

Telegraph models of financial markets

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ABSTRACT. In this paper we develop a financial market model based on continuous time random motions with alternating constant velocities and with jumps occurring when the velocity switches. If jump directions are in the certain correspondence with the velocity directions of the underlying random motion with respect to the interest rate, the model is free of arbitrage and complete. Memory effects of this model are discussed.

Keywords. Jump telegraph process, european option pricing, perfect hedging, self-financing strategy, fundamental equation, historical volatility.

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RESUMEN. En este artículo introducimos un modelo de mercado financiero basado en movimientos aleatorios con la alternancia de velocidades y con saltos que ocurren cuando la velocidad se cambia. Este modelo es libre del arbitraje si las direcciones de saltos están en cierta correspondencia con las direcciones de velocidades del movimiento subyacente. Suponemos que la tasa de interés depende del estado de mercado. Las estrategias reproducibles para opciones son construidas en detalles. Se obtienen las fórmulas de forma cerrada para los precios de opción.

1. Introduction

Option pricing models based on the geometric Brownian motion have well known limitations. These models have infinite propagation velocities, independent log-returns increments on separated time intervals and others. Moreover it is widely accepted that financial time series are not Gaussian.

Various authors proposed to apply random motions with finite velocities for option pricing models (see e.g. [3], [4], [6]). These models exploit in various aspects Markov processes with continuous time and few states.

We discuss here the model, which was proposed in [6]. It is based on (inhomogeneous) telegraph process [5], which is a continuous time random motion with constant velocities alternating at independent and exponentially distributed time intervals. We assume the log-price of risky asset follows this process with jumps at the times of trend changes. As a basis for building the model, we take a Markov process $\sigma(t)$ with values ± 1 and transition probability intensities λ_{\pm} . Using these, we define processes $c_{\sigma(t)} = c_{\pm}$, $h_{\sigma(t)} = h_{\pm} > -1$ and $r_{\sigma(t)} = r_{\pm}$, $r_{\pm} > 0$. Let us introduce $X^{\sigma}(t) = \int_0^t c_{\sigma(s)} ds$ and a pure jump process $J^{\sigma} = J^{\sigma}(t)$ with alternating jumps of sizes h_{\pm} . The evolution of the risky asset $S^{\sigma}(t)$ is determined by a stochastic exponent of the sum $X^{\sigma} + J^{\sigma}$. The risk-free asset is given by the usual exponent of the process $Y^{\sigma} = Y^{\sigma}(t) = \int_0^t r_{\sigma(s)} ds$. Here and below the superscript σ indicates the starting value $\sigma = \sigma(0)$ of $\sigma(t)$. If $(r_{\pm} - c_{\pm})/h_{\pm} > 0$, then this model is complete and arbitrage-free.

Our model uses parameters c_{\pm} to capture bullish and bearish trends in a market evolution, and values h_{\pm} to describe sizes of possible crashes and jumps. Thus, we study a model that is both realistic and general enough to enable us to incorporate different trends and extreme events. This approach looks rather natural. It explains processes on overcashd and undercashd markets. Moreover the underlying process converges to Brownian motion under suitable rescaling [6].

The proposed model have some curious properties. For example, historical volatility is nonconstant, that corresponds to memory effects of such kind of models. We discuss here the historical volatility problem.

This paper is organized as follows. In the section 2 we describe the data of the model. This section provides no-arbitrage criterium and fundamental equation. Section 3 defines and discusses the notion of historical volatility in the framework of this model.

2. Dynamics of the risky asset and the martingale measure

We assume the price of risky asset follows the equation

$$dS^{\sigma}(t) = S^{\sigma}(t-)d(X^{\sigma}(t) + J^{\sigma}(t)), \quad t > 0. \quad (2.1)$$

Here the process $S^{\sigma}(t)$, $t \geq 0$ is right-continuous, $\sigma = \pm 1$ indicates the initial state of the market. This equation can be solved in terms of stochastic exponential \mathcal{E}_t (see [2]).

Integrating (2.1) we obtain

$$S^{\sigma}(t) = S_0 \mathcal{E}_t(X^{\sigma} + J^{\sigma}) = S_0 e^{X^{\sigma}(t)} \kappa^{\sigma}(t),$$

where $S_0 = S^{\sigma}(0)$ and

$$\kappa^{\sigma}(t) = \prod_{s \leq t} (1 + \Delta J^{\sigma}(s)).$$

The bond price changes by the form

$$B(t) = e^{Y^\sigma(t)}, \quad Y^\sigma(t) = \int_0^t r_{\sigma(s)} ds, \quad r_-, r_+ > 0. \quad (2.2)$$

It means that the current interest rate depends on a market state.

We assume the following restrictions to the parameters of the model

$$\frac{r_- - c_-}{h_-} > 0, \quad \frac{r_+ - c_+}{h_+} > 0. \quad (2.3)$$

It could be demonstrated directly (see [6]) that $X^\sigma + J^\sigma$ is a martingale if and only if $h_- = -c_-/\lambda_-$, $h_+ = -c_+/\lambda_+$. Since the process σ is the unique source of randomness, the market model (2.1)-(2.2) can not have more than one martingale measure.

