1^1, \cdots, u_1^n\right)\right| \le L\left(\left|u_2^n - u_1^n\right| + \cdots + \left|u_2^n - u_1^n\right|\right), \qquad (5)$$

because $i = 1, \cdots, n$ and the functions f_i, g_i are locally Lipschitz continuous. Therefore

$$\begin{aligned} \left|T_i\left(u^1, \cdots, u^n\right)\right| \le & M + \int_0^t \int_{-\infty}^{+\infty} \left|f_1\left(u^1, \cdots, u^n\right)\right| \left|G_\xi^\varepsilon(x-\xi, t-\tau)\right| d\xi d\tau \\ & + Kt \le M + 2K\sqrt{\frac{t}{\pi\varepsilon}} + Kt \quad (i = 1, \cdots, n), \end{aligned}$$

since $\int_{-\infty}^{\infty} G^{\varepsilon}(x-\xi,t)d\xi = 1$ $(t>0)$ and

$$\begin{aligned}\int_0^t \int_{-\infty}^{+\infty} & \left|G_\xi^\varepsilon(x-\xi,t-\tau)\right| d\xi d\tau \\ &= \int_0^t \int_{-\infty}^{+\infty} \frac{|x-\xi|}{4\varepsilon(t-\tau)\sqrt{\pi\varepsilon(t-\tau)}} e^{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}} d\xi d\tau \\ &= \int_0^t \frac{1}{\sqrt{\pi\varepsilon(t-\tau)}} \int_{-\infty}^{+\infty} \left|\eta e^{-\eta^2}\right| d\eta d\tau = 2\sqrt{\frac{t}{\pi\varepsilon}}.\end{aligned}$$

In view of the inequalities (4), (5), we have

$$\begin{aligned}&\left|T_i\left(u_2^1,\cdots,u_2^n\right) - T_i\left(u_1^1,\cdots,u_1^n\right)\right| \\ &\qquad \le \left(2L\sqrt{\frac{t}{\pi\varepsilon}} + Lt\right)\left(\left\|u_2^1 - u_1^1\right\|_{L^\infty} + \cdots + \|u_2^n - u_1^n\|_{L^\infty}\right).\end{aligned}$$

where $i = 1, \cdots, n$. Thus

$$\begin{aligned}&\left\|\mathbf{T}\left(u_2^1,\cdots,u_2^n\right) - \mathbf{T}\left(u_1^1,\cdots,u_1^n\right)\right\|_B \\ &\qquad \le n\left(2L\sqrt{\frac{t}{\pi\varepsilon}} + Lt\right)\left(\|u_2^1 - u_1^1\|_{L^\infty} + \cdots + \|u_2^n - u_1^n\|_{L^\infty}\right).\end{aligned}$$

If we choose $\tau_0 > 0$ such that

$$\frac{2K}{\sqrt{\pi\varepsilon}}\sqrt{\tau_0} + K\tau_0 \le M, \quad n\left(\frac{2L}{\sqrt{\pi\varepsilon}}\sqrt{\tau_0} + L\tau_0\right) < 1,$$

then $\mathbf{T}\left(u^1,\cdots,u^n\right) \in B_{\tau_0}$ and $\mathbf{T}$ is a contractive mapping. By the Brouwer-Schauder's fixed point theorem, there is a unique $\left(u^{1\varepsilon},\cdots,u^{n\varepsilon}\right) \in B_{\tau_0}$ so that

$$\mathbf{T}\left(u^{1\varepsilon},\cdots,u^{n\varepsilon}\right) = \left(u^{1\varepsilon},\cdots,u^{n\varepsilon}\right),$$

i.e., the Cauchy problem (1)-(2) has a local smooth solution $\left(u^{1\varepsilon},\cdots,u^{n\varepsilon}\right) \in C^\infty(R\times(0,\tau_0))$, and

$$|u^{1\varepsilon}(x,t)| \le 2M, \cdots, |u^{n\varepsilon}(x,t)| \le 2M, \qquad \forall (x,t) \in R\times[0,\tau_0].$$

(ii) If the solution $\left(u^1(x,t),\cdots,u^n(x,t)\right)$ has an *a-priori* estimate

$$\left\|u^1(x,t)\right\|_{L^\infty} \le M(T), \cdots, \|u^n(x,t)\|_{L^\infty} \le M(T) \qquad (\forall t \in [0,T]),$$

then

$$|u_0^1(x)| \le M(T), \cdots, |u_0^n(x)| \le M(T).$$

Thus from the proof of (i) , there is a small $\tau > 0$ which depends only on $M(T)$ such that the Cauchy problem has a unique solution $\left(u^{1\varepsilon}(x,t),\cdots,u^{n\varepsilon}(x,t)\right)$ on $R\times[0,\tau]$ and

$$\left|u^{1\varepsilon}(x,t)\right| \le 2M(T), \cdots, |u^{n\varepsilon}(x,t)| \le 2M(T) \qquad \forall t \in [0,\tau].$$

