Revista Colombiana de Matemáticas
Volumen 41 (2007), páginas 91–106

Conservation laws II: weak solutions

Ming Tao
Zhixin Cheng
Jin Yan
University of Science and Technology of China, Hefei

Abstract. In this paper, we apply the maximum principle and the compensated compactness method to get the existence of weak solutions to the Cauchy problems for the nonlinear hyperbolic conservation laws of quadratic flux and the LeRoux system with sources.

Keywords. Weak solution, maximum principle, entropy-entropy flux pair, compensated compactness, Dirac measure.

2000 Mathematics Subject Classification. Primary: 35B40. Secondary: 35L65.

Resumen. En este artículo aplicamos el principio del máximo y el método de la compactificación compensada para obtener soluciones débiles a los problemas de Cauchy para las leyes de conservación hiperbólica, no lineal, de flujo cuadrático y el sistema LeRoux con fuentes.

1. Introduction

In this paper, we study the Cauchy problem for two nonlinear hyperbolic conservation laws, one is related to a system of quadratic flux

$$\begin{cases} u_t + \frac{1}{2}\big(3u^2 + v^2\big)_x + g_1(u,v) = 0, \\ v_t + (uv)_x + g_2(u,v) = 0, \end{cases}$$

and the other is related to the LeRoux system

$$\begin{cases} u_t + \big(u^2 + v\big)_x + f(u,v) = 0, \\ v_t + (uv)_x + g(u,v) = 0. \end{cases}$$

By applying the compensated compactness method and the maximum principle, we get the existence of weak solutions to them.

Next, we introduce some basic lemmas which are very useful later in the present paper.

Lemma 1.1. *Suppose that $v^\varepsilon(x,t)$ satisfies the parabolic equation*

$$v_t + \big(vf(u,v)\big)_x + g(u,v) = \varepsilon v_{xx}, \tag{1.1}$$

and $v(x,0) = v_0(x) \geq \delta > 0$. Where $f(u,v) \in C^1(R^2)$, $g(u,v)$ is locally Lipchitz continuous and $g(u,v) = vh(u,v)$, $h(u,v) \in C(R)$. If $|u(x,t)| \leq M(\varepsilon,\delta,T)$, $|v^\varepsilon(x,t)| \leq M(\varepsilon,\delta,T)$ on $R\times[0,T]$, then the solution $v^\varepsilon(x,t) \geq c(t,\delta,\varepsilon) > 0$ on $R\times[0,T]$, where $c(t,\delta,\varepsilon)$ could tend to zero as δ,ε tend to zero or t tends to infinity.

Proof. We rewrite equation (1.1) as follows:

$$w_t + f(u,v)w_x + f(u,v)_x + h(u,v) = \varepsilon\big(w_{xx} + w_x^2\big), \tag{1.2}$$

where $w = \log v$. Then

$$w_t = \varepsilon w_{xx} + \varepsilon\left(w_x - \frac{f(u,v)}{2\varepsilon}\right)^2 - f(u,v)_x - \frac{f^2(u,v)}{4\varepsilon} - h(u,v).$$

The solution w of (1.2) with initial data $w_0(x) = \log\big(v_0(x)\big)$ can be represented by a Green function $G^\varepsilon(x-y,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\exp\left\{-\frac{(x-y)^2}{4\varepsilon t}\right\}$:

$$\begin{aligned} w = &\int_{-\infty}^{\infty} G^\varepsilon(x-y,t)w_0(y)dy \\ &+\int_0^t \int_{-\infty}^{\infty}\left[\varepsilon\left(w_x - \frac{f(u,v)}{2\varepsilon}\right)^2 - f(u,v)_x - \frac{f^2(u,v)}{4\varepsilon} - h(u,v)\right] \\ &\times G^\varepsilon(x-y,t-s)dyds. \end{aligned} \tag{1.3}$$

Since

$$\int_{-\infty}^{\infty} G^\varepsilon(x-\xi,t)d\xi = 1, \quad \int_0^t\int_{-\infty}^{+\infty}|G_y^\varepsilon(x-y,t-s)|dyds = 2\sqrt{\frac{t}{\pi\varepsilon}}\ (t>0),$$

it follows from (1.3) that

$$\begin{aligned} w &\geq \log\delta + \int_0^t\int_{-\infty}^{\infty}(-f(u,v)_x - \frac{f^2(u,v)}{4\varepsilon} - h(u,v))G^\varepsilon(x-y,t-s)dyds \\ &= \log\delta + \int_0^t\int_{-\infty}^{\infty}\Bigg[f(u,v)G_y^\varepsilon(x-y,t-s) - \left(\frac{f^2(u,v)}{4\varepsilon} + h(u,v)\right) \\ &\quad \times G^\varepsilon(x-y,t-s)\Bigg]dyds \\ &\geq \log\delta - 2M\sqrt{\frac{t}{\pi\varepsilon}} - M_1 t \geq -C(t,\delta,\varepsilon) > -\infty. \end{aligned}$$

Thus $v^\varepsilon(x,t)$ has a positive lower bound $c(t,\delta,\varepsilon)$. ☑

Lemma 1.2. *Assume that $u(x,t)$ satisfies the parabolic equation*

$$u_t + a(u,x,t)u_x + g(u,x,t) = u_{xx}, \tag{1.4}$$

and $|u(x,0)| \leq M$, $|g(u,x,t)| \leq C|u| + \tilde{C}$, where $C, \tilde{C} > 0$ and $a(u,x,t)$ is bounded. Then for any $T > 0$, there exists $M(T) > 0$ such that $|u(x,t)| \leq M(T)$ on $R \times [0,T]$.

Proof. Multiplying equation (1.4) by $2u$, we have

$$\begin{aligned}\left(u^2\right)_t + a(u,x,t)\left(u^2\right)_x &= 2uu_{xx} - 2ug(u,x,t) \\ &\leq (2uu_x)_x - 2u_x^2 + 2|u|\left(C|u| + \tilde{C}\right) \\ &\leq (u^2)_{xx} + (2C+1)u^2 + \tilde{C}^2.\end{aligned}$$

Let $v = u^2 e^{-(2C+1)}$. Then direct calculations show that

$$v_t + av_x \leq v_{xx} + \tilde{C}^2 e^{-(2C+1)t}.$$

Set $w = v + \frac{\tilde{C}^2}{2C+1} e^{-(2C+1)t}$. Then $w_t + a(u,x,t)w_x \leq w_{xx}$ and

$$w|_{t=0} = \left(u|_{t=0}\right)^2 + \frac{\tilde{C}^2}{2C+1} \leq M^2 + \frac{\tilde{C}^2}{2C+1}.$$

Thus we have $w(x,t) \leq M^2 + \frac{\tilde{C}^2}{2C+1}$ by the maximum principle and hence

$$|u(x,t)| \leq \left[\left(M^2 + \frac{\tilde{C}^2}{2C+1}\right) e^{(2C+1)t}\right]^{\frac{1}{2}} \leq M(T).$$

☑

From the proof of Lemma 1.2, we get

Corollary 1.1. *Assume that $u(x,t) \geq (\leq) 0$ satisfies*

$$u_t + a(u,x,t)u_x + g(u,x,t) \leq (\geq) u_{xx}, \tag{1.5}$$

and $|u(x,0)| \leq M$, $g(u,x,t) \geq (\leq) Cu + \tilde{C}$, where $C, \tilde{C} \in R$ and $a(u,x,t)$ is bounded. Then for any $T > 0$, there exists $M(T) > 0$ such that $u(x,t) \leq M(T)(u(x,t) \geq -M(T))$ on $R \times [0,T]$.

