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Nontrivial solutions for a Robin problem with a nonlinear term asymptotically linear at $-\infty$ and superlinear at $+\infty$

Soluciones no triviales para un problema de Robin con un término no lineal asintótico en $-\infty$ y superlineal en $+\infty$

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Abstract. In this paper we study the existence of solutions for a Robin problem, with a nonlinear term with subcritical growth respect to a variable.

Key words and phrases. Robin problems, weak solutions, Sobolev spaces, functional, Palais-Smale condition, critical points.

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Resumen. En este artículo estudiamos la existencia de soluciones de un problema de Robin, con término no lineal con crecimiento subcrítico respecto a una variable.

Palabras y frases clave. Problemas de Robin, soluciones débiles, espacios de Sobolev, funcionales, condición de Palais-Smale, puntos críticos.

1. Introduction

In this paper, we study the existence of nontrivial solutions of the following problem with the real parameter $\alpha \neq 0$:

$$(\mathbb{P}) \quad \left\{ \begin{array}{rcll} u & \in & H^1(\Omega, -\Delta)\,, & \\ -\Delta u & = & f(x, u(x))\,, & \text{in} \quad \Omega\,, \\ \gamma_1 u + \alpha \gamma_0 u & = & 0\,, & \text{on} \quad \partial\Omega\,. \end{array} \right.$$

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Here Δ is the Laplace operator, Ω is a bounded domain in $\mathbb{R}^n (n \geq 2)$ simply connected and with smooth boundary $\partial\Omega$. The case $\alpha = 0$ was studied by Arcoya and Villegas in [1].

The function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, satisfies the following conditions:

f_0) The function f is continuous.

f_1) $|f(x,s)| \leq c\,(1 + |s|^\sigma)$, $\forall x \in \overline{\Omega}$ and $\forall s \in \mathbb{R}$, where the exponent σ is a constant such that

$$\begin{array}{lll} 1 < \sigma < \frac{n+2}{n-2} & \text{if} & n \geq 3\,, \\ 1 < \sigma < \infty & \text{if} & n = 2\,. \end{array}$$

f_2) There exists $\lambda > 0$ such that

$$\lim_{s \to -\infty} [f(x,s) - \lambda s] = 0\,, \quad \text{uniformly in} \quad x \in \overline{\Omega}\,.$$

f_3) There exist $s_0 > 0$ and $\theta \in \left(0, \frac{1}{2}\right)$ such that

$$0 < F(x,s) \leq \theta s f(x,s)\,, \quad \forall x \in \overline{\Omega}\,, \quad \forall s \geq s_0\,,$$

where $F(x,s) = \int_0^s f(x,t)dt$ is a primitive of f.

The boundary condition $\gamma_1 u + \alpha\gamma_0 u = 0$ involves the trace operators: $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ and $\gamma_1 : H^1(\Omega, -\Delta) \to H^{-1/2}(\partial\Omega)$, where $H^1(\Omega, -\Delta) = \left\{u \in H^1(\Omega) : -\Delta u \in L^2(\Omega)\right\}$ with the norm

$$\|u\|_{H^1(\Omega,-\Delta)} = \left(\|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{1/2}\,,$$

for each $u \in H^1(\Omega, -\Delta)$, $\gamma_1 u \in H^{-1/2}(\partial\Omega)$ and $\gamma_0 u \in H^{1/2}(\partial\Omega)$. Identifying the element $\gamma_0 u$ with the functional $\gamma_0^* u \in H^{-1/2}(\partial\Omega)$ defined by

$$\langle \gamma_0^* u\,, w\rangle = \int_{\partial\Omega} (\gamma_0 u) w ds\,, \quad \forall w \in H^{1/2}(\partial\Omega)\,,$$

the boundary condition makes sense in $H^{-1/2}(\partial\Omega)$. The mathematical difficulties that arise by involving this type of boundary conditions are in the Condition of Palais -Smale.

2. **Preliminary results**

To get the results of existence Theorems 4.2 and 4.1 we will use the following Theorem.

Theorem 2.1 (Theorem of Silva, E. A.). *Let $X = X_1 \oplus X_2$ be a real Banach space, with $\dim(X_1) < +\infty$. If $\Phi \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition and the following conditions:*

I) $\Phi(u) \leq 0\,, \quad \forall u \in X_1\,.$

II) There exists $\rho_0 > 0$ such that $\Phi(u) \geq 0, \forall u \in \partial B_{\rho_0}(0) \cap X_2$.