Since the solution has the *a priori* estimate (3), we have

$$\left|u^{1\varepsilon}(x,\tau)\right| \leq M(T), \cdots, |u^{n\varepsilon}(x,\tau)| \leq M(T).$$

If we consider τ as the initial time, then a similar treatment shows that the solution also exists on $R \times [\tau, 2\tau]$ and

$$\left|u^{1\varepsilon}(x,t)\right| \leq 2M(T), \cdots, |u^{n\varepsilon}(x,t)| \leq 2M(T) \qquad \forall t \in [\tau, 2\tau].$$

Because of the *a priori* estimate (3), we have

$$\left|u^{1\varepsilon}(x,2\tau)\right| \leq M(T), \cdots, |u^{n\varepsilon}(x,2\tau)| \leq M(T).$$

Therefore, the local time τ can be extended to T step by step since the step time depends only on $M(T)$. In particular, if the solution has the *a priori* estimate

$$\left\|u^{1\varepsilon}(x,t)\right\|_{L^\infty(R\times[0,+\infty))} \leq N, \cdots, \|u^{n\varepsilon}(x,t)\|_{L^\infty(R\times[0,+\infty))} \leq N,$$

then the solution exists on $R \times [0, +\infty)$ from the above analysis. ☑

2. Applications

In this section, we apply Theorem 1.1 to two special parabolic systems, where one is related to the hyperbolic system of quadratic flux (6) and another related to the Le Roux system (10). First we consider the following Cauchy problem for the parabolic system related to a hyperbolic system of quadratic flux [1,2] with sources:

$$\begin{cases} u_t + \frac{1}{2}\left(3u^2 + v^2\right)_x + g_1(u,v) = \varepsilon u_{xx} \\ v_t + (uv)_x + g_2(u,v) = \varepsilon v_{xx} \end{cases} \tag{6}$$

with the bounded measurable initial data

$$\big(u(x,0), v(x,0)\big) = \big(u_0(x), v_0(x)\big), \tag{7}$$

where the functions $g_1(u,v), g_2(u,v)$ are locally Lipschitz continuous.

Proposition 2.1. *Suppose that $g_1(u,v), g_2(u,v)$ satisfy*

$$-\frac{vg_2(u,v)}{\sqrt{u^2+v^2}+u} \leq g_1(u,v) \leq \frac{vg_2(u,v)}{\sqrt{u^2+v^2}-u}. \tag{8}$$

Then the Cauchy problem (6)-(7) has a unique solution $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ on $R \times [0,+\infty)$, and there exists $M > 0$ such that

$$|u^\varepsilon(x,t)| \leq M, \quad |v^\varepsilon(x,t)| \leq M \qquad \forall (x,t) \in R \times [0,+\infty).$$

There are many functions $g_1(u,v), g_2(u,v)$ which satisfy the condition (8). For instance, $g_1(u,v) = \alpha u^2 v^2$, $g_2 = \beta u^2 v\left(\sqrt{u^2+v^2} + |u|\right)$ where $(|\alpha| \leq \beta)$.

Proof. Let F be the mapping from R^2 into R^2 defined by

$$F : (u,v) \to \left(\frac{3}{2}u^2 + \frac{1}{2}v^2, uv\right),$$

then

$$dF = \begin{pmatrix} 3u & v \\ v & u \end{pmatrix}.$$

Thus the eigenvalues of system (6) are

$$\lambda_1 = 2u - s^{\frac{1}{2}}, \qquad \lambda_2 = 2u + s^{\frac{1}{2}},$$

and the two Riemann invariants are

$$W(u,v) = u + s^{\frac{1}{2}}, \qquad Z(u,v) = u - s^{\frac{1}{2}},$$

where $s = u^2 + v^2$.