Lemma 1.3. *(Lu [1]) Let $\phi_1(r), \phi_2(r)$ be the solutions of the Fuchsian equation*

$$\phi'' - \left(1 + \frac{c}{r^2}\right)\phi = 0, \tag{1.6}$$

where c is a constant. If $\phi_1(r) > 0$, $\phi_1'(r) > 0$ for $r > 0$, then

$$\frac{\phi_1'(r)}{\phi_1(r)} = 1 + O\left(\frac{1}{r^2}\right), \qquad c_1\phi_1(r)e^{-r} = 1 + O\left(\frac{1}{r}\right); \tag{1.7}$$

as r approaches infinity;

If $\phi_2(r) > 0, {\phi_2}'(r) < 0$ *for* $r > 0$*, then*

$$\frac{{\phi_2}'(r)}{\phi_2(r)} = -1 + O\left(\frac{1}{r^2}\right), c_2\phi_2(r)e^r = 1 + O\left(\frac{1}{r}\right); \tag{1.8}$$

as r *approaches infinity, where* c_1, c_2 *are two suitable, positive constants.*

2. A system of quadratic flux with sources

Let us consider the following Cauchy problem for the nonlinear conservation laws of quadratic flux [1,2] with sources:

$$\begin{cases} u_t + \frac{1}{2}\big(3u^2 + v^2\big)_x + g_1(u,v) = 0, \\ v_t + (uv)_x + g_2(u,v) = 0, \end{cases} \tag{2.1}$$

with the bounded measurable initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \geq 0, \tag{2.2}$$

where $g_1(u,v)$ and $g_2(u,v)$ are locally Lipchitz continuous functions.

By simple calculations, the two eigenvalues of system (2.1) are

$$\lambda_1 = 2u - s^{\frac{1}{2}}, \qquad \lambda_2 = 2u + s^{\frac{1}{2}};$$

and the two Riemann invariants are

$$W(u,v) = u + s^{\frac{1}{2}}, \qquad Z(u,v) = u - s^{\frac{1}{2}}.$$

Here and below $s = u^2 + v^2$.

We now study the Cauchy problem (2.1)-(2.2) by using the maximum principle and the compensated compactness method to obtain the following main result:

Theorem 2.1. *Suppose that* $g_1(u,v), g_2(u,v)$ *have the property: there exist four constants* $C_1, C_2, C_3, C_4 \in R$ *such that*

$$g_1W_u + g_2W_v \geq C_1W + C_2, g_1Z_u + g_2Z_v \leq C_3Z + C_4, \tag{2.3}$$

and also $g_2(u,v) = vh(u,v)$*, where* $h(u,v)$ *is continuous. Then the Cauchy problem (2.1)-(2.2) has a weak solution in the sense of distribution.*

Proof. First consider the Cauchy problem for the related parabolic system

$$\begin{cases} u_t + \frac{1}{2}\big(3u^2 + v^2\big)_x + g_1(u,v) = \varepsilon u_{xx}, \\ v_t + (uv)_x + g_2(u,v) = \varepsilon v_{xx}, \end{cases} \tag{2.4}$$

with initial data

$$\big(u^\varepsilon(x,0), v^\varepsilon(x,0)\big) = \big(u_0(x), v_0(x) + \varepsilon\big) * G^\varepsilon, \tag{2.5}$$

where G^ε is a mollifier.

We assert that the viscosity solutions $\big(u^\varepsilon(x,t), v^\varepsilon(x,t)\big)$ of the Cauchy problem (2.4)-(2.5) exist and satisfy that for any $T > 0$, $|u^\varepsilon(x,t)| \leq M(T)$, $0 < c(\varepsilon,t) \leq v^\varepsilon(x,t) \leq M(T)$ on $R \times [0,T]$, where $M(T)$ is a positive constant which independent of ε, $c(\varepsilon, t)$ is a positive function which could tend to zero as ε tends to zero or t tends to infinity.

In fact, by simple calculations, we have

$$W_u = 1 + \frac{u}{\sqrt{s}},\ W_v = \frac{v}{\sqrt{s}},\ W_{uu} = \frac{v^2}{s^{\frac{3}{2}}},\ W_{uv} = -\frac{uv}{s^{\frac{3}{2}}},\ W_{vv} = \frac{u^2}{s^{\frac{3}{2}}};$$

$$Z_u = 1 - \frac{u}{\sqrt{s}},\ Z_v = -\frac{v}{\sqrt{s}},\ Z_{uu} = -\frac{v^2}{s^{\frac{3}{2}}},\ Z_{uv} = \frac{uv}{s^{\frac{3}{2}}},\ Z_{vv} = -\frac{u^2}{s^{\frac{3}{2}}}.$$

Multiplying the first equation and the second in (2.1) by W_u and W_v respectively and adding the result, we have

$$W_t + \lambda_2 W_x = \varepsilon W_{xx} - \Big(W_{uu}{u_x}^2 + 2W_{uv}u_x v_x + W_{vv}{v_x}^2\Big) - \big[g_1(u,v)W_u + g_2(u,v)W_v\big];$$

similarly,

$$Z_t + \lambda_1 Z_x = \varepsilon Z_{xx} - (Z_{uu}{u_x}^2 + 2Z_{uv}u_x v_x + Z_{vv}{v_x}^2) - [g_1(u,v)Z_u + g_2(u,v)Z_v].$$

Thus in terms of the inequalities (2.3), we obtain

$$W_t + \lambda_2 W_x + C_1 W + C_2 \le \varepsilon W_{xx},\quad Z_t + \lambda_1 Z_x + C_3 z + C_4 \ge \varepsilon Z_{xx}. \tag{2.6}$$

If we consider (2.6) as inequalities about the variables W and Z, then we can obtain that for any $T > 0$, $W(u^\varepsilon, v^\varepsilon) \le N(T)$, $Z(u^\varepsilon, v^\varepsilon) \ge -N(T)$ on $R \times [0, T]$ by Corollary 1.1, where N(T) is independent of ε. Thus we have the estimates $|u^\varepsilon(x,t)| \le M(T)$, $0 < c(\varepsilon, t) \le v^\varepsilon(x,t) \le M(T)$ in light of Lemma 1.1 and hence the assertion by Paper I. Therefore, there exists a subsequence (still labeled) $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ such that

$$w^\star - \lim(u^\varepsilon(x,t), v^\varepsilon(x,t)) = (u(x,t), v(x,t)).$$

Now we construct four families of entropy-entropy fluxes of Lax type of system (2.1). Any entropy-entropy flux pair $\big(\bar\eta(u,v), \bar q(u,v)\big)$ of system (2.1) satisfies

$$\bar q_u = 3u\bar\eta_u + v\bar\eta_v, \bar q_v = v\bar\eta_u + u\bar\eta_v. \tag{2.7}$$

