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A note on Banach algebras that are not isomorphic to a group algebra

Una nota sobre álgebras de Banach no isomorfas a una álgebra de grupos

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Abstract. It is proved in this paper that several classical Banach algebras are not isomorphic to a group algebra. These algebras includes $C(K)$ algebras where K is a compact Hausdorff space. In the case of amalgams, we give conditions for an amalgam to be a group algebra.

Key words and phrases. Amalgams, Dunford-Pettis property, Radon-Nikodym property.

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Resumen. En este artículo se prueba que algunas álgebras de Banach clásicas no son isomorfas a un álgebra de grupo. Estas álgebras incluyen a las álgebras $C(K)$ donde K es un espacio de Hausdorff Compacto. En el caso de las amalgamas, damos condiciones para que una amalgama sea un álgebra de grupo.

Palabras y frases clave. Amalgamas, propiedad de Dunford-Pettis, propiedad de Radon-Nikodym.

1. Introduction

In [13] the authors posed and answered the question when an amalgam is a convolution algebra, which led to the question when these algebras are a group algebra, indeed, when these algebras are isomorphic to $L^1(G)$, where G is a locally compact group equipped with a Haar measure. In this paper we answer this question and furthermore we prove that many classical infinite dimensional

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Banach algebras are not isomorphic to $L^1(G)$ for any locally compact topological group G. This goal is achieved by using some Banach spaces techniques like weak completeness as well as the Radon Nikodym property and the Dunford Pettis Property ([5], [8], and [9]).

The paper is organized as follows: Section 2 contains some definitions and preliminary facts necessary for developing the remainder of the paper. In Section 3 we give necessary and sufficient conditions for an amalgam to be a group algebra, while Section 4 deals with the case of several classical Banach algebras.

2. Preliminaries

In this section we present some definitions and preliminary facts which will be useful in next sections. Some omitted material regarding Banach algebras are classical and may be found in [1], [5], [6].

A Banach space X is called *weakly sequentially complete* if and only if every weakly Cauchy sequence is convergent. Examples of Banach spaces which are not weakly sequentially complete are both c_0 and $C(K)$ for K an infinite compact Hausdorff space [5], while $L^1(\mu)$ spaces, and the spaces of countably additive scalar measures are examples of weakly complete Banach spaces [4], [8], [10].

A Banach space X has the *Dunford Pettis Property* if for every Banach space Y, every weakly compact operator $T : X \longrightarrow Y$ is completely continuous [7]. It is well known that reflexive complemented subspaces of Banach spaces with the Dunford Pettis Property are finite dimensional. It is also well known that $L^1(\mu)$ has the Dunford Pettis Property. We recall that a Banach space X has the *Radon Nikodym Property* if the Radon Nikodym theorem holds for the Bochner integral [9]. Among Banach spaces without the Radon Nikodym Property are c_0, $C(K)$ and $L^1(\mu)$ with μ a non-atomic countably additive scalar measure. A useful Banach space with the Radon Nikodym Property is $\ell_1(\Gamma)$ for any set Γ. Omitted terms and definitions on Banach algebras are classical and may be found in the cited literature.

3. Amalgams

Let G_1 and G_2 be locally compact topological groups with G_2 containing a compact open subgroup H. Let m be the Haar measure on G_2 normalized such that $m(H) = 1$. Denote by μ the Haar measure of G_1 and suppose additionally that G_1 contains an open set V with compact closure $\overline{V}$ such that

$$\mu\left(\overline{V}\backslash V\right) = 0, \quad G_1 = \bigcup_{n=-\infty}^{\infty} (t_n V), \quad t_n \in G_1, \quad \mu\big(t_n\overline{V} \cap t_m\overline{V}\big) = 0 \ \ \forall n \neq m.$$

Put $U = \overline{V} \times H$ and $U_\alpha = \alpha + U$ with α of the form (t_n, s), where $s \in G_2$ and t_n as above. Then we can write

$$G = G_1 \times G_2 = \bigcup_{\alpha \in J} U_\alpha, \text{ where } J = \bigcup_{n \in \mathbb{Z}} \{t_n\} \times G_2.$$

Using the above representation and following [2], [11], [12], [13], we define the amalgam $(L^p, l^q)(G)$ to be the space of functions f which are locally in $L^p(\mu)$ such that

$$\|f\|_{p,q} = \left[\sum_{\alpha \in J} \left[\int_{V_\alpha} |f(x)|^p \right]^{q/p} \right]^{1/p} < \infty \,.$$

For $p = \infty$ we have

$$\|f\|_{\infty,q} = \left[\sum_{\alpha \in J} \sup_{x \in V_\alpha} |f_\alpha(x)|^q \right]^{1/q} .$$

As in [14] $f_\alpha := f \mathcal{X}_{\overline{V}_\alpha}$ where $\mathcal{X}_{\overline{V}_\alpha}$ denotes the characteristic function of $\overline{V}_\alpha$. It is noticed in [13] that for G compact, $(L^p, l^q)(G) = L^p(G)$, while $(L^p, l^q)(G) = l^q(G)$ for $G-$discrete.

It is the aim of this section to study the case of when an amalgam is a group algebra.