By simple calculations, we have

$$W_u = 1 + \frac{u}{\sqrt{s}},\ W_v = \frac{v}{\sqrt{s}},\ W_{uu} = \frac{v^2}{s^{\frac{3}{2}}},\ W_{uv} = -\frac{uv}{s^{\frac{3}{2}}},\ W_{vv} = \frac{u^2}{s^{\frac{3}{2}}};$$

$$Z_u = 1 - \frac{u}{\sqrt{s}},\ Z_v = -\frac{v}{\sqrt{s}},\ Z_{uu} = -\frac{v^2}{s^{\frac{3}{2}}},\ Z_{uv} = \frac{uv}{s^{\frac{3}{2}}},\ Z_{vv} = -\frac{u^2}{s^{\frac{3}{2}}}.$$

It is clear that $W(u,v)$ is convex and $Z(u,v)$ is concave, i.e.,

$$W_{uu}a^2 + 2W_{uv}ab + W_{vv}b^2 \geq 0, \quad Z_{uu}a^2 + 2Z_{uv}ab + Z_{vv}b^2 \leq 0 \qquad \forall (a,b) \in R^2.$$

Multiplying the first and the second equation in (6) by W_u and W_v respectively and add them up, we have

$$\begin{aligned} W_t + \lambda_2 W_x = \varepsilon W_{xx} &- \left(W_{uu}{u_x}^2 + 2W_{uv}u_x v_x + W_{vv}{v_x}^2\right) \\ &- \left[g_1(u,v)W_u + g_2(u,v)W_v\right]; \end{aligned}$$

similarly,

$$Z_t + \lambda_1 Z_x = \varepsilon Z_{xx} - \left(Z_{uu}{u_x}^2 + 2Z_{uv}u_x v_x + Z_{vv}{v_x}^2\right) - \left[g_1(u,v)Z_u + g_2(u,v)Z_v\right].$$

Therefore, in light of the assumption (8), we obtain

$$g_1(u,v)W_u + g_2(u,v)W_v \geq 0, \quad g_1(u,v)Z_u + g_2(u,v)Z_v \leq 0$$

and hence

$$W_t + \lambda_2 W_x \leq \varepsilon W_{xx}, \quad Z_t + \lambda_1 Z_x \geq \varepsilon Z_{xx}. \tag{9}$$

If we consider (9) as the inequalities about the variables w and z, then we can get the estimates $W(u^\varepsilon, v^\varepsilon) \leq N,\ Z(u^\varepsilon, v^\varepsilon) \geq -N$ by the maximum principle and thereby $|u^\varepsilon(x,t)| \leq M, \quad |v^\varepsilon(x,t)| \leq M$ for two positive constants M, N which depend only on the L^∞ norm of the initial data. So we end the proof according to Theorem 1.1 (ii). ☑

Next we study the following Cauchy problem for the parabolic system related to the LeRoux system [3] with sources:

$$\begin{cases} u_t + \left(u^2 + v\right)_x + f(u,v) = \varepsilon u_{xx}, \\ v_t + (uv)_x + g(u,v) = \varepsilon v_{xx}, \end{cases} \tag{10}$$

with the continuously differential initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \geq 0, \tag{11}$$

where the functions $f(u,v), g(u,v)$ are locally Lipschitz continuous, $u_0(x)$, $v_0(x)$, $u_0'(x)$ and $v_0'(x)$ are bounded.

Proposition 2.2. *Suppose that $f(u,v), g(u,v)$ satisfy*

$$g(u,v) \geq \frac{-u-\sqrt{u^2+4v}}{2} f(u,v), \quad g(u,v) \geq \frac{-u+\sqrt{u^2+4v}}{2} f(u,v) \qquad (12),$$

and $g(u,v) = vh(u,v)$, where $h(u,v)$ is a continuous function. Then the Cauchy problem (10)-(11) has a unique solution $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ on $R\times[0,+\infty)$ and

$$|u^\varepsilon(x,t)| \leq M, \quad 0 \leq v^\varepsilon(x,t) \leq M \qquad \forall (x,t) \in R\times[0,+\infty).$$

There are many functions $f(u,v), g(u,v)$ which satisfy the assumption of the proposition, since the inequalities (12) is equivalent to

$$g(u,v) \geq -uf(u,v) + \sqrt{u^2+4v}\,|f(u,v)|.$$

For instance,

$$f(u,v) = \alpha uv\sqrt{u^2+4v}, \quad g(u,v) = \alpha v|u|\left(u^2+4v\right) \qquad (\forall \alpha > 0).$$

To prove the Proposition 2.2 we need the following two Lemmas.

