

Network tomography

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*Dedicado a la memoria de un gran matemático y una excelente persona,
Chepe Escobar.*

ABSTRACT. While conventional tomography is associated to the Radon transform in Euclidean spaces, electrical impedance tomography, or EIT, is associated to the Radon transform in the hyperbolic plane. We discuss some recent work on network tomography that can be associated to a problem similar to EIT on graphs and indicate how in some sense it may be also associated to the Radon transform on trees.

Keywords. Network monitoring, inverse problems in finite graphs and trees, Radon transform.

2000 Mathematics Subject Classification. Primary: 05C40, 35R30. Secondary: 94C12.

RESUMEN. Mientras la tomografía convencional es asociada a la transformada de Radon en espacios Euclideos, la tomografía de impedancia eléctrica, o EIT, es asociada a la transformada de Radón en el plano hiperbólico. Aquí exponemos nuestro más reciente trabajo sobre tomografía de redes que puede ser asociado a un problema similar a EIT en grafos y también, en cierto sentido, a la transformada de Radon en árboles .

1. Introduction

In a number of beautiful papers [28, 29, 30], Escobar studied very interesting questions about Riemannian metrics whose scalar curvatures or other scalar functions of the curvature tensor are prescribed. This type of questions are similar to the relation between the Pompeiu problem [10, 11], a problem in integral geometry, and the inverse conductivity problem, a classical inverse

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This paper is based on research partially supported by grant NSF-DMS-0400698.

problem on PDE. In the recent past the authors have been considering a discrete version of this type of questions in the context of communication networks and we hope to give the reader a glimpse into this very interesting mathematical subject which is related to a number of significant questions about monitoring the integrity of communication networks. For instance, the early detection and, hopefully, prevention of breakdowns in such networks. In particular, we are interested in trying to identify and thwart a particular kind of malicious attack, the saturation of the network by a very large number of incoming messages.

The mathematical model is as follows. Let G be a finite connected graph with boundary ∂G . The edges of the graph have either endpoints which are nodes in G or one of them in G and the other one in ∂G . (Note that in this model we don't consider "boundary edges" where both nodes lie in ∂G). To simplify, the reader could think that G is a finite connected regular lattice. If $\{a, b\}$ is an edge, the corresponding weight $\omega(a, b)$ is a non-negative number that represents the traffic along the edge $\{a, b\}$, e.g., the number of messages exchanged between a and b at a given moment. (For simplicity, it may be easier to consider the weights as a percentage of the total number of some sufficiently large bound, thus, their values could equally be non-negative real numbers.)

In the continuous case, the corresponding problem is the inverse conductivity problem, which is akin to the work of Escobar in [28, 29]. Namely, let D be the unit disk in the plane, α a non-negative function on D , and consider the input-output map defined by the Neumann-to-Dirichlet boundary value problem:

$$(\diamond) \quad \begin{cases} \Delta U + \alpha U = 0 & \text{in } D \\ \frac{\partial U}{\partial n} = u & \text{on } \partial D \end{cases}$$

given an input function u on ∂D we have a corresponding solution U of the problem (\diamond) , which is unique for $\alpha > 0$.

Returning to the context of networks, given the corresponding graph G , its collections of nodes V and edges E , and a weight ω , we can define a corresponding Laplace operator Δ_ω in terms of the degree $d_\omega x$ associated to every vertex x via the formula

$$d_\omega x = \sum_{y \in V} \omega(x, y)$$

Namely, for a function f ,

$$\Delta_\omega f(x) := \sum_{y \in V} [f(y) - f(x)] \cdot \frac{\omega(x, y)}{d_\omega x}, \quad x \in V$$

For a subgraph S of a graph G we define the boundary ∂S by

$$\partial S = \{z \in V \mid z \notin S \text{ and } z \sim y \text{ for some } y \in S\}$$

where $z \sim y$ means that the two nodes z and y are connected by an edge in E . Also, by \bar{S} we denote a graph whose nodes and edges are in $S \cup \partial S$. The