Eliminating $\bar q$ from (2.7), we get

$$v(\bar\eta_{vv} - \bar\eta_{uu}) + 2u\bar\eta_{uv} = 0. \tag{2.8}$$

Let $\bar\eta(u,v) = \eta(u,s), \bar q(u,v) = q(u,s)$. Then by simple calculations, the entropy equation (2.8) is changed to the following simple equation:

$$\eta_{ss} = \frac{1}{4s}\eta_{uu}, \tag{2.9}$$

and the entropy flux q corresponding to the entropy η satisfies

$$q_u = 2u\eta_u + 2s\eta_s. \tag{2.10}$$

If k denotes a constant, then $\eta = h(s)e^{ku}$ solves (2.9) provided that

$$h''(s) - \frac{k^2}{4s}h(s) = 0.$$

Let $a(s)=s^{\frac{1}{4}}$, $r=ks^{\frac{1}{2}}$, $h(s)=a(s)\phi(r)$. Then

$$\phi''(r)-\left(1+\frac{3}{4r^2}\right)\phi(r)=0, \tag{2.11}$$

which is the standard Fuchsian equation. We can find a series solution of (2.11) with the following form:

$$\phi_1(r)=r^{\frac{3}{2}}\sum_{n=0}^{\infty}c_nr^{2n}=r^{\frac{3}{2}}g(r). \tag{2.12}$$

Then the coefficients c_n must satisfy

$$c_n=\frac{c_{n-1}}{\left(2n+\frac{3}{2}\right)\left(2n+\frac{1}{2}\right)-\frac{3}{4}}, \qquad \text{for } n\geq 1$$

and c_0 could be any positive constant. Thus

$$\phi_2(r)=r^{\frac{3}{2}}g(r)\int_r^{\infty}\left(r^3g^2(r)\right)^{-1}dr, \tag{2.13}$$

is another independent solution of (2.11).

If $\eta_k=a(s)\phi(r)e^{ku}$, we have from (2.10)

$$(q_k)_u=2ku\eta_k+\left(\frac{1}{2}+r\frac{\phi'(r)}{\phi(r)}\right)\eta_k \tag{2.14}$$

and hence, one entropy flux q_k corresponding to η_k is

$$q_k=\eta_k\left(2u+s^{\frac{1}{2}}+\frac{r}{k}\left(\frac{\phi'(r)}{\phi(r)}-1\right)-\frac{3}{2k}\right). \tag{2.15}$$

Let $\eta_{-k}=a(s)\phi(r)e^{-ku}$. Then one entropy flux q_{-k} corresponding to η_{-k} is

$$q_{-k}=\eta_{-k}\left(2u-s^{\frac{1}{2}}-\frac{r}{k}\left(\frac{\phi'(r)}{\phi(r)}-1\right)+\frac{3}{2k}\right). \tag{2.16}$$

It is clear that ϕ_1, ϕ_2 given in (2.12) and (2.13) satisfy that $\phi_1(r)>0$, $\phi_1'(r)>0$ and $\phi_2(r)>0$ for all $s>0$. The strict positivity of $\phi_2''(r)$ gives $\phi_2'(r)<0$ as $s>0$ because $\lim\limits_{r\to\infty}\phi_2(r)=0$, $\lim\limits_{r\to\infty}\phi_2'(r)=0$.

Applying the estimates in Lemma 1.3 to $\phi_1(r),\phi_2(r)$, we have

$$\eta_k^1=a(s)\phi_1(r)e^{ku}=e^{kw}\left(a(s)+O\left(\frac{1}{r}\right)\right)=e^{kw}\left(a(s)+O\left(\frac{1}{k}\right)\right) \tag{2.17}$$

on any compact subset of $s>0$ since $r=ks^{\frac{1}{2}}$,

$$q_k^1=\eta_k^1\left(2u+s^{\frac{1}{2}}+\frac{r}{k}\left(\frac{\phi_1'(r)}{\phi_1(r)}-1\right)-\frac{3}{2k}\right)=\eta_k^1\left(\lambda_2+O\left(\frac{1}{k}\right)\right) \tag{2.18}$$

on $s\geq 0$ by the fact that factor $r(\frac{\phi_1'(r)}{\phi_1(r)}-1)$ is uniformly bounded. Furthermore

$$q_k^1=\eta_k^1\left(\lambda_2-\frac{3}{2k}+O\left(\frac{1}{k^2}\right)\right), \tag{2.19}$$

on any compact subset of $s>0$. Similarly,

$$\left\{\begin{array}{llll} \eta^1_{-k} &= e^{-kz}\left(a(s)+O\left(\frac{1}{k}\right)\right), & q^1_{-k} &= \eta^1_{-k}\left(\lambda_1+\frac{3}{2k}+O\left(\frac{1}{k^2}\right)\right); \\ \eta^2_{k} &= e^{kz}\left(a(s)+O\left(\frac{1}{k}\right)\right), & q^2_{k} &= \eta^2_{k}\left(\lambda_1-\frac{3}{2k}+O\left(\frac{1}{k^2}\right)\right); \\ \eta^2_{-k} &= e^{-kw}\left(a(s)+O\left(\frac{1}{k}\right)\right), & q^2_{-k} &= \eta^2_{-k}\left(\lambda_2+\frac{3}{2k}+O\left(\frac{1}{k^2}\right)\right) \end{array}\right. \tag{2.20}$$

on any compact subset of $s>0$, and

$$q^1_{-k}=\eta^1_{-k}\left(\lambda_1+O\left(1/k\right)\right),\, q^2_{k}=\eta^2_{k}\left(\lambda_1+O\left(1/k\right)\right),\, q^2_{-k}=\eta^2_{-k}\left(\lambda_2+O\left(1/k\right)\right) \tag{2.21}$$

on $s\geq 0$. These estimates about the entropy-entropy flux pairs will be used to reduce the Young measure ν.

Next we verify the compactness of η_t+q_x in H^{-1}_{loc}. It is obvious that system (2.1) has a strictly convex entropy $\eta^\star=\frac{u^2+v^2}{2}$ and the corresponding entropy flux $q^\star=u^3+uv^2$. Multiplying the first equation in (2.4) by u and the second by v, then adding the result, we have

$$\eta^\star(u^\varepsilon,v^\varepsilon)_t+q^\star(u^\varepsilon,v^\varepsilon)_x=\varepsilon\eta^\star_{xx}-\varepsilon\left(\left(u^\varepsilon_x\right)^2+\left(v^\varepsilon_x\right)^2\right)-\left(\eta^\star_u g_1+\eta^\star_v g_2\right). \tag{2.22}$$

Noticing that the term $(\eta^\star_u g_1+\eta^\star_v g_2)\in L^\infty\big(R\times[0,T]\big)$, $\forall\ T>0$ and hence is bounded in $L^1_{loc}\big(R\times R^+\big)$, we can easily obtain that $\varepsilon\left(u^\varepsilon_x\right)^2$ and $\varepsilon\left(v^\varepsilon_x\right)^2$ are bounded in L^1_{loc}. For simplicity, we will drop the superscript ε.