III) There exist $e \in X_2 - \{0\}$ *and a constant* M *such that*

$$\Phi(v + te) \leq M\,, \quad \forall v \in X_1 \quad \text{and} \quad \forall t > 0\,.$$

Then Φ *has at least a critical point different from zero.*

Proof. See [6, Lemma 1.13, p. 460]. ☑

We use the decomposition of $H^1(\Omega)$ as orthogonal sum of two subspaces established in [3]. We denote the sequence of eigenvalues of the problem

$$\begin{cases} -\Delta u &= \mu u, \quad \text{in} \quad \Omega, \\ \gamma_1 u + \alpha\gamma_0 u &= 0, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

in the case $\alpha < 0$ with $\{\mu_j\}_{j=1}^{\infty}$, where

$$\mu_1 = \inf_{\substack{u\neq 0 \\ u\in H^1(\Omega)}} \frac{\int_\Omega |\nabla u|^2 + \alpha\int_{\partial\Omega}(\gamma_0 u)^2 ds}{\int_\Omega u^2} < 0\,. \tag{2.2}$$

With X_1 we denote the space associated to the first eigenvalue μ_1, and with $X_2 = X_1^{\perp}$ the orthogonal complement of X_1 respect to the inner product defined by

$$(u,v)_k = \int_\Omega \nabla u\cdot\nabla v + \alpha\int_{\partial\Omega}(\gamma_0 u)(\gamma_0 v)ds + k\int_\Omega uv\,, \quad u,v\in H^1(\Omega)\,, \tag{2.3}$$

where k is a positive constant suitably selected in [3]. Then

$$H^1(\Omega) = X_1 \oplus X_2\,, \tag{2.4}$$

and

$$\int_\Omega |\nabla\varphi|^2 + \alpha\int_{\partial\Omega}(\gamma_0\varphi)^2 ds = \mu_1\int_\Omega \varphi^2\,, \quad \forall\varphi\in X_1\,. \tag{2.5}$$

In the case $\alpha > 0$, the constant k in (2.3) is positive and arbitrary. We denote with $\{\beta_j\}_{j=1}^{\infty}$ the eigenvalues of Problem (2.1), in particular, we have

$$\beta_1 = \inf_{\substack{u\neq 0 \\ u\in H^1(\Omega)}} \frac{\int_\Omega |\nabla u|^2 + \alpha\int_{\partial\Omega}(\gamma_0 u)^2 ds}{\int_\Omega u^2} > 0\,. \tag{2.6}$$

With Y_1 we denote the space associated to β_1 and $Y_2 = Y_1^{\perp}$ the orthogonal complement of Y_1 with respect to the inner product defined by the formula (2.3). Then

$$H^1(\Omega) = Y_1 \oplus Y_2\,, \tag{2.7}$$

and

$$\int_\Omega |\nabla\varphi|^2 + \alpha\int_{\partial\Omega}(\gamma_0\varphi)^2 ds = \beta_1\int_\Omega \varphi^2\,, \quad \forall\varphi\in Y_1\,. \tag{2.8}$$

3. Condition of Palais-Smale

Following Arcoya - Villegas [1], Figueiredo [4], and using theorems 3.1, 3.2 and 3.3 of [3] we establish the conditions under which the functional

$$\Phi(u)=\frac{1}{2}\int_\Omega |\nabla u|^2+\frac{\alpha}{2}\int_{\partial\Omega}(\gamma_0 u)^2 ds-\int_\Omega F(x,u)\,, \quad \forall u\in H^1(\Omega)\,,$$

satisfies the Palais-Smale condition. We prove the cases $\alpha>0$ and $\alpha<0$. The condition of Palais-Smale $(P.S)$ affirms: any sequence $\{u_n\}_{n=1}^\infty$ in $H^1(\Omega)$ such that $|\Phi(u_n)|\leq c$ and $\lim_{n\to\infty}\Phi'(u_n)=0$ in $H^{-1}(\Omega)$, contains a convergent subsequence in the norm of $H^1(\Omega)$. In virtue of the density of $C^\infty\left(\overline{\Omega}\right)$ in $H^1(\Omega)$ and by the continuity of the operator $\gamma_0: H^1(\Omega)\to L^2(\partial\Omega)$, we have the following lemma:

Lemma 3.1. *Let us suppose $\Omega\subset\mathbb{R}^n$ bounded with boundary of class C^1. If $u\in H^1(\Omega)$, $u^+(x)=\max\limits_{x\in\overline{\Omega}}\{u(x),0\}$ and $u^-(x)=\max\limits_{x\in\overline{\Omega}}\{-u(x),0\}$ then*

$$\int_{\partial\Omega}\gamma_0\left(u^+\right)\gamma_0\left(u^-\right)ds=0\,. \tag{3.1}$$