(outward) normal derivative $\frac{\partial f}{\partial n_\omega}(z)$ at $z \in \partial S$ is defined to be

$$\frac{\partial f}{\partial n_\omega}(z) = \sum_{y \in S} [f(z) - f(y)] \cdot \frac{\omega(z, y)}{d'_\omega z},$$

where $d'_\omega z = \sum_{y \in S} \omega(z, y)$

In this model, there are two kinds of disruptions of traffic data that could arise. In one of them, disruptions occurs when an edge “ceases” to exist, in this case the “topology” of the graph has changed, and we refer to the important work of Fan Chung and her collaborators which offers crucial insights into this question. (See, for instance, [19], [20] and [21]). We would also like to point out here Colin de Verdiere’s book [46] for early work on this subject which is rarely mentioned in the literature. In the other, the weights change because of “increase” of traffic, that is, the network configuration remains the same but the weights have either increased or remained the same. In this second situation, we can appeal to the following theorem, whose proof appears in [12].

Theorem 1.1. [12] *Let ω_1 and ω_2 be weights with $\omega_1 \leq \omega_2$ on $\bar{S} \times \bar{S}$, and $f_1, f_2 : \bar{S} \rightarrow \mathbb{R}$ be functions satisfying for $j = 1, 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, & x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \Phi(z), & z \in \partial S \\ \int_S f_j d\omega_j = K \end{cases}$$

for any given function $\Phi : \partial S \rightarrow \mathbb{R}$ with $\int_{\partial S} \Phi = 0$, and a given constant K with $K > m_0$, where $m_0 = \max_{j=1,2} |m_j| \cdot \text{vol}(S, \omega_j)$, $m_j = \min_{z \in \partial S} f_j(z)$, $j = 1, 2$ and

$\text{vol}(S, \omega_j) = \sum_{x \in S} d_{\omega_j} x$. If we assume that

$$\begin{aligned} (i) & \omega_1(z, y) = \omega_2(z, y) \text{ on } \partial S \times \overset{\circ}{\partial S} \\ (ii) & f_1|_{\partial S} = f_2|_{\partial S}, \end{aligned}$$

then we have

$$f_1 \equiv f_2$$

and

$$\omega_1(x, y) = \omega_2(x, y)$$

for all x and y in \bar{S} .

That is, the theorem allows us to decide whether there is an increase of traffic somewhere in the network or not. While this is only a uniqueness theorem, nevertheless, one can effectively compute the actual weights from the knowledge of the Dirichlet data for convenient choices of the input Neumann data in a way similar to that done in [24] and [26] for lattices. Similarly, the Green function of this Neumann boundary value problem can be represented by an explicit matrix.

Let us now discuss the relationship between the above results to the problem of understanding a large network like the internet.

One way to make more concrete the problem of visualizing the internet appears in [37] and [38]. It indicates that the natural domain might be a hyperbolic space of dimension higher than 2 although, large subsets of the internet could be modelled using trees. One can see that this suggestion leads to a question closely resembling EIT, and it is natural to consider it as a problem in hyperbolic tomography [6], [7]. On the other hand, Theorem 1 represents a significant result on the inversion of the Neumann-Dirichlet problem by studying it directly on “weighted graphs”. The corresponding case of the Radon transform in the hyperbolic plane has been studied in [6], [7], and [33]. Also, experimental evidence indicates that at least locally, the network could be modelled as being part of a tree and therefore it can be visualized using 2-dimensional hyperbolic geometry. As a consequence, a different way to study locally this kind of networks would be by using the Radon transform on trees. As it turns out, an inversion formula for the Radon transform on trees is already known and it can be found in [8].

For the sake of completeness, we will describe here a simplified version of the Radon transform on trees and its inversion formula. As explained below, this seems to be enough to deal with the network problems one is often interested in.