The first class of entropy-entropy flux pair of Lax type related to the function ϕ_1, denoted by $\eta^1_{\pm k}$ are clearly smooth function of (u,v). In fact

$$\eta^1_{\pm k}=k^{\frac{3}{2}}\sum_{n=0}^{\infty}c_n\left(k^2 s\right)^n e^{\pm ku};$$

it is easy to see that $(\eta^1_{\pm k})_t+(q^1_{\pm k})_x$ is compact in H^{-1}_{loc}.

However the second order derivatives of the second class of entropy-entropy flux pair of Lax type related to the function ϕ_2, denoted by $\eta^2_{\pm k}$, are singular at the point $(u,v)=(0,0)$. In fact

$$\eta^2_{\pm k}=k^{-\frac{1}{2}}e^{\pm ku}r^2 g(r)\int_r^\infty\left(r^3 g^2(r)\right)^{-1}dr,$$

where

$$\int_r^\infty\left(r^3 g^2(r)\right)^{-1}dr=O\left(\frac{1}{r^2}\right),\qquad \text{as } r\to 0$$

and hence for any fixed $k>0$, $\eta^2_{\pm k}$ and $q^2_{\pm k}$ are uniformly bounded from (2.20). Moreover,

$$\eta^2_{\pm k}=k^{-\frac{1}{2}}e^{\pm ku}\left(\frac{1}{2g(r)}-r^2 g(r)\int_r^\infty\frac{g'(r)}{r^2 g^3(r)}dr\right), \tag{2.23}$$

where

$$g'(r)=\sum_{n=1}^{\infty}2nc_n r^{2n-1}\leq\sum_{n=1}^{\infty}c_{n-1}r^{2n-1}=rg(r),$$

thus

$$r^2 g(r) \int_r^\infty \frac{g^{'}(r)}{r^2 g^3(r)} dr = O\big(r^2 \log r\big), \qquad \text{as } r \to 0.$$

This implies that for any fixed $k > 0$, the first order derivatives of $\eta^2_{\pm k}$ are uniformly bounded. It is clear that the first part $I_1 = k^{-\frac{1}{2}} e^{\pm ku} \frac{1}{2g(r)}$ in $\eta^2_{\pm k}$ is smooth; its second order derivatives are bounded. But the second part in $\eta^2_{\pm k}$ can be written as $-r^2 I_2$, where

$$I_2 = k^{-\frac{1}{2}} e^{\pm ku} g(r) \int_r^\infty \frac{g^{'}(r)}{r^2 g^3(r)} dr,$$

its second order derivatives are singular at the point $(0,0)$. In fact, all derivatives of second order of function $r^2 I_2$ are bounded except the terms $\big(r^2\big)_{uu} I_2$, $\big(r^2\big)_{vv} I_2 = 2k^2 I_2$, but they are positive.

Therefore, multiplying system (2.4) by $\nabla \eta^2_{\pm k}$, we have

$$\begin{aligned}
\big(\eta^2_{\pm k}\big)_t + \big(q^2_{\pm k}\big)_x &= \varepsilon \big(\eta^2_{\pm k}\big)_{xx} - \varepsilon\Big(\big(\eta^2_{\pm k}\big)_{uu} u_x^2 + 2\big(\eta^2_{\pm k}\big)_{uv} u_x v_x + \big(\eta^2_{\pm k}\big)_{vv} v_x^2\Big) \\
&\quad - \Big(\big(\eta^2_{\pm k}\big)_u g_1 + \big(\eta^2_{\pm k}\big)_v g_2\Big) \\
&= \varepsilon \big(\eta^2_{\pm k}\big)_{xx} - \varepsilon\Big(A(u,v) u_x^2 + B(u,v) u_x v_x + C(u,v) v_x^2\Big) \\
&\quad - 2k^2 \varepsilon I_2 \big(u_x^2 + v_x^2\big) - \Big(\big(\eta^2_{\pm k}\big)_u g_1 + \big(\eta^2_{\pm k}\big)_v g_2\Big),
\end{aligned} \tag{2.24}$$

where $A(u,v)$, $B(u,v)$, $C(u,v)$ are the regular derivatives of second order of $\eta^2_{\pm k}$.

Let $K \subset R \times R^+$ be an arbitrary compact set and choose $\phi \in C_0^\infty\big(R \times R^+\big)$ such that $\phi_K = 1$, $0 \le \phi \le 1$ and write $S = \text{supp}\phi$.

Multiplying (2.24) by ϕ and integrating over $R \times R^+$, we have

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty 2k^2 \varepsilon I_2 \big(u_x^2 + v_x^2\big) \phi \, dx dt \\
&\qquad = \int_0^\infty \int_{-\infty}^\infty -\varepsilon\Big(A(u,v) u_x^2 + B(u,v) u_x v_x + C(u,v) v_x^2\Big)\phi \\
&\qquad\quad + \eta^2_{\pm k}\phi_t + q^2_{\pm k}\phi_x + \varepsilon \eta^2_{\pm k}\phi_{xx} - \big(\big(\eta^2_{\pm k}\big)_u g_1 + \big(\eta^2_{\pm k}\big)_v g_2\big)\phi \, dx dt \\
&\qquad \le M(\phi),
\end{aligned}$$

where the last inequality follows from the boundedness of viscosity solutions, the local boundedness in L^1_{loc} of the regular part $A(u,v)u_x^2 + B(u,v)u_x v_x + C(u,v)v_x^2$ and $\big(\eta^2_{\pm k}\big)_u g_1 + \big(\eta^2_{\pm k}\big)_v g_2$.

Considering (2.24) again, we see that the parts

$$\varepsilon\Big(\big(\eta^2_{\pm k}\big)_{uu} u_x^2 + 2\big(\eta^2_{\pm k}\big)_{uv} u_x v_x + (\eta^2_{\pm k})_{vv} v_x^2\Big), \qquad \big(\eta^2_{\pm k}\big)_u g_1 + \big(\eta^2_{\pm k}\big)_v g_2$$

are both bounded in L^1_{loc} and hence, compact in $W^{-1,\alpha}_{loc}$ for a constant $\alpha \in (1,2)$. The part $\varepsilon(\eta^2_{\pm k})_{xx}$ is clearly compact in $W^{-1,2}_{loc}$ because of the boundedness of derivatives of the first order of $\eta^2_{\pm k}$, and the L^1_{loc} estimates for εu_x^2 and εv_x^2. Noticing the boundedness of $\big(\eta^2_{\pm k}\big)_t + \big(q^2_{\pm k}\big)_x$ in $W^{-1,\infty}$, we get the compactness of $\big(\eta^2_{\pm k}\big)_t + \big(q^2_{\pm k}\big)_x$ in H^{-1}_{loc} by Theorem 2.3.2 of [1].

Finally, we shall prove that the family of the Young measures $\nu_{x,t}$, determined by the sequence of viscosity solutions $\big(u^\varepsilon(x,t), v^\varepsilon(x,t)\big)$ of the Cauchy problem (2.4)-(2.5), must be Dirac measures. Since the viscosity solutions $\big(u^\varepsilon(x,t), v^\varepsilon(x,t)\big)$ of Cauchy problem (2.4)-(2.5) are bounded in $L^\infty\big(R\times[0,T]\big)$ for any $T>0$, by Theorem 2.2.1 in Reference [1], we consider the family of compactly supported probability measure $\nu_{x,t}$. Without loss of generality we may fix $(x,t)\in R\times R^+$ and consider only one measure ν.