Proof. Let $\{u_n\}_{n=1}^\infty$ be a sequence in $C^\infty\left(\overline{\Omega}\right)$ such that $u_n\longrightarrow u$ in $H^1(\Omega)$ then $u_n^+\longrightarrow u^+$ and $u_n^-\longrightarrow u^-$ in $H^1(\Omega)$, see [2]. By the continuity of the operator $\gamma_0: H^1(\Omega)\to L^2(\partial\Omega)$ we have $\gamma_0\left(u_n^+\right)\to\gamma_0\left(u^+\right)$ and $\gamma_0\left(u_n^-\right)\longrightarrow\gamma_0\left(u^-\right)$ in $L^2(\partial\Omega)$ then:

$$\begin{aligned}\int_{\partial\Omega}\gamma_0\left(u^+\right)\gamma_0\left(u^-\right)ds&=\lim_{n\to\infty}\int_{\partial\Omega}\gamma_0\left(u_n^+\right)\gamma_0\left(u_n^-\right)ds\\&=\lim_{n\to\infty}\int_{\partial\Omega}u_n^+u_n^-ds\\&=0\,.\end{aligned}$$

☑

From (3.1) we have:

$$\int_{\partial\Omega}(\gamma_0 u)\left(\gamma_0 u^+\right)ds=\int_{\partial\Omega}\left(\gamma_0 u^+\right)^2 ds\,, \tag{3.2}$$

$$\int_{\partial\Omega}(\gamma_0 u)\left(\gamma_0 u^-\right)ds=-\int_{\partial\Omega}\left(\gamma_0 u^-\right)^2 ds\,. \tag{3.3}$$

Lemma 3.2 (Condition of Palais-Smale)**.** *If $\alpha<0$ we suppose $(f_0),(f_1),(f_2)$ and (f_3). In the case $\alpha>0$, moreover, we also suppose the following conditions*

S_1) *The number λ of condition (f_2) is not an eigenvalue of the operator $-\Delta$ with boundary condition $\gamma_1 u+\alpha\gamma_0 u=0$.*

S_2) *The numbers σ and θ of the conditions (f_1) and (f_3) are such that*

$$\begin{aligned}&\sigma\theta\leq\frac{1}{2}+\frac{1}{n}\quad\text{if } n\geq 3\quad\text{and}\\&\sigma\theta<1\quad\text{if } n=2\,.\end{aligned}$$

Then $\forall u \in H^1(\Omega)$ the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha}{2}\int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_\Omega F(x,u)\,,$$

satisfies the condition of Palais-Smale (P.S.).

Proof. Let $\{u_n\}_{n=1}^\infty$ be a sequence in $H^1(\Omega)$ such that

$$|\Phi(u_n)| = \left|\frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{\alpha}{2}\int_{\partial\Omega} (\gamma_0 u_n)^2 ds - \int_\Omega F(x,u_n)\right| \le C\,, \tag{3.4}$$

and $\forall v \in H^1(\Omega)$

$$|\langle \Phi'(u_n), v\rangle| = \left|\int_\Omega \nabla u_n \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 u_n)(\gamma_0 v) ds - \int_\Omega f(x,u_n)v\right| \le \varepsilon_n \|v\|\,, \tag{3.5}$$

for some constant $C > 0$ and $\varepsilon_n \to 0^+$.

To show that $\{u_n\}_{n=1}^\infty$ has a convergent subsequence it is enough to prove that $\{u_n\}_{n=1}^\infty$ is bounded.

Case $\alpha < 0$**.** We argue by contradiction. Let us consider a subsequence of $\{u_n\}_{n=1}^\infty$, which we denote in the same way, such that

$$\lim_{n\to\infty} \|u_n\| = +\infty\,.$$

Let $z_n = \frac{u_n}{\|u_n\|}$. Then there exists a subsequence of $\{z_n\}$ which we denote in the same way, such that

$$\begin{array}{rclccc} z_n & \rightharpoonup & z_0 & \text{weakly in} & H^1(\Omega)\,, & z_0 \in H^1(\Omega)\,, \\ z_n & \to & z_0 & \text{in} & L^2(\Omega)\,, & \\ \gamma_0 z_n & \to & \gamma_0 z_0 & \text{in} & L^2(\partial\Omega)\,, & \\ z_n(x) & \to & z_0(x) & \text{a.e.} & x \in \Omega\,, & \\ |z_n(x)| & \le & q(x) & \text{a.e.} & x \in \Omega\,, & q \in L^2(\Omega)\,. \end{array} \tag{3.6}$$

Dividing the terms of (3.5) by $\|u_n\|$ and taking the limit $\forall v \in H^1(\Omega)$ we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{\|u_n\|} v = \int_\Omega \nabla z_0 \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 z_0)(\gamma_0 v) ds\,. \tag{3.7}$$

From (3.7) with $v = 1$ in $\overline{\Omega}$, we get

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{\|u_n\|} = \alpha \int_{\partial\Omega} \gamma_0 z_0 ds < +\infty\,. \tag{3.8}$$

We obtain the desired contradiction in three steps.