Lemma 2.1. *Let $(u(x,t), v(x,t)) \in C^\infty(R\times(0,T])$ be the local solution of the cauchy problem (10)-(11). Then $u_x(x,t)$ and $v_x(x,t)$ are bounded on $R\times[0,T]$.*

Proof. The solution $u(x,t), v(x,t)$ of the Cauchy problem (10)-(11) can be represented by the Green function $G^\varepsilon(x-y,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\exp\left\{-\frac{(x-y)^2}{4\varepsilon t}\right\}$ (see [4]) as follows:

$$\begin{aligned} u(x,t) = &\int_{-\infty}^{\infty} G^\varepsilon(x-y,t)u_0(y)dy + \int_0^t\int_{-\infty}^{\infty}(u^2(y,s)+v(y,s))G_y^\varepsilon(x-y,t-s) \\ &- f\big(u(y,s),v(y,s)\big)G^\varepsilon(x-y,t-s)dyds, \\ v(x,t) = &\int_{-\infty}^{\infty} G^\varepsilon(x-y,t)v_0(y)dy + \int_0^t\int_{-\infty}^{\infty} uv(y,s)G_y^\varepsilon(x-y,t-s) \\ &- g\big(u(y,s),v(y,s)\big)G^\varepsilon(x-y,t-s)dyds. \end{aligned}$$

Thus

$$\begin{aligned} u_x(x,t) = &\int_{-\infty}^{\infty} G^\varepsilon(x-y,t)u_0'(y)dy + \int_0^t\int_{-\infty}^{\infty}(2uu_y(y,s)+v_y(y,s)) \\ &G_y^\varepsilon(x-y,t-s) - f(u(y,s),v(y,s))G_y^\varepsilon(x-y,t-s)dyds, \qquad (13) \\ v_x(x,t) = &\int_{-\infty}^{\infty} G^\varepsilon(x-y,t)v_0'(y)dy + \int_0^t\int_{-\infty}^{\infty}(uv_y(y,s)+vu_y(y,s)) \\ &G_y^\varepsilon(x-y,t-s) - g(u(y,s),v(y,s))G_y^\varepsilon(x-y,t-s)dyds. \qquad (14) \end{aligned}$$

Define

$$\omega_1(t) = \sup_{x\in R}|u_x(x,t)|, \quad \omega_2(t) = \sup_{x\in R}|v_x(x,t)|,$$

then it follows from (13), (14) that

$$\omega_1(t) \le M + C_1\sqrt{t} + \int_0^t \frac{C_2}{\sqrt{t-s}}\big(w_1(s)+w_2(s)\big)ds,$$

$$\omega_2(t) \le M + C_1\sqrt{t} + \int_0^t \frac{C_2}{\sqrt{t-s}}\big(w_1(s)+w_2(s)\big)ds$$

for two suitable large positive constants C_1, C_2. Thus

$$\omega_1(t)+\omega_2(t) \le C(T) + \int_0^t \frac{C}{\sqrt{t-s}}\big(w_1(s)+w_2(s)\big)ds.$$

Hence

$$\omega_1(t)+\omega_2(t) \le C(T)e^{2C\sqrt{t}}$$

by Gronwall's inequality and this completes the proof of the lemma. ☑

Lemma 2.2. *Assume that $g(u,v) = vh(u,v)$ and $h(u,v)$ is continuous. Then the local solution $(u(x,t), v(x,t)) \in C^\infty(R\times(0,T])$ of the cauchy problem (10)-(11) has the property $v(x,t) \ge 0$ on $R\times[0,T]$.*

Proof. Let $|u(x,t)| \le N(T)$, $|v(x,t)| \le N(T)$. Then there exists $\alpha > 0$ such that $|u_x(x,t) + h(u,v)| \le \alpha$. Set

$$v(x,t) = \left[w(x,t) - \frac{N(T)(x^2+CRt)}{R^2}\right]e^{\alpha t}.$$

Direct calculations show that

$$w(x,0) = v_0(x) + \frac{N(T)x^2}{R^2} \ge 0,$$
$$w(R,t) = v(R,t)e^{-\alpha t} + N(T) + \frac{CN(T)Rt}{R^2} \ge 0,$$
$$w(-R,t) = v(-R,t)e^{-\alpha t} + N(T) + \frac{CN(T)Rt}{R^2} \ge 0,$$

and

$$\begin{aligned} w_t + uw_x - w_{xx} = {} & \frac{N(T)}{R^2}(CR + 2xu - 2) \\ & + \left[-w(x,t) + \frac{N(T)(x^2+CRt)}{R^2}\right][\alpha + u_x + h(u,v)]. \end{aligned} \tag{15}$$