For any entropy-entropy flux pairs(η_i, q_i), $(i=1,2)$ of system (2.1) satisfying the compactness of $\eta(u^\varepsilon, v^\varepsilon)_t + q(u^\varepsilon, v^\varepsilon)x$ in H^{-1}_{loc}, we have

$$\begin{aligned}\overline{\eta_1(u^\varepsilon, v^\varepsilon)}\cdot\overline{q_2(u^\varepsilon, v^\varepsilon)} \quad &- \quad \overline{\eta_2(u^\varepsilon, v^\varepsilon)}\cdot\overline{q_1(u^\varepsilon, v^\varepsilon)}\\ &= \quad \overline{\eta_1(u^\varepsilon, v^\varepsilon)q_2(u^\varepsilon, v^\varepsilon) - \eta_2(u^\varepsilon, v^\varepsilon)q_1(u^\varepsilon, v^\varepsilon)}.\end{aligned}$$

Here and below we use the notation $\overline{\eta(u^\varepsilon, v^\varepsilon)} = w^\star - \lim \eta(u^\varepsilon, v^\varepsilon)$. Then in light of the Young measure representation theorem, we get the measure equation

$$\langle\nu,\eta_1\rangle\langle\nu,q_2\rangle - \langle\nu,\eta_2\rangle\langle\nu,q_1\rangle = \langle\nu,\eta_1 q_2 - \eta_2 q_1\rangle. \tag{2.25}$$

Let Q denote the smallest characteristic rectangle:

$$Q = \{(u,v): w_- \le w \le w_+, \quad z_- \le z \le z_+, \quad v \ge 0\}.$$

We now prove that suppν is either contained in the point (0,0) or in another point $(w^\star, z^\star)$.

Assume that suppν is not the unique point (0,0), then $\langle\nu,\eta^1_k\rangle > 0$ and $\langle\nu,\eta^2_{-k}\rangle > 0$, where η^1_k, η^2_{-k} are given in (2.17), (2.20).

We introduce two new probability measures μ^+_k, μ^-_k on Q, defined by

$$\big\langle\mu^+_k, h\big\rangle = \frac{\big\langle\nu, h\eta^1_k\big\rangle}{\big\langle\nu,\eta^1_k\big\rangle}, \qquad \big\langle\mu^-_k, h\big\rangle = \frac{\big\langle\nu, h\eta^2_{-k}\big\rangle}{\big\langle\nu,\eta^2_{-k}\big\rangle},$$

where $h = h(u,v)$ denotes an arbitrary continuous functions. Clearly μ^+_k, μ^-_k are uniformly bounded with respect to k. Then as a consequence of weak-star compactness, there exist probability measures $\mu^\pm$ on Q such that

$$\langle\mu^\pm, h\rangle = \lim_{k\to\infty}\langle\mu^\pm_k, h\rangle$$

after the selection of an appropriate subsequence. Moreover,

$$\text{supp}\mu^+ = Q\cap\{(u,v): w = w_+\}, \quad \text{supp}\mu^- = Q\cap\{(u,v): w = w_-\}. \tag{2.26}$$

In fact, for any function $h(w,z)\in C_0(Q)$, satisfying

$$\text{supp}h(w,z)\subset Q\cap\{(u,v): w\le w_0\},$$

where $w_0 < w_+$ is any number, as $k \to \infty$, we have

$$\frac{\left|\left\langle \nu, h\eta_k^1\right\rangle\right|}{\left|\left\langle \nu, \eta_k^1\right\rangle\right|} = \frac{\left|\left\langle \nu, he^{kw}\left(a(s)+O\left(\frac{1}{k}\right)\right)\right\rangle\right|}{\left|\left\langle \nu, e^{kw}\left(a(s)+O\left(\frac{1}{k}\right)\right)\right\rangle\right|} \le \frac{c_1 e^{k(w_0+\delta)}}{c_2 e^{k(w_+-\delta)}} \to 0,$$

where c_1, c_2 are two suitable positive constants and $\delta > 0$ satisfies $2\delta < w_+ - w_0$, since Q is the smallest characteristic rectangle of ν. Thus we get the first equality of (2.26). A similar treatment gives the second one.

Let $(\eta_1, q_1) = \left(\eta_k^1, q_k^1\right)$ in (2.25). Then

$$\langle \nu, q_2\rangle - \langle \nu, \eta_2\rangle \frac{\left\langle \nu, q_k^1\right\rangle}{\left\langle \nu, \eta_k^1\right\rangle} = \frac{\left\langle \nu, \eta_k^1 q_2 - \eta_2 q_k^1\right\rangle}{\left\langle \nu, \eta_k^1\right\rangle}. \tag{2.27}$$

Using the estimate (2.19) and letting $k \to \infty$ in (2.27), we have

$$\langle \nu, q_2\rangle - \langle \nu, \eta_2\rangle \left\langle \mu^+, \lambda_2\right\rangle = \left\langle \mu^+, q_2 - \lambda_2\eta_2\right\rangle. \tag{2.28}$$

Similarly, let $(\eta_1, q_1) = \left(\eta_{-k}^2, q_{-k}^2\right)$, we have

$$\langle \nu, q_2\rangle - \langle \nu, \eta_2\rangle \left\langle \mu^-, \lambda_2\right\rangle = \left\langle \mu^-, q_2 - \lambda_2\eta_2\right\rangle. \tag{2.29}$$

Let $(\eta_1, q_1) = \left(\eta_k^1, q_k^1\right)$, $(\eta_2, q_2) = \left(\eta_{-k}^2, q_{-k}^2\right)$ in (2.25), we have

$$\frac{\left\langle \nu, q_{-k}^2\right\rangle}{\left\langle \nu, \eta_{-k}^2\right\rangle} - \frac{\left\langle \nu, q_k^1\right\rangle}{\left\langle \nu, \eta_k^1\right\rangle} = \frac{\left\langle \nu, \eta_k^1 q_{-k}^2 - \eta_{-k}^2 q_k^1\right\rangle}{\left\langle \nu, \eta_{-k}^2\right\rangle \left\langle \nu, \eta_k^1\right\rangle}. \tag{2.30}$$

We assert $w_- = w_+$. If not, choose $\delta_0 > 0$ such that $2\delta_0 < w_+ - w_-$, then

$$\left\langle \nu, \eta_{-k}^2\right\rangle \ge c_1 e^{-k(w_-+\delta_0)}, \qquad \left\langle \nu, \eta_k^1\right\rangle \ge c_2 e^{k(w_+-\delta_0)}$$

for two suitable positive constants c_1, c_2 and hence, the right-hand side of (2.30) satisfies

$$\frac{\left\langle \nu, \eta_k^1 q_{-k}^2 - \eta_{-k}^2 q_k^1\right\rangle}{\left\langle \nu, \eta_{-k}^2\right\rangle \left\langle \nu, \eta_k^1\right\rangle} = O\left(\frac{1}{k}\right) e^{-k(w_+-w_--2\delta_0)} \to 0, \qquad \text{as } k \to \infty,$$

resulting from the estimates given by (2.17), (2.18) and (2.20). Letting $k \to \infty$ in (2.30), we have $\langle \mu^+, \lambda_2\rangle = \langle \mu^-, \lambda_2\rangle$. Combining this with (2.28)-(2.29) gives the relation:

$$\left\langle \mu^+, q - \lambda_2\eta\right\rangle = \left\langle \mu^-, q - \lambda_2\eta\right\rangle \tag{2.31}$$

for any (η, q) satisfying that $\eta_t + q_x$ is compact in H_{loc}^{-1}.