First step. We shall prove

$$z_0(x) = 0 \quad \text{a.e.} \quad x \in \Omega, \quad \text{and} \quad \gamma_0 z_0(x) = 0 \quad \text{a.e.} \quad x \in \partial\Omega\,. \tag{3.9}$$

First we prove

$$z_0(x) \leq 0 \quad \text{a.e.} \quad x \in \Omega, \quad \text{and} \quad \gamma_0 z_0(x) \leq 0 \quad \text{a.e.} \quad x \in \partial\Omega. \tag{3.10}$$

Let $\Omega^+ = \{x \in \Omega : z_0(x) > 0\}$ and $|\Omega^+|$ be the measure of Lebesgue of Ω^+. Choosing $v = z_0^+$ in (3.7) we obtain

$$\lim_{n\to\infty} \int_{\Omega^+} \frac{f(x,u_n)}{\|u_n\|} z_0 = \int_{\Omega^+} |\nabla z_0|^2 + \alpha \int_{\partial\Omega} \left(\gamma_0 z_0^+\right)^2 ds < \infty\,. \tag{3.11}$$

Using conditions (f_3) and (f_2), for $x \in \Omega^+$ we obtain

$$\frac{f(x,u_n(x))z_0(x)}{\|u_n\|} \geq -(\lambda q(x) + K_1) z_0(x)\,. \tag{3.12}$$

Indeed, condition (f_3) implies the existence of a constant $c > 0$ such that

$$f(x,s) \geq c s^{\frac{1}{\theta}-1}\,, \quad \forall s \geq s_0\,. \tag{3.13}$$

Then we can choose $s^* > s_0$ such that

$$f(x,s) \geq \lambda s\,, \quad \forall s \geq s^*\,. \tag{3.14}$$

On the other hand, by (f_2), for $\varepsilon > 0$ there is $s' < 0$ such that

$$|f(x,s) - \lambda s| \leq \varepsilon\,, \quad \forall s \leq s' \quad \text{and} \quad \forall x \in \overline{\Omega}\,, \tag{3.15}$$

by the continuity of the function f there exists a constant K_1 such that

$$|f(x,s) - \lambda s| \leq K_1\,, \quad \forall s \in (-\infty, s^*] \quad \text{and} \quad \forall x \in \overline{\Omega}\,. \tag{3.16}$$

From (3.14) and (3.16) we get

$$f(x,s) \geq \lambda s - K_1 \quad \forall s \in \mathbb{R}, \quad \forall x \in \overline{\Omega}. \tag{3.17}$$

Now, using (3.17) with $x \in \Omega^+$ we have

$$\begin{aligned}\frac{f(x,u_n(x))z_0(x)}{\|u_n\|} &\geq \frac{(\lambda u_n(x) - K_1)}{\|u_n\|} z_0(x)\\ &\geq (\lambda z_n(x) - K_1) z_0(x)\\ &\geq -(\lambda q(x) + K_1) z_0(x)\,.\end{aligned}$$

From (3.6) we have $\lim_{n\to\infty} u_n(x) = +\infty$ for a.e. $x \in \Omega^+$ and using (3.13) the superlinearity of f in $+\infty$ we have for a. e. $x \in \Omega^+$

$$\lim_{n\to+\infty} \frac{f(x,u_n)z_0(x)}{\|u_n\|} = \lim_{n\to\infty} \frac{f(x,u_n)}{u_n(x)} z_n(x) z_0(x) = +\infty\,.$$

If $|\Omega^+| > 0$, by the Fatou's Lemma, we get

$$\begin{aligned} +\infty &= \int_{\Omega^+} \underline{\lim}_{n\to\infty} \frac{f(x,u_n(x))}{\|u_n\|} z_0(x) \\ &\le \underline{\lim}_{n\to\infty} \int_{\Omega^+} \frac{f(x,u_n)}{\|u_n\|} z_0(x)\,, \end{aligned}$$

then

$$\lim_{n\to+\infty} \int_{\Omega^+} \frac{f(x,u_n(x))}{\|u_n\|} z_0(x) = +\infty\,,$$

in contradiction with (3.11). Hence $|\Omega^+| = 0$ and $z_0(x) \le 0$ a.e. $x \in \Omega$. If $y \in \partial\Omega$, then

$$\gamma_0 z_0(y) = \lim_{r\to 0} \frac{1}{|B(y,r)\cap\Omega|} \int_{B(y,r)\cap\Omega} z_0(x)dx \le 0\,.$$