Theorem 2.1. ([6]) *Let $Z(t) = \mathcal{E}_t(X^* + J^*)$, $t \geq 0$ with $h_\sigma^* = -c_\sigma^*/\lambda_\sigma$ be the density of probability \mathbf{P}^* relative to \mathbf{P} .*

The process $(B(t)^{-1}S(t))_{t \geq 0}$ is the \mathbf{P}^ -martingale if and only if*

$$c_\sigma^* = \lambda_\sigma + \frac{c_\sigma - r_\sigma}{h_\sigma}, \quad \sigma = \pm 1.$$

Under the probability \mathbf{P}^ the Poisson process N is driven by the parameters*

$$\lambda_\sigma^* = \frac{r_\sigma - c_\sigma}{h_\sigma} > 0, \quad \sigma = \pm 1.$$

Consider the function

$$F(t, x, \sigma) = \mathbf{E}^* \left[e^{-Y(T-t)} f \left(x e^{X(T-t)} \kappa(T-t) \right) \mid \sigma(0) = \sigma \right],$$

$$\sigma = \pm 1, \quad 0 \leq t \leq T,$$

where \mathbf{E}^* denotes the expectation with respect to martingale measure \mathbf{P}^* , which is defined in Theorem 2.1 by the density $Z(t) = \mathcal{E}_t(X^* + J^*)$. Function $F_t = F(t, S(t), \sigma(t)) = \varphi_t S(t) + \psi_t B(t)$ is the strategy value at time t of the option with the claim $f(S_T)$ at the maturity time T .

Function F solves the following difference-differential equation (see [6]), which have a sense of fundamental Black-Scholes equation in the classic model based on geometric Brownian motion:

$$\begin{aligned} & \frac{\partial F}{\partial t}(t, x, \sigma) + c_\sigma x \frac{\partial F}{\partial x}(t, x, \sigma) \\ & = \left(r_\sigma + \frac{r_\sigma - c_\sigma}{h_\sigma} \right) F(t, x, \sigma) - \frac{r_\sigma - c_\sigma}{h_\sigma} F(t, x(1 + h_\sigma), -\sigma), \\ & \sigma = \pm 1, \end{aligned} \quad (2.4)$$

with the terminal condition $F_{t \uparrow T} = f(x)$.

3. Historical volatility

Seeking for simplicity, we consider the particular case of $\lambda_- = \lambda_+ = \lambda$.

Theorem 3.1. *Let $f = f(x)$ and $\alpha_{\pm} = \alpha_{\pm}(t)$, $t \geq 0$ be smooth functions. Then $u_{\sigma} = \mathbb{E}_{\sigma} f(x - \alpha_{\sigma}(t) + X(t) + J(t))$, $\sigma = \pm 1$ form a solution of the following system*

$$\frac{\partial u_{\sigma}}{\partial t} - \left[c_{\sigma} - \frac{d\alpha_{\sigma}}{dt} \right] \frac{\partial u_{\sigma}}{\partial x} = -\lambda u_{\sigma}(x, t) + \lambda u_{-\sigma}(x + \beta_{\sigma}(t), t) \quad (3.1)$$

with $\beta_{\sigma}(t) = h_{\sigma} - (\alpha_{\sigma}(t) - \alpha_{-\sigma}(t))$, $\sigma = \pm 1$.

The proof follows from the following lemma.

Lemma 3.1. *Let $p_{\sigma} = p_{\sigma}(x, t)$ be generalized probability density of $X(t) + J(t)$. Then*

$$\frac{\partial p_{\sigma}}{\partial t} + c_{\sigma} \frac{\partial p_{\sigma}}{\partial x} = -\lambda p_{\sigma}(x, t) + \lambda p_{-\sigma}(x - h_{\sigma}, t) \quad (3.2)$$

with initial condition $p_{\sigma}|_{t=0} = \delta(x)$.

Let $m_{\sigma}(t) = \mathbb{E}(X^{\sigma}(t) + J^{\sigma}(t))$ and $s_{\sigma}(t) = \text{Var}(X^{\sigma}(t) + J^{\sigma}(t))$. Applying Theorem 3.1 with functions $f(x) = x$, $\alpha_{\pm} = 0$ and $f(x) = x^2$, $\alpha_{\pm} = m_{\pm}(t)$ one can obtain that

$$\frac{dm_{\sigma}}{dt} = -\lambda(m_{\sigma} - m_{-\sigma}) + c_{\sigma} + \lambda h_{\sigma}, \quad (3.3)$$

$$\frac{ds_{\sigma}}{dt} = -\lambda(s_{\sigma} - s_{-\sigma}) + \lambda(h_{\sigma} + m_{-\sigma} - m_{\sigma})^2, \quad \sigma = \pm 1 \quad (3.4)$$

with zero initial conditions. Hence

$$\mathbf{m}(t) = \int_0^t e^{(t-\tau)\Lambda} (\mathbf{c} + \lambda \mathbf{h}) d\tau, \quad \mathbf{s}(t) = \lambda \int_0^t e^{(t-\tau)\Lambda} (\mathbf{h} - \Delta \mathbf{m}(\tau))^2 d\tau. \quad (3.5)$$