Let (η, q) in (2.31) be $\left(\eta_{-k}^2, q_{-k}^2\right)$. If $w_+ - w_- > 2\delta_0$, we get from the left-hand side of (2.31) that

$$\left|\left\langle \mu^+, q - \lambda_2\eta\right\rangle\right| \ge \frac{c_1}{k} e^{k(w_+-\delta_0)},$$

and from the right-hand side of (2.31)

$$\left|\left\langle \mu^-, q - \lambda_2\eta\right\rangle\right| \le \frac{c_2}{k} e^{-k(w_+-\delta_0)},$$

for two positive constants c_1, c_2. This is impossible, hence $w_+ = w_-$. Similarly we can prove $z_+ = z_-$ by using entropy-entropy flux pairs $\left(\eta_k^2, q_k^2\right)$, $\left(\eta_{-k}^1, q_{-k}^1\right)$.

Thus the support set of ν is either $(0,0)$ or another point $(u^\star, v^\star)$. This completes the proof of Theorem 2.1 according to the compensated compactness method. ☑

Remark 2.1. *There are many functions $g_1(u,v), g_2(u,v)$ satisfying the assumptions of Theorem 2.1. For instance,*

$$\begin{aligned} g_1(u,v) &= \alpha\sqrt{u^2+v^2}+\beta, \\ g_2(u,v) &= \frac{v}{v+1}\left(\theta\left(\sqrt{u^2+v^2}-|u|\right)+\gamma\right) \qquad \alpha,\beta,\gamma,\theta \in R. \end{aligned}$$

In fact, $|W_u| = 1+\frac{u}{\sqrt{s}} \le 2$, $|W_v| = \frac{|v|}{\sqrt{s}} \le 1$ and $\sqrt{s}-|u| \le W$. Thus

$$\begin{aligned} g_1 W_u &= \alpha W + \left(1+\frac{u}{\sqrt{s}}\right)\beta \ge \alpha W - 2|\beta|, \\ g_2 W_v &= \frac{v}{v+1}\left(\theta\left(\sqrt{s}-|u|\right)+\gamma\right)\frac{v}{\sqrt{s}} \ge -|\theta|W - |\gamma|, \end{aligned}$$

and hence $g_1W_u + g_2W_v \ge (\alpha-|\theta|)W - (2|\beta|+|\gamma|)$.

Similarly, $g_1Z_u + g_2Z_v \le (-\alpha-|\theta|)Z + (2|\beta|+|\gamma|)$. Besides, since $(g_1)_u = \alpha\frac{u}{\sqrt{s}}$, $(g_1)_v = \alpha\frac{v}{\sqrt{s}}$ are bounded, $g_1(u,v)$ is locally Lipchitz continuous, so is $g_2(u,v)$.

3. The LeRoux system with sources

In this section, we consider the following Cauchy problem for the LeRoux system [3,4] with sources:

$$\begin{cases} u_t + \left(u^2+v\right)_x + f(u,v) = 0, \\ v_t + (uv)_x + g(u,v) = 0, \end{cases} \tag{3.1}$$

with the bounded measurable initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \ge 0, \tag{3.2}$$

where $f(u,v)$ and $g(u,v)$ are locally Lipchitz continuous functions.

By simple calculations, the two eigenvalues of system (3.1) are

$$\lambda_1 = \frac{3u}{2} - \frac{D}{2}, \qquad \lambda_2 = \frac{3u}{2} + \frac{D}{2}$$

and the two Riemann invariants are

$$w(u,v) = u + D, \qquad z(u,v) = u - D.$$

Here and hereafter $D = \sqrt{u^2+4v}$.

The main result about the existence of weak solution of the Cauchy problem (3.1)-(3.2) is given as follows:

Theorem 3.1. *Suppose that $f(u,v), g(u,v)$ have the property: there exist four constants $c_1,\ c_2,\ c_3,\ c_4 \in R$ such that*

$$w_u f + w_v g \geq c_1 w + c_2, \qquad z_u f + z_v g \leq c_3 z + c_4, \tag{3.3}$$

and also $g(u,v) = vh(u,v)$, where $h(u,v)$ is continuous. Then the Cauchy problem (3.1)-(3.2) has a weak solution in the sense of distribution.

Proof. First consider the Cauchy problem for the related parabolic system

$$\begin{cases} u_t + \left(u^2 + v\right)_x + f(u,v) = \varepsilon u_{xx}, \\ v_t + (uv)_x + g(u,v) = \varepsilon v_{xx}, \end{cases} \tag{3.4}$$

with initial data

$$\left(u^\varepsilon(x,0), v^\varepsilon(x,0)\right) = \left(u_0(x), v_0(x) + \varepsilon\right) * G^\varepsilon, \tag{3.5}$$

where G^ε is a mollifier.

By simple calculations, we have

$$w_u = 1 + \frac{u}{D},\ w_v = \frac{2}{D},\ w_{uu} = \frac{4v}{D^3},\ w_{uv} = \frac{-2u}{D^3},\ w_{vv} = -\frac{4}{D^3};$$
$$z_u = 1 - \frac{u}{D},\ z_v = -\frac{2}{D},\ z_{uu} = -\frac{4v}{D^3},\ z_{uv} = \frac{2u}{D^3},\ z_{vv} = \frac{4}{D^3}.$$

Multiplying system (3.4) by (w_u, w_v) and (z_u, z_v) respectively to obtain

$$\begin{aligned} w(u,v)_t &+ \lambda_2 w(u,v)_x + (w_u g_1 + w_v g_2) \\ &= \varepsilon w(u,v)_{xx} - \varepsilon\left(w_{uu} u_x^2 + 2 w_{uv} u_x v_x + w_{vv} v_x^2\right) \\ &= \varepsilon w(u,v)_{xx} - \frac{\varepsilon}{D} w(u,v)_x z(u,v)_x \end{aligned}$$

and

$$\begin{aligned} z(u,v)_t &+ \lambda_1 z(u,v)_x + (z_u g_1 + z_v g_2) \\ &= \varepsilon z(u,v)_{xx} - \varepsilon\left(z_{uu} u_x^2 + 2 z_{uv} u_x v_x + z_{vv} v_x^2\right) \\ &= \varepsilon z(u,v)_{xx} + \frac{\varepsilon}{D} w(u,v)_x z(u,v)_x. \end{aligned}$$

Thus, in view of the inequalities (3.3), we get

$$w_t + \left(\lambda_2 + \frac{\varepsilon}{D} z_x\right) w_x + c_1 w + c_2 \leq \varepsilon w_{xx}; \tag{3.6}$$

$$z_t + \left(\lambda_1 - \frac{\varepsilon}{D} w_x\right) z_x + c_3 z + c_4 \geq \varepsilon z_{xx}. \tag{3.7}$$