See [5, p. 143]. Below we prove that

$$\int_\Omega z_0(x)dx = 0 = \int_{\partial\Omega} \gamma_0 z_0(s)ds\,. \tag{3.18}$$

Let $v = \frac{1}{2}u_n$ in (3.5) and subtracting this identity from (3.4), we obtain

$$\left| \int_\Omega \left\{ \frac{f(x,u_n)}{2} u_n - F(x,u_n) \right\} \right| \le \frac{\varepsilon_n}{2} \|u_n\| + C\,. \tag{3.19}$$

Dividing this inequality by $\|u_n\|$ and passing to the limit, we get

$$\lim_{n\to\infty} \int_\Omega \frac{\frac{f(x,u_n)}{2} u_n - F(x,u_n)}{\|u_n\|}\, dx = 0\,. \tag{3.20}$$

On the other hand, given $\varepsilon > 0$, conditions (f_0) and (f_2) imply the existence of a constant $k_\varepsilon > 0$ such that

$$\left| \frac{1}{2} f(x,s)s - F(x,s) \right| \le \varepsilon|s| + k_\varepsilon\,, \quad \forall s \le s^*\,. \tag{3.21}$$

Using (3.21) we have

$$\begin{aligned} \left| \int_{u_n(x)\le s^*} \frac{\frac{f(x,u_n)}{2} u_n - F(x,u_n)}{\|u_n\|} \right| &\le \varepsilon \int_\Omega \frac{|u_n|}{\|u_n\|} + \frac{K_\varepsilon}{\|u_n\|}|\Omega| \\ &\le \varepsilon c + \frac{K_\varepsilon}{\|u_n\|}|\Omega| \end{aligned}$$

and, since ε is arbitrary,

$$\lim_{n\to\infty} \int_{u_n(x)\le s^*} \frac{\frac{f(x,u_n)}{2} u_n - F(x,u_n)}{\|u_n\|} = 0\,. \tag{3.22}$$

The identities (3.20) and (3.22) show that

$$\lim_{n\to\infty}\int_{u_n(x)>s^*}\frac{\frac{f(x,u_n)}{2}u_n-F(x,u_n)}{\|u_n\|}=0\,. \tag{3.23}$$

Using (3.16) and condition (f_3), we obtain

$$\begin{aligned}\int_{u_n(x)>s^*}\frac{\frac{f(x,u_n)}{2}u_n-F(x,u_n)}{\|u_n\|}&\geq\left(\frac{1}{2}-\theta\right)s^*\int_{u_n(x)>s^*}\frac{f(x,u_n(x))}{\|u_n\|}\\&\geq\left(\frac{1}{2}-\theta\right)s^*\left\{\int_\Omega\frac{f(x,u_n)}{\|u_n\|}-\int_{u_n(x)\leq s^*}\frac{f(x,u_n)}{\|u_n\|}\right\}\\&\geq\left(\frac{1}{2}-\theta\right)s^*\left\{\int_\Omega\frac{f(x,u_n)}{\|u_n\|}-\lambda\int_\Omega\chi_nz_n-\frac{K_1}{\|u_n\|}|\Omega|\right\},\end{aligned}$$

where

$$\chi_n(x)=\begin{cases}1 & \text{if} \quad u_n(x)\leq s^*\,,\\ 0 & \text{otherwise.}\end{cases}$$

Using (3.8), (3.20) and getting the limit we have

$$\begin{aligned}0\ &\geq\ \lim_{n\to\infty}\left(\frac{1}{2}-\theta\right)s^*\left\{\int_\Omega\frac{f(x,u_n)}{\|u_n\|}-\lambda\int_\Omega\mathcal{X}_nz_ndx-\frac{K_1|\Omega|}{\|u_n\|}\right\}\\&=\ \left(\frac{1}{2}-\theta\right)s^*\left\{\alpha\int_{\partial\Omega}\gamma_0z_0ds-\lambda\int_\Omega z_0\right\}\\&=\ \left(\frac{1}{2}-\theta\right)s^*\left\{-\alpha\int_{\partial\Omega}|\gamma_0z_0|ds+\lambda\int_\Omega|z_0|\right\}\geq 0\,.\end{aligned}$$