2. The Radon transform on homogeneous trees

Let us now remind the reader what do we mean by a tree T . A tree T is a finite or countable collection V of vertices $\{v_j, j = 0, 1, \dots\}$ and a collection E of edges $e_{jk} = (v_j, v_k)$, in other words, pairs of vertices. We orient the edge e_{jk} by thinking that v_j is the first node and v_k the second node. We always include the edges e_{kj} in this collection, which have the reverse orientation. Given two vertices u and v , we say they are neighbors if (u, v) is an edge and write $u \smile v$ in this case. A geodesic γ from u_0 to u_l is a collection $u_0, u_1, \dots, u_{l-1}, u_l$ of pairwise distinct vertices such that $u_0 \smile u_1, u_1 \smile u_2, \dots, u_{l-1} \smile u_l$. If it turns out that $u_0 \smile u_l$ then we consider the closed geodesic path $\tilde{\gamma}$ by adding the edge (u_l, u_0) to γ . Unless explicitly mentioned, our geodesics will not be closed. To simplify the notation, for any geodesic $\gamma = u_0 \smile u_1 \smile u_1 \smile u_2 \smile \dots \smile u_{l-1} \smile u_l$ open or closed, we denote by $-\gamma$ the geodesic with the opposite orientation, i.e., $-\gamma = u_l \smile u_{l-1} \smile \dots \smile u_0$. The collection of all (open) geodesics is denoted by Γ . If T is infinite, then a complex valued function is defined to be in $L^1(T)$ if

$$\sum_{v \in V} |f(v)| < \infty$$

The Radon transform R of a function $f \in L^1(T)$ is simply the bounded function Rf on Γ defined by

$$Rf(\gamma) = \sum_{v \in \gamma} f(v)$$

Given a node v we denote by $\nu(v)$ the number of edges that contain v as an endpoint. This number is usually called the degree of the node. We will assume throughout that we always have $\nu(v) \geq 3$ to ensure that the Radon transform is injective. (In our applications this is only needed for nodes v that lie in $\text{supp}(f)$). In the terminology of [8] we are assuming there are neither black holes nor flat points in T . Under these conditions, the Radon transform in a tree is invertible. In fact, the explicit inversion formula resembles that of the inversion for the Radon transform in the Euclidean plane [9], [13], [14], and [33]. Unfortunately, even in this case, we need to introduce a significant amount of auxiliary notations. For the purpose of illustration we describe the inversion formula here only for the case of homogeneous trees and we refer to [8] for the general case.

3. Inversion of the Radon transform on homogeneous trees

Consider a homogeneous tree T in which each vertex touches $q + 1$ edges with $q \geq 2$. If n is a nonnegative integer, let $v(n)$ be the number of vertices of T at distance n from a fixed vertex of T . It follows that

$$v(n) = \begin{cases} 1 & \text{if } n = 0 \\ (q + 1)q^{n-1} & \text{if } n \geq 1 \end{cases}$$

We give the following definitions. Let v, w be two vertices in T that are connected by a path $(v = v_0, \dots, v_m = w)$, then the *distance* between v and w is the nonnegative integer $|v, w| = m$. Also, for $f \in L^1(T)$, let μ_n be the average operator defined by

$$\mu_n f(v) = \frac{1}{v(n)} \sum_{|v,w|=n} f(w), \text{ for } v \in T$$

It can be seen that μ_n is basically a convolution with the radial kernel

$$h_n(v, w) = \begin{cases} \frac{1}{v(n)} & \text{if } |v, w| = n \\ 0 & \text{if } |v, w| \neq n \end{cases}$$

Let $\beta = q/(2(q + 1))$ and R^* be the dual Radon transform defined for $\Phi \in L^\infty(\Gamma)$ by

$$R^* \Phi(v) = \int_{\Gamma_v} \Phi(\gamma) d\rho_v(\gamma)$$

for each vertex $v \in T$, with respect to a suitable family $\{\rho_v : v \in T\}$ of measures on Γ_v , where Γ_v is the set of all of the geodesics containing the vertex v .

In order to obtain the inversion of R we observe that R^*R acts as a convolution operator given by the radial kernel $\beta h_0 + \sum_{n=1}^{\infty} 2\beta h_n$.

Proposition 3.1. *The identity*

$$R^*R = \beta\mu_0 + \sum_{n=1}^{\infty} 2\beta\mu_n \quad \text{on } L^1(T),$$

holds in $L^1(T)$, where the series is absolutely convergent in the convolution operator norm on $L^2(T)$, thus providing a bounded extension of R^*R to $L^2(T)$.

Theorem 3.2. *The unique bounded extension to $L^2(T)$ of the operator R^*R is invertible on $L^2(T)$, and its inverse is the operator*

$$E = \frac{2(q+1)^3}{q(q-1)^2} \left[\mu_0 + \sum_{n=1}^{\infty} (-1)^n 2\mu_n \right]$$

which acts as the convolution operator with the radial kernel

$$\frac{2(q+1)^3}{q(q-1)^2} [h_0 + \sum_{n=1}^{\infty} (-1)^n 2h_n].$$

As above, this series converges absolutely in the convolution operator norm on $L^2(T)$. In particular, E is bounded.