Here

$$\Lambda = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} m_+ \\ m_- \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} s_+ \\ s_- \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix},$$

$$\mathbf{h} = \begin{pmatrix} h_+ \\ h_- \end{pmatrix}, \quad \Delta \mathbf{m} = \begin{pmatrix} m_+ - m_- \\ m_- - m_+ \end{pmatrix}.$$

Integrating in (3.5) we have

$$m_{\pm}(t) = [A + \lambda B \pm (a + \lambda b)\Phi_{\lambda}(t)]t, \quad (3.6)$$

where $A = (c_+ + c_-)/2$, $a = (c_+ - c_-)/2$, $B = (h_+ + h_-)/2$, $b = (h_+ - h_-)/2$ and $\Phi_{\lambda}(t) = (1 - e^{-2\lambda t})/(2\lambda t)$. Functions m_{\pm}/t converge to $A + \lambda B$ as $t \rightarrow \infty$. Further $M/t = (m_+ + m_-)/(2t) \equiv A + \lambda B$ and $m_{\pm}/t|_{t=0} = c_{\pm} + \lambda h_{\pm}$. Moreover

$$s_{\pm}(t) = [a^2/\lambda + \lambda B^2 + \psi(t) \pm \varphi(t)]t, \quad (3.7)$$

where

$$\psi(t) = (a + \lambda b) [(a + \lambda b)\Phi_{2\lambda}(t) - 2a\Phi_{\lambda}(t)] / \lambda$$

and

$$\varphi(t) = 2B [(a + \lambda b)e^{-2\lambda t} - a\Phi_\lambda(t)].$$

Notice, that

$$\lim_{t \rightarrow +0} \varphi(t) = 2\lambda bB, \quad \lim_{t \rightarrow +0} \psi(t) = (\lambda^2 b^2 - a^2)/\lambda$$

and

$$\lim_{t \rightarrow \infty} \varphi(t) = \lim_{t \rightarrow \infty} \psi(t) = 0.$$

We now define historical volatility

$$HV_\sigma(t-s) = \sqrt{\frac{\text{Var}_\sigma \left\{ \log \left(\frac{S(t)}{S(s)} \right) \right\}}{t-s}}, \quad t > s \geq 0. \quad (3.8)$$

We have $HV_\sigma(t) = f_\sigma(t)$, where the function f_σ is given by $f_\sigma(t) = \sqrt{s_\sigma(t)}/t$. Here $s_\pm(t)$ is defined by (3.7) with $\ln(1+h_\pm)$ instead of h_\pm . Hence

$$\begin{aligned} f_\pm(t) &= \sqrt{a^2/\lambda + \lambda B^2 + \psi(t) \pm \varphi(t)} \\ &= \sqrt{\sigma^2 + \kappa^2 \Phi_{2\lambda}(t)/\lambda + \gamma_\pm \Phi_\lambda(t) \pm 2B\kappa e^{-2\lambda t}} \end{aligned}$$

with $\sigma^2 = a^2/\lambda + \lambda B^2$, $\kappa = a + \lambda b$, $\gamma_\pm = -2a(\kappa \pm \lambda B)/\lambda$ and

$$\lim_{t \rightarrow +0} f_\pm(t) = \sqrt{\lambda} |h_\pm| = \sqrt{\lambda} |\ln(1+h_\pm)|, \quad (3.9)$$

$$\lim_{t \rightarrow +\infty} f_\pm(t) = \sigma = \sqrt{a^2/\lambda + \lambda ([\ln(1+h_+)(1+h_-)]/2)^2}. \quad (3.10)$$

In particular case $\kappa = 0$

$$f_\pm(t) = \sqrt{\sigma^2 + \gamma_\pm \Phi_\lambda(t)}$$

with $\gamma_\pm = \mp 2aB$.

Remark 3.1. *It is interesting that model (2.2) has some features of models with memory. The simplest form of a model with memory uses an ARCH-type equation for a log-price process:*

$$\log S(t)/S(0) = at + \sigma w(t) - \sigma \int_0^t ds \int_{-\infty}^s p e^{-(p+q)(s-u)} dw(u),$$

where σ , q , $p+q > 0$ and $w = w(t)$, $t \geq 0$ is a standard Brownian motion. In this case the historical volatility is given by

$$f(t) = \frac{\sigma}{2\lambda} \sqrt{q^2 + p(2q+p)\Phi_\lambda(t)},$$

with $2\lambda = p+q$ (see examples 4.3 and 4.5 in [1]).

In the framework of our model historical volatility function $f(t)$ has the same structure.

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