If we apply Corollary 1.1 to (3.6), (3.7), then we can obtain that for any $T > 0$, $w(u^\varepsilon, v^\varepsilon) \leq N(T)$, $z(u^\varepsilon, v^\varepsilon) \geq -N(T)$ on $R \times [0,T]$, where N(T) is independent of ε. Thus we have the estimates $|u^\varepsilon(x,t)| \leq M(T)$, $0 < c(\varepsilon,t) \leq v^\varepsilon(x,t) \leq M(T)$ in light of Lemma 1.2. Hence there exists a subsequence (still labeled) $\left(u^\varepsilon(x,t), v^\varepsilon(x,t)\right)$ such that

$$w^\star - \lim\left(u^\varepsilon(x,t), v^\varepsilon(x,t)\right) = \left(u(x,t), v(x,t)\right).$$

Now we construct four families of entropy-entropy flux pair of Lax type of system (2.1). Let $\rho = D^3$, $\theta = \frac{3}{2}u$. Then for smooth solutions, the LeRoux system is equivalent to the following system:

$$\begin{cases} \rho_t + (\rho\theta)_x = 0, \\ \theta_t + \left(\frac{\theta^2}{2} + \frac{3}{8}\rho^{\frac{2}{3}}\right)_x = 0. \end{cases} \tag{3.8}$$

Any entropy-entropy flux pair (η, q) as functions of variables (ρ, θ) satisfies

$$(q_\rho, q_\theta) = \left(\theta\eta_\rho + \frac{1}{4}\rho^{-\frac{1}{3}}\eta_\theta, \rho\eta_\rho + \eta_\theta\right). \tag{3.9}$$

Eliminating the q from (3.9), we have the entropy equation

$$\eta_{\rho\rho} = \frac{1}{4}\rho^{-\frac{4}{3}}\eta_{\theta\theta}. \tag{3.10}$$

If k denotes a constant, then the function $\eta = h(\rho)e^{k\theta}$ solves (3.10) provided that

$$h'' = \frac{1}{4}k^2\rho^{-\frac{4}{3}}h(\rho).$$

Let $h(\rho) = \rho^{\frac{1}{3}}\phi(s)$, $s = \frac{3}{2}k\rho^{\frac{1}{3}}$. Then ϕ solves the Fuchsian equation

$$\phi'' - \left(1 + \frac{2}{s^2}\right)\phi = 0. \tag{3.11}$$

Thus, one solution ϕ_1 of (3.11) is

$$\phi_1 = s^2\sum_{n=0}^{\infty} c_{2n}s^{2n} = s^2 g(s), \tag{3.12}$$

where

$$g(s) = \sum_{n=0}^{\infty} c_{2n}s^{2n}, \qquad c_{2n} = \frac{c_{2(n-1)}}{(2+2n)(1+2n)+2},$$

c_0 is an arbitrary positive constant, and

$$\phi_2 = s^2 g(s)\int_s^{\infty}\left(s^4 g^2(s)\right)^{-1}ds \tag{3.13}$$

is another independent solution of (3.11).

It is clear that ϕ_1, ϕ_2 satisfy that $\phi_1(s) > 0$, $\phi_1'(s) > 0$ and $\phi_2(s) > 0$ for $s \geq 0$. The strict positivity of ϕ_2'' gives $\phi_2' < 0$ as $s > 0$ because $\lim_{s\to\infty}\phi_2(s) = 0$, $\lim_{s\to\infty}\phi_2'(s) = 0$.

A simple calculation shows that the two eigenvalues of (3.8) are

$$\lambda_1 = \theta - \frac{\rho^{\frac{1}{3}}}{2}, \qquad \lambda_2 = \theta + \frac{\rho^{\frac{1}{3}}}{2},$$

with the corresponding two Riemann invariants

$$z = \theta - \frac{3}{2}\rho^{\frac{1}{3}}, \qquad w = \theta + \frac{3}{2}\rho^{\frac{1}{3}}.$$

Thus $\theta_w = \theta_z = \frac{1}{2}$, $\rho_z = \rho^{\frac{2}{3}}$, $\rho_w = -\rho^{\frac{2}{3}}$, and

$$\begin{cases} q_w = \frac{1}{2}q_\theta + \rho^{\frac{2}{3}}q_\rho, \quad & q_z = \frac{1}{2}q_\theta - \rho^{\frac{2}{3}}q_\rho; \\ \eta_w = \frac{1}{2}\eta_\theta + \rho^{\frac{2}{3}}\eta_\rho, & \eta_w = \frac{1}{2}\eta_\theta - \rho^{\frac{2}{3}}\eta_\rho. \end{cases} \tag{3.14}$$

Since $q_w = \lambda_2\eta_w$, $q_z = \lambda_1\eta_z$ from the definition of entropy, we have

$$q_\theta = q_w + q_z = \theta\eta_\theta + \rho\eta_\rho. \tag{3.15}$$

Let $\eta_k = \rho^{\frac{1}{3}}\phi(s)e^{k\theta}$, $\eta_{-k} = \rho^{\frac{1}{3}}\phi(s)e^{-k\theta}$, we have from (3.15) that

$$\begin{cases} q_k = \eta_k\left(\theta - \frac{2}{3k} + \frac{\rho^{\frac{1}{3}}\phi'(s)}{2\phi(s)}\right), \\ q_{-k} = \eta_{-k}\left(\theta + \frac{2}{3k} - \frac{\rho^{\frac{1}{3}}\phi'(s)}{2\phi(s)}\right). \end{cases} \tag{3.16}$$

Let $\eta^i_{\pm k} = \rho^{\frac{1}{3}}\phi_i(s)e^{\pm k\theta}$. Then the estimates in Lemma 1.3 give

$$\begin{cases} \eta^1_k = \rho^{\frac{1}{3}}e^{kw}\left(1+O\left(\frac{1}{k}\right)\right), & q^1_k = \eta^1_k\left(\lambda_2 - \frac{2}{3k} + O\left(\frac{1}{k^2}\right)\right); \\ \eta^2_k = \rho^{\frac{1}{3}}e^{kz}\left(1+O\left(\frac{1}{k}\right)\right), & q^1_k = \eta^2_k\left(\lambda_1 - \frac{2}{3k} + O\left(\frac{1}{k^2}\right)\right); \\ \eta^1_{-k} = \rho^{\frac{1}{3}}e^{-kz}\left(1+O\left(\frac{1}{k}\right)\right), & q^1_{-k} = \eta^1_{-k}\left(\lambda_1 + \frac{2}{3k} + O\left(\frac{1}{k^2}\right)\right); \\ \eta^2_{-k} = \rho^{\frac{1}{3}}e^{-kw}\left(1+O\left(\frac{1}{k}\right)\right), & q^2_{-k} = \eta^2_{-k}\left(\lambda_2 + \frac{2}{3k} + O\left(\frac{1}{k^2}\right)\right) \end{cases} \tag{3.17}$$

on any compact subset of $s > 0$, and

$$\begin{cases} q^1_k = \eta^1_k\left(\lambda_2 + O\left(\frac{1}{k}\right)\right), & q^2_k = \eta^2_k\left(\lambda_1 + O\left(\frac{1}{k}\right)\right), \\ q^1_{-k} = \eta^1_{-k}\left(\lambda_1 + O\left(\frac{1}{k}\right)\right), & q^2_{-k} = \eta^2_{-k}\left(\lambda_2 + O\left(\frac{1}{k}\right)\right), \end{cases} \tag{3.18}$$

on $s \geq 0$.