Corollary 3.1. *The Radon transform $R:L^1(T) \rightarrow L^\infty(\Gamma)$ is inverted by*

$$ER^*Rf = f.$$

References

- [1] J. BARAS, C. A. BERENSTEIN & F. GAVILÁNEZ, Local monitoring of the internet network. Available at http://techreports.isr.umd.edu/TechReports/ISR/2003/TR_2003-7/TR_2003-7.phtml.
- [2] J. BARAS, C. A. BERENSTEIN & F. GAVILÁNEZ, Discrete and continuous inverse conductivity problems, *AMS Contemporary Math.*, **362** (2004), 33–51.
- [3] D. C. BARBER AND B. H. BROWN, Inverse problems in partial differential equations (Arcata, CA, 1989), 151–164, SIAM, Philadelphia, PA, 1990.
- [4] A. BENSOUSSAN AND J. L. MENALDI, Difference Equations on weighted graphs, *J. Convex Anal.* **12** 2005, 13–44.
- [5] C. A. BERENSTEIN, F. GAVILÁNEZ & J. BARAS, Network Tomography, *AMS Contemporary Math.*, **405** 2006, 11–17.
- [6] C. A. BERENSTEIN & E. CASADIO TARABUSI, The inverse conductivity problem and the hyperbolic Radon transform, “75 years of Radon Transform”, S. Gindikin and P. Michor, editors. International Press, 1994.
- [7] C. A. BERENSTEIN AND E. CASADIO TARABUSI, Integral geometry in hyperbolic spaces and electrical impedance tomography, *SIAM J. Appl. Math.* **56** (1996), 755–764.
- [8] C. A. BERENSTEIN ET AL, Integral Geometry on Trees, *American Journal of Mathematics* **113** (1991), 441–470.
- [9] C. A. BERENSTEIN, Radon transforms, wavelets, an applications, Lecture notes in Mathematics **1684** (1998), Springer-Verlag, New York, 1–33.
- [10] C. A. BERENSTEIN & P. YANG, *Advances in Math.* **44** (1982), 1–17
- [11] C. A. BERENSTEIN & P. YANG, *J. reine angew. Math.* **382** (1987), 1–21
- [12] C. A. BERENSTEIN & S-Y. CHUNG, ω -Harmonic functions and inverse conductivity problems on networks, *SIAM J. Appl. Math.*, **65** (2005), 1200–1226.