Hence

$$\left(\frac{1}{2}-\theta\right)s^*\left\{-\alpha\int_{\partial\Omega}|\gamma_0z_0|ds+\lambda\int_\Omega|z_0|\right\}=0\,. \tag{3.24}$$

Then

$$\int_{\partial\Omega}|\gamma_0z_0|ds=0=\int_\Omega|z_0|\,.$$

Using (3.10) we have (3.9). Now, the limit (3.7) is

$$\lim_{n\to\infty}\int_\Omega\frac{f(x,u_n)}{\|u_n\|}v=0\,,\quad\forall v\in H^1(\Omega)\,. \tag{3.25}$$

Second step. We shall prove now that

$$\lim_{n\to\infty}\sup\int_\Omega\frac{f(x,u_n(x))}{\|u_n\|}z_n\leq 0\,. \tag{3.26}$$

We denote:

$$I_1 = \int_{u_n(x)<0} \frac{f(x,u_n(x))}{\|u_n\|} z_n \,,$$
$$I_2 = \int_{0\le u_n(x)\le s_0} \frac{f(x,u_n(x))}{\|u_n\|} z_n \,,$$
$$I_3 = \int_{u_n(x)>s_0} \frac{f(x,u_n)}{\|u_n\|} z_n \,.$$

Let us prove that

$$\lim_{n\to\infty} I_1 = 0\,. \tag{3.27}$$

From condition (f_2), we have $\lim_{s\to-\infty} \frac{sf(x,s)-\lambda s^2}{s} = 0$, so, given $\varepsilon > 0$ by the continuity of f there exists a constant $c_\varepsilon > 0$ such that

$$\left|f(x,s)s - \lambda s^2\right| \le c_\varepsilon + \varepsilon|s|\,, \quad \forall s \le 0\,, \tag{3.28}$$

then

$$\begin{aligned}\left|\int_{u_n<0} f(x,u_n)u_n\right| &\le c_\varepsilon \int_{u_n<0} dx + \varepsilon \int_{u_n<0} |u_n| + \lambda \int_{u_n<0} u_n^2 \\ &\le c + (c+\lambda)\int_\Omega u_n^2\,.\end{aligned}$$

Dividing the last inequality by $\|u_n\|^2$ and getting the limit yields (3.27), because $z_n \to z_0$ in $L^2(\Omega)$ and $z_0(x) = 0$ a.e. $x \in \Omega$. Let us see that

$$\lim_{n\to\infty} I_2 = 0\,. \tag{3.29}$$

If $L = \max\left\{|f(x,s)| : (x,s) \in \overline{\Omega} \times [0,s_0]\right\}$ then

$$\begin{aligned}\left|\int_{0\le u_n(x)\le s_0} \frac{f(x,u_n)}{\|u_n\|} z_n\right| &\le \int_{0\le u_n\le s_0} \frac{|f(x,u_n)|}{\|u_n\|^2}|u_n(x)| \\ &\le \frac{Ls_0}{\|u_n\|^2}|\Omega|\,.\end{aligned}$$

Therefore, $\lim_{n\to\infty} I_2 = 0$. To prove that $\lim_{n\to\infty} I_3 = 0$, first we see that

$$\lim_{n\to\infty} \sup I_3 \le 0. \tag{3.30}$$

From (3.19) and (3.21) we have

$$\begin{aligned}&\left|\int_{u_n>s_0}\left\{F(x,u_n)-\frac{1}{2}f(x,u_n)u_n\right\}\right|\\&\leq\int_{u_n\leq s_0}\left|\frac{1}{2}f(x,u_n)u_n-F(x,u_n)\right|+c+\frac{\varepsilon_n}{2}\|u_n\|\\&\leq\int_{u_n\leq s_0}(\varepsilon|u_n|+k_\varepsilon)+c+\frac{\varepsilon_n}{2}\|u_n\|\\&\leq\int_\Omega(\varepsilon|u_n|+k_\varepsilon)+c+\frac{\varepsilon_n}{2}\|u_n\|\\&\leq c\varepsilon\|u_n\|+c+\frac{\varepsilon_n}{2}\|u_n\|\,.\end{aligned}$$

On the other hand, condition (f_3) implies

$$\left(\frac{1}{2}-\theta\right)\int_{u_n>s_0}f(x,u_n)u_n\leq\int_{u_n>s_0}\left\{\frac{1}{2}f(x,u_n)u_n-F(x,u_n)\right\}.$$