Theorem 1. *Let G be a locally compact group and $p, q \in [1, \infty]$. The following statements hold.*

1) $\left(L^1, l^1\right)(G)$ *is always a group algebra.*
2) $\left(L^p, l^p\right)(G)$, $p > 1$ *is a group algebra if and only if G is finite.*
3) $\left(L^p, l^1\right)(G)$, $p > 1$ *is a group algebra if G is discrete.*
4) $\left(L^p, l^q\right)(G)$, $p \geq 1$, $q > 1$ *is a group algebra if G is finite.*
5) $\left(L^\infty, l^q\right)(G)$, $q > 1$ *is not a group algebra.*

Proof. In [13] was proved that

$$\|f * g\|_{p,1} \leq C \|f\|_{p,1} . \|g\|_{p,1},$$

which proves that $(L^p, l^1)(G)$ is a Banach algebra under convolution.

We want to know for which case $\left(L^p, l^q\right)(G)$ is a group algebra under convolution; indeed, under which conditions there is a locally compact group G' such that $\left(L^p, l^q\right)(G)$ is a Banach algebra isomorphic to $L^1\left(G'\right)$.

In the case $p = q$ we have that $\left(L^p, l^q\right)(G) = L^p(G)$. Which implies that $\left(L^1, l^1\right)(G)$ is always a convolution algebra, so statement 1) is obtained.

Now, for $p > 1$, it is known that $L^p(G)$ is algebra if and only if G is compact. So, if there is a locally compact group G' such that $L^p(G)$ is isomorphic to $L^1(G')$, it implies that $L^1(G')$ is reflexive, because so is $L^p(G)$ (recall $p > 1$). Therefore $L^1(G')$ is finite dimensional, which implies that G' a finite group and consequently we have the following conclusion: for $p > 1$, $(L^p, l^p)(G)$ is a group algebra if and only if G is finite therefore statement 2) is proved.

On the other hand, for $p > 1,\ q = 1,\ \left(L^p, l^1\right)(G)$ is always an algebra and

$$\left(L^p, l^1\right)(G) = \sum_{n\in\mathcal{Z}} \left[\int_{V_n} |f(x)|^p\right]^{1/p},$$

which contains a complemented copy of $L^p(V_1)$. This implies that $L^p(V_1)$ is complemented in $L^1(G')$. Since $L^1(G')$ enjoys the Dunford Pettis Property, being $p > 1,\ L^p(V_1)$ is a reflexive complemented subspace of $L^1(G')$, which implies $L^p(V_1)$ finite dimensional. Therefore V_1 is a finite set. Since V_1 is open, we conclude that G is discrete. In this way we obtain 3).

Now we consider the case $p \geq 1$ and $q > 1$. In this case we have $(L^p, l^q)(G)$ is an algebra if G is compact (see [15]). Since

$$\left(L^p, l^q\right)(G) = \left\{ f : G \longrightarrow \mathbb{C} \text{ measurable} : \sum_{n=1}^{\infty} \left[\int_{V_n} |f(x)|^p\right]^{1/p} < \infty \right\},$$

we see that $\left(L^p, l^q\right)(G)$ contains a complemented copy of $L^p(V_1)$. So applying the same argument as in the case of $\left(L^p, l^1\right)(G),\ p > 1$, we see that $L^p(V_1)$ is finite dimensional and therefore G is discrete. G being discrete and compact, it is a finite group, i.e. the conclusion in 4).

Finally, we will consider $p = \infty$. From [11] we know that $\left(L^\infty, l^q\right)(G) \supset L^\infty(V)$ which is not weakly sequentially complete; since $L^1(G')$ is weakly sequentially complete we conclude that $\left(L^\infty, l^q\right)(G)$ cannot be isomorphic to $L^1(G')$ for which the theorem is done. ☑

Remark 1. The case $p < 1,\ q < 1$, is out of the range of this paper because the amalgams $\left(L^p, l^q\right)(G)$ are not Banach algebras.

4. The case of other Banach algebras

Proposition 2. *c_0 is not isomorphic to a group algebra.*

Proof. It is well known that c_0 is not weakly sequentially complete. Since for every G', $L^1(G')$ is weakly sequentially complete, then there is not any isomorphism between c_0 an $L^1(G)$. ☑

Several consequences from above proposition:

Theorem 3. *None of the algebras $L^\infty(\mu)$, the disk algebra $A(D)$, $C(K)$ with K an infinite Hausdorff compact set, commutative C^*-algebras, H^∞, and $B(H)$, the algebra of bounded linear operator on a Hilbert space H, are group algebras.*
Proof.

a) $L^\infty(\mu)$, the disk algebra $A(D)$ nor $C(K)$, K an infinite compact set are group algebras because all of them contain copy of c_0.
b) Commutative C^*-algebras are not group algebras, because they are isomorphic to spaces of continuous functions.

c) H^{∞} is not a group algebra because according to [15, III.E.4], it contains a copy of l^{∞}, which contains a copy of c_0.

d) $B(H)$ is not a group algebra. In fact, if $T \in B(H)$ and T^* denotes the adjoint operator of T, then the Banach algebra generated by $\{I, T, T^*\}$ is isomorphic to a C^*-algebra.

☑

Examples of C^*-algebras which are important in Modern Harmonic Analysis may be found in [3].

We denote by $R(G)$ the algebra obtained by adding a unit to $L^1(G)$ and by $B(G)$ the convolution algebra of all Borel measures on G [1] . With these notations we have the following result:

Theorem 4. *$R(G)$ and $B(G)$ are group algebras if and only if G is discrete.*

Proof. $R(G)$ is a group algebra if and only if $R(G)$ is isomorphic to $L^1(G')$ for some locally compact group G'. Since $R(G)$ has an unit [1], so does $L^1(G')$. This happens if and only if G is discrete, if and only if G' is discrete. The proof for $B(G)$ is similar, because it has an unit. ☑

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