Now we prove

$$w(x,t) \ge 0 \qquad \forall (x,t) \in R\times[0,T].$$

If not, there exists $(x_0,t_0) \in (-R,R)\times(0,T]$ such that $w(x_0,t_0)$ is the minimum on $[-R,R]\times[0,T]$, then $w(x_0,t_0) < 0$ and

$$w_t(x_0,t_0) \le 0, \quad w_x(x_0,t_0) = 0, \quad w_{xx}(x_0,t_0) \ge 0,$$
$$w_t(x_0,t_0) + u(x_0,t_0)w_x(x_0,t_0) - w_{xx}(x_0,t_0) \le 0.$$

Whereas the value of the righthand side of (15) at (x_0, t_0) is positive if C is large enough, this is impossible and hence $w(x,t) \geq 0$ on $R \times [0,T]$. Thus $v(x,t) \geq 0$ on $R \times [0,T]$ by letting $R \to \infty$. ☑

Proof of Proposition 2.2. Let F be the mapping from R^2 into R^2 defined by

$$F : (u,v) \to \left(u^2 + v, uv\right),$$

then

$$dF = \begin{pmatrix} 2u & 1 \\ v & u \end{pmatrix}.$$

Thus the eigenvalues of system (10) are

$$\lambda_1 = \frac{1}{2}(3u - D), \quad \lambda_2 = \frac{1}{2}(3u + D);$$

and the two Riemann invariants are

$$W(u,v) = u + D, \quad Z(u,v) = u - D,$$

where $D = \sqrt{u^2 + 4v}$. By simple calculations, we have

$$W_u = 1 + \frac{u}{D},\ W_v = \frac{2}{D},\ W_{uu} = \frac{4v}{D^3},\ W_{uv} = \frac{-2u}{D^3},\ W_{vv} = -\frac{4}{D^3};$$

$$Z_u = 1 - \frac{u}{D},\ Z_v = -\frac{2}{D},\ Z_{uu} = -\frac{4v}{D^3},\ Z_{uv} = \frac{2u}{D^3},\ Z_{vv} = \frac{4}{D^3}.$$

we multiply system (10) by $\nabla W(u,v)$ and $\nabla Z(u,v)$ respectively to obtain

$$\begin{aligned} W(u,v)_t + \lambda_2 W(u,v)_x &= \varepsilon W(u,v)_{xx} - \varepsilon\left(W_{uu}{u_x}^2 + 2W_{uv}u_x v_x + W_{vv}{v_x}^2\right) \\ &\quad - [f(u,v)W_u + g(u,v)W_v] \\ &= \varepsilon W_{xx} - \varepsilon \frac{1}{\sqrt{u^2+4v}} W(u,v)_x Z(u,v)_x \\ &\quad - [f(u,v)W_u + g(u,v)W_v] \end{aligned}$$

and

$$\begin{aligned} Z(u,v)_t + \lambda_1 Z(u,v)_x &= \varepsilon Z(u,v)_{xx} - \varepsilon\left(Z_{uu}{u_x}^2 + 2Z_{uv}u_x v_x + Z_{vv}{v_x}^2\right) \\ &\quad - [f(u,v)Z_u + g(u,v)Z_v] \\ &= \varepsilon Z_{xx} + \varepsilon \frac{1}{\sqrt{u^2+4v}} W(u,v)_x Z(u,v)_x \\ &\quad - [f(u,v)Z_u + g(u,v)Z_v]. \end{aligned}$$

In view of the inequalities (12), we have

$$f(u,v)W_u + g(u,v)W_v \geq 0, \quad f(u,v)Z_u + g(u,v)Z_v \leq 0$$

and hence

$$W(u,v)_t + \lambda_2 W(u,v)_x \leq \varepsilon W(u,v)_{xx} - \varepsilon \frac{1}{\sqrt{u^2+4v}} W(u,v)_x Z(u,v)_x, \quad (16)$$

$$Z(u,v)_t + \lambda_1 Z(u,v)_x \geq \varepsilon Z(u,v)_{xx} + \varepsilon \frac{1}{\sqrt{u^2+4v}} W(u,v)_x Z(u,v)_x. \quad (17)$$

Therefore, in light of Lemma 2.2, we can get $W(u^\varepsilon, v^\varepsilon) \leq N,\ Z(u^\varepsilon, v^\varepsilon) \geq -N$ by applying the maximum principle to (16),(17) and thereby $|u^\varepsilon(x,t)| \leq M,\ 0 \leq v^\varepsilon(x,t) \leq M$ for two positive constants M, N which depend only on the L^∞ norm of the initial data $u_0(x), v_0(x)$. So we end the proof according to Theorem 1.1 (ii). ☑

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