So,

$$\left(\frac{1}{2}-\theta\right)\int_{u_n>s_0}f(x,u_n)u_n\leq c+\left(c\varepsilon+\frac{\varepsilon_n}{2}\right)\|u_n\|\,.$$

Dividing by $\|u_n\|^2$, we obtain

$$\int_{u_n>s_0}\frac{f(x,u_n)}{\|u_n\|}z_n\leq\frac{c}{\|u_n\|^2}+\left(c\varepsilon+\frac{\varepsilon_n}{2}\right)\frac{1}{\|u_n\|}\,,$$

then $\lim_{n\to\infty}\sup I_3\leq 0$. Hence

$$\lim_{n\to\infty}\sup\int_\Omega\frac{f(x,u_n)}{\|u_n\|}z_n=\lim_{n\to\infty}\sup\{I_1+I_2+I_3\}\leq 0\,.$$

Third step. Finally we prove

$$\lim_{n\to\infty}\sup\int_\Omega\frac{f(x,u_n)}{\|u_n\|}z_n=1\,,\tag{3.31}$$

which contradicts (3.26). From (3.5) with $v=z_n$ and dividing by $\|u_n\|$, we get

$$\frac{\varepsilon_n}{\|u_n\|}\leq\int_\Omega z_n^2-1-\alpha\int_{\partial\Omega}(\gamma_0 z_n)^2ds+\int_\Omega\frac{f(x,u_n)}{\|u_n\|}z_n\leq\frac{\varepsilon_n}{\|u_n\|}\,.$$

By taking superior limit we obtain (3.31).

Case $\alpha > 0$. In this case we have that

$$(u,v)_* = \int_\Omega \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) ds\,, \quad \forall u, v \in H^1(\Omega)\,,$$

defines an inner product in $H^1(\Omega)$ and the norm $\|u\|_* = \sqrt{(u,u)_*}$ is equivalent to the usual norm $\|\cdot\|$ of $H^1(\Omega)$. As a matter of fact, from (2.6) we get $\int_\Omega u^2 \leq \beta_1^{-1}\|u\|_*^2$, then

$$\|u\|^2 \leq (1+\beta_1^{-1})\|u\|_*^2 = d_1\|u\|_*^2\,, \quad \forall u \in H^1(\Omega)\,.$$

On the other hand, the inequality $\|\gamma_0 u\|_{L^2(\partial\Omega)} \leq c_1\|u\|$ implies,

$$\|u\|_*^2 \leq (1+\alpha c_1^2)\|u\|^2 = d_2\|u\|^2\,, \quad \forall u \in H^1(\Omega)\,.$$

Then

$$(d_1)^{-1/2}\|u\|^2 \leq \|u\|_*^2 \leq d_2\|u\|^2\,, \quad \forall u \in H^1(\Omega)\,.$$

Henceforth, we denote the constants with the same letter c and expressions of the form $c\varepsilon_n$ with ε_n. Using the inner product previously defined and its associated norm, the inequalities (3.4) and (3.5) take the form

$$|\Phi(u_n)| = \left|\frac{1}{2}\|u_n\|_*^2 - \int_\Omega F(x,u_n)\right| \leq C\,, \tag{3.32}$$

$$\begin{aligned} |\langle \Phi'(u_n), v\rangle| &= \left|(u_n,v)_* - \int_\Omega f(x,u_n)v\right| \\ &\leq \varepsilon_n'\|v\| \leq \sqrt{d_1}\varepsilon_n'\|v\|_* = \varepsilon_n\|v\|_*, \end{aligned} \tag{3.33}$$

where, $\lim_{n\to\infty}\varepsilon_n = 0$ and $v \in H^1(\Omega)$.

Next we shall prove that the sequence $\{u_n\}_{n=1}^\infty$ is bounded. With this purpose first we establish the inequality $\|u_n^+\|_*^2 \leq c + c\,\|u_n^-\|_*$ and second, we prove that $\|u_n^-\|_*$ is bounded. The desired result will follow from the equality $\|u\|_*^2 = \|u^+\|_*^2 + \|u^-\|_*^2$, $\forall u \in H^1(\Omega)$.