Next we verify the compactness of $\eta_t + q_x$ in H^{-1}_{loc} for the entropy-entropy flux pair constructed above. We only prove for $(\eta, q) = (\eta^2_k, q^2_k)$. A similar treatment gives the proofs for the others.

Obviously, system (3.1) has a convex entropy $\eta^\star = \frac{u^2}{2} + \int_0^v \log v dv$ and the corresponding entropy flux $q^\star = \frac{2u^3}{3} + uv\log v$, so we easily get the boundedness of $\varepsilon\big(u^\varepsilon_x\big)^2$ and $\varepsilon\frac{(v^\varepsilon_x)^2}{v^\varepsilon}$ in L^1_{loc}. For simplicity, we will drop the superscript ε.

Multiplying system (3.4) by (η_u, η_v), we have

$$\eta_t + q_x + (\eta_u f + \eta_v g) = \varepsilon\eta_{xx} - \varepsilon\left(\eta_{uu}u_x^2 + 2\eta_{uv}u_x v_x + \eta_{vv}v_x^2\right).$$

Because

$$\int_s^\infty \frac{ds}{s^4 g^2(s)} = O\left(\frac{1}{s^3}\right), \qquad \text{as } s \to 0, \tag{3.19}$$

for any fixed $k > 0$ we have that

$$\eta = \frac{2}{3k}s^3 g(s)\int_s^\infty \frac{ds}{s^4 g^2(s)}e^{k\theta}$$

and q are both bounded on $R \times [0, T]$ $(\forall T > 0)$. Thus $\eta(u, v)_t + q(u, v)_x$ is bounded in $W^{-1,\infty}$, $\eta_u f + \eta_v g$ is bounded on $R \times [0, T]$ $(\forall T > 0)$ and hence bounded in L^1_{loc}. Because

$$\eta = \frac{2e^{k\theta}}{k}\left(\frac{1}{g(s)} - 2s^3 g(s)\int_s^\infty \frac{g'(s)ds}{s^3 g^3(s)}\right) = I_1 - \frac{4}{k}e^{k\theta} I, \tag{3.20}$$

and $g'(s)/s \le g(s)$, we have

$$\int_s^\infty \frac{g'(s)ds}{s^3 g^3(s)} = O\left(\frac{1}{s}\right), \qquad \text{as } s \to 0. \tag{3.21}$$

Thus η_s and η_θ are both bounded on $R\times[0,T]$ $(\forall T > 0)$ and $\eta_s = O(s),$ as $s \to 0$. Since

$$s_u = \frac{3ku}{2}\big(u^2+4v\big)^{-\frac{1}{2}}, \qquad s_v = 3k\big(u^2+4v\big)^{-\frac{1}{2}},$$

then $\varepsilon\eta_x = O\left(\varepsilon\left(|u_x| + |v_x|\right)\right)$. Hence $\varepsilon\eta_{xx}$ is compact in H^{-1}_{loc} from the boundedness of εu_x^2 and $\varepsilon\frac{v_x^2}{v}$ in L^1_{loc}.

Since $\eta = I_1 - \frac{4}{k}e^{k\theta}I$, where $I_1 = \frac{2}{kg(s)}e^{k\theta}$ is bounded in C^2, we only need to show the boundedness of L in L^1_{loc}, where

$$L = \varepsilon\left(I_{uu}u_x^2 + 2I_{uv}u_x v_x + I_{vv}v_x^2\right).$$

Let $L = L_1 + L_2$, where

$$\begin{cases} L_1 &= \varepsilon I_{ss}\left((s_u)^2 u_x^2 + 2s_u s_v u_x v_x + (s_v)^2 v_x^2\right), \\ L_2 &= \varepsilon I_s\left(s_{uu}u_x^2 + 2s_{uv}u_x v_x + s_{vv}v_x^2\right). \end{cases}$$

Noticing that $g^{'}(s)/s \le g(s)$, $I_s = O(s)$ $(s \to 0)$ and I_{ss} is bounded, we have that L_1 and L_2 are controlled by $\varepsilon\left(O\left(\frac{v_x^2}{|v|}\right) + O\left(u_x^2\right)\right)$ and hence bounded in L^1_{loc}. Thus the term $\varepsilon\left(\eta_{uu}u_x^2 + 2\eta_{uv}u_x v_x + \eta_{vv}v_x^2\right)$ is bounded in L^1_{loc}. Therefore, by a standard argument, $\left(\eta_k^2\right)_t + \left(q_k^2\right)_x$ is compact in H^{-1}_{loc}.

Finally, we can prove the Young measure must be a Dirac mass by using the same method as in the proof of Theorem 2.1. So we end the proof of Theorem 3.1. ☑

Remark 3.1. *There are many functions $f(u,v), g(u,v)$ which satisfy the hypotheses of Theorem 3.1. For example, $f(u,v) = a\sqrt{u^2+4v+1} + b$,*

$$g(u,v) = \frac{v}{\sqrt{v+1}}\left(c\left(\sqrt{u^2+4v} - |u|\right) + d\right) \qquad (a,b,c,d \in R).$$

In fact, $|w_u| = 1 + \frac{u}{D} \le 2$, $w_v = \frac{2}{D}$ and $D - |u| \le min(w, -z)$. Thus

$$w_u f \ge -|a|(D+1)\left(1+\frac{u}{D}\right) - 2|b| \ge -|a|w - 2\big(|a|+|b|\big), \quad w_v g \ge -2|c|w - 2|d|$$

and hence $w_u f + w_v g \ge \big(-|a| - 2|c|\big)w - 2\big(|a|+|b|+|d|\big)$. Similarly, $z_u f + z_v g \le -\big(|a|+2|c|\big)z + 2\big(|a|+|b|+|d|\big)$. In addition, it is easy to see that $f(u,v), g(u,v)$ are locally Lipchitz continuous by the fact that $v\sqrt{u^2+4v}$ is locally Lipchitz continuous.

References

[1] Y. G. Lu, *Hyperbolic conservation laws and the compensated compactess method*, Vol. **128**, Chapman and Hall, New York, 2002.

[2] Y. G. Lu, Convergence of the viscosity method for a nonstrictly hyperbolic system, *Acta Math. Sci.*, **12** (1992), 349–360.

[3] Y. G. Lu, I. Mantilla & L. Rendón, Convergence of approximated solutions to a nonstrictly hyperbolic system, *Advanced Nonlinear Studies*, **1** (2001), 65–79

[4] A. Y. LeRoux, Numerical stability for some equations of gas dynamics, *Mathematics of Computation*, **37** (1981), 435–446.

(Recibido en enero de 2007. Aceptado en abril de 2007)

Department of Mathematics
University of Science and Technology of China
Hefei 230026, P. R. China

e-mail: mtao@mail.ustc.edu.cn
e-mail: czx@mail.ustc.edu.cn
e-mail: yanjin@mail.ustc.edu.cn