- [13] C. A. BERENSTEIN, Local tomography and related problems, *AMS Contemporary Mathematics*, **278** (2001), 3–14.
- [14] C. A. BERENSTEIN, & D. WALNUT, Local inversion of the Radon transform in even dimensions using wavelets, “75 years of Radon transform”, S. Gindikin and P. Michor editors, 1994, International Press, Somerville MA, 45–69.
- [15] C. A. BERENSTEIN, & D. WALNUT, Wavelets and local tomography, “Wavelets in Medicine and Biology”, A. Aldroubi and M. Unser, editors, CRC Press, 1996, 231–261.
- [16] R. M. BROWN, & R. TORRES, Uniqueness in the inverse conductivity problem for conductivities with $\frac{3}{2}$ -derivatives in L^p , $p > 2n$, *JFAA* **9** (2003), 563–574.
- [17] R. W. BROWN, E. HAACKE, M. THOMPSON & R. VENKATESAN, *Magnetic Resonance Imaging: Physical Principles and Sequence Design*. Wiley & Sons, New York, 1999.
- [18] A. P. CALDERON, On an inverse boundary value problem, Seminar on numerical analysis and its applications to continuum physics, Soc. Brz. Math. 1980, 65–73.
- [19] F. CHUNG, & R. ELLIS, A chip-firing game and Dirichlet eigenvalues, *Discrete Math.* **257** (2002), 341–355.
- [20] F. CHUNG, *Spectral Graph Theory*, CBMS Series 92, AMS, 1997.
- [21] F. CHUNG, & K. ODEN, Weighted Graph Laplacians and Isoperimetric Inequalities, *Pacific J. math* **192** (2000), 257–273.
- [22] M. COATES, A. HERO III, R. NOWAK, & B. YU, Internet tomography, *IEEE Signal processing magazine*, New York, 2002.
- [23] E. B. CURTIS, T. INGERMAN, & J. A. MORROW, Circular planar graphs and resistors networks, *Linear algebra and its applications* **283** (1998), 115–150.
- [24] E. B. CURTIS, & J. A. MORROW, Determining the resistors in a network, *SIAM J. Appl. Math.* **50** (1990), 918–930.
- [25] E. B. CURTIS, & J. A. MORROW, Inverse problems for electrical networks, *Series on Applied Mathematics*, 13, World Scientific, Hackensack NJ, 2000.
- [26] E. B. CURTIS, & J. A. MORROW, The Dirichlet to Neumann map for a resistor network, *SIAM J. Appl. Math.* **51** (1991), 1011–1029.
- [27] T. DAUBECHIES, *Ten lectures in wavelets*. SIAM/CBMS, Philadelphia PA, 1992.
- [28] J. F. ESCOBAR & R. SCHOEN, Conformal metrics with prescribed scalar curvature, *Invent Math.*, **86** (1986), 243–254.
- [29] J. F. ESCOBAR, Uniqueness theorems on conformal deformations of metrics, Sobolev inequalities and an eigenvalue, *Comm. Pure Appl. Math.*, **43** (1990), 857–883.
- [30] J. F. ESCOBAR, Conformal deformations of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, *Ann. of Math.*, **136** (1992), 1–50.
- [31] T. EVANS & D. WALNUT, Wavelets in hyperbolic geometry. MS dissertation, Department of Mathematics, George Mason University, Fairfax: 1999, 1–51.
- [32] F. GAVILÁNEZ, Multiresolution Analysis on the Hyperbolic plane, Master’s thesis, University of Maryland, College Park, 2002.
- [33] S. HELGASON, *The Radon transform*, Birkhauser, Boston, 1999.
- [34] A. I. KATSEVICH & A. G. RAMM, *The Radon transform and local tomography*, CRC Press, Boca Raton FL, 1996.
- [35] S. LISSIANOI . & I. PONOMAREV, On the inversion of the geodesic Radon transform on the hyperbolic plane, *Journal of Inverse problems* **13** (1997), 1053–1062.
- [36] S. MALLAT, *A wavelet tour of signal processing*. Academic Press, San Diego CA, 2001.
- [37] T. MUNZNER, Exploring large graphs in 3D hyperbolic space, *IEEE Computer Graphics and Applications* **18** 4 (1998), 18–23.
- [38] T. MUNZNER & P. BURCHARD, Visualizing the structure of the world wide web in 3D hyperbolic space, *Proceedings of VRML ’95*, (San Diego, California, December 1995), special issue of *Computer Graphics*, ACM SIGGRAPH, New York, 1995, 33–38.
- [39] A.I. NACHMAN, Global uniqueness for a two-dimensional inverse boundary problem, *Annals Math.* **143** (1996), 71–96.

- [40] F. NATTERER, & F. WUBBELING, *Mathematical methods in image reconstruction*. SIAM, Philadelphia PA, 2001.
- [41] F. NATTERER, *The mathematics of computerized tomography*. SIAM, Philadelphia PA, 2001.
- [42] F. SANTOSA, & M. VOGELIUS, A backprojection algorithm for electrical impedance imaging. *SIAM J. Appl. Math.* **50** (1990), 216–243.
- [43] J. SYLVESTER, & G. UHLMANN, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.*, **125** 2 (1987), 153-169.
- [44] G. UHLMAN, *Developments in inverse problems since Calderon’s foundational paper*, in *Harmonic analysis and partial differential equations*, U. Chicago Press, 1999, 295-345.
- [45] Y-C. DE VERDIERE, I. GITLER, & D. VERTIGAN, *Reseaux électriques planaires II*, *Comment. Math. Helvetici*, **71** (1996), 144–167.
- [46] Y-C. DE VERDIERE, *Spectrs de graphes*, Soc. Math. France, 1998.

(Recibido en abril de 2006. Aceptado en septiembre de 2006)

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