First step. We shall prove

$$\int_{u_n \geq s_0} F(x,u_n)dx \leq c + \varepsilon_n \|u_n\|_* + c\left\|u_n^-\right\|_{L^1}\,. \tag{3.34}$$

From (f_3) we have

$$\int_{u_n\geq s_0} F(x,u_n) \leq \left(\frac{1}{\theta} - 2\right)^{-1} \int_{u_n \geq s_0} \{f(x,u_n)u_n - 2F(x,u_n)\}\,dx\,. \tag{3.35}$$

From (3.32) and (3.33) we get

$$\left|\int_\Omega \{f(x,u_n)u_n - 2F(x,u_n)\}dx\right| \leq c + \varepsilon_n\|u_n\|_*\,. \tag{3.36}$$

Hence

$$\int_{u_n \geq s_0} \{f(x,u_n)u_n - 2F(x,u_n)\}dx \leq c + \varepsilon_n \|u_n\|_* + \int_{u_n < s_0} |2F(x,u_n) - f(x,u_n)u_n| \,. \tag{3.37}$$

Conditions (f_0) and (f_2) imply

$$|2F(x,s) - f(x,s)s| \leq c + c|s| \,, \quad s < 0 \,, \quad \forall x \in \overline{\Omega} \,, \tag{3.38}$$

from (3.37) and (3.38) we get

$$\left| \int\limits_{u_n \geq s_0} \{f(x,u_n)u_n - 2F(x,u_n)\}\, dx \right| \leq c + \varepsilon_n \|u_n\|_* + c \left\|u_n^-\right\|_{L^1} \,. \tag{3.39}$$

Now, from (3.35) and (3.39), we obtain (3.34).

Second step. We shall prove now that

$$\left| \, \|u_n^-\|_*^2 - \int_{u_n<0} 2F(x,u_n) \right| \leq c + \varepsilon_n \|u_n^-\|_* + c \left\|u_n^-\right\|_{L^1} \,. \tag{3.40}$$

Making $v(x) = u_n^-(x)$ in (3.33) we have

$$\left| \, \|u_n^-\|_*^2 - \int_{u_n<0} f(x,u_n)u_n \right| \leq \varepsilon_n \left\|u_n^-\right\|_* \,. \tag{3.41}$$

From (3.38) and (3.41) we obtain

$$\begin{aligned} \left| \, \|u_n^-\|_*^2 - \int_{u_n<0} 2F(x,u_n) \right| &= \Bigg| \, \|u_n^-\|_*^2 - \int_{u_n<0} f(x,u_n)u_n \\ &\qquad + \int_{u_n<0} f(x,u_n)u_n - \int_{u_n<0} 2F(x,u_n) \Bigg| \\ &\leq \left| \, \|u_n^-\|_*^2 - \int_{u_n<0} f(x,u_n)u_n \right| \\ &\qquad + \left| \int_{u_n<0} \{f(x,u_n)u_n - 2F(x,u_n)\}\, dx \right| \\ &\leq \varepsilon_n \left\|u_n^-\right\|_* + \int_{u_n<0} |f(x,u_n)u_n - 2F(x,u_n)|\, dx \\ &\leq c + \varepsilon_n \left\|u_n^-\right\|_* + c \left\|u_n^-\right\|_{L^1} \,. \end{aligned}$$

Third step. Next we shall verify the inequality

$$\left\|u_n^+\right\|_*^2 \leq c + c\left\|u_n^-\right\|_* . \tag{3.42}$$

From (3.32) we have

$$\begin{aligned}\left| \ \left\|u_n^+\right\|_*^2 - \int_{u_n\geq 0} 2F(x,u_n)\right| - \left|\int_{u_n<0} 2F(x,u_n) - \left\|u_n^-\right\|_*^2\right| \\ \leq \left| \ \left\|u_n^+\right\|_*^2 + \left\|u_n^-\right\|_*^2 - \int_\Omega 2F(x,u_n)\right| \\ = \left| \ \|u_n\|_*^2 - \int_\Omega 2F(x,u_n)\right| \leq c\,,\end{aligned}$$

and with (3.40) we get

$$\begin{aligned}\left| \ \left\|u_n^+\right\|_*^2 - \int_{u_n\geq 0} 2F(x,u_n)\right| \leq c + \left|\int_{u_n<0} 2F(x,u_n) - \left\|u_n^-\right\|_*^2\right| \\ \leq c + c\left\|u_n^-\right\|_* .\end{aligned}$$

Then the above inequality and (3.34) give

$$\begin{aligned}\left\|u_n^+\right\|_*^2 &\leq c + c\left\|u_n^-\right\|_* + \left|\int_{u_n\geq 0} 2F(x,u_n)\right| \\ &\leq c + c\left\|u_n^-\right\|_* + \left|\int_{0\leq u_n(x)\leq s_0} 2F(x,u_n)\right| \\ &\quad + \left|\int_{u_n>s_0} 2F(x,u_n)\right| \\ &\leq c + c\left\|u_n^-\right\|_* + \varepsilon_n\left\|u_n^+\right\|_* .\end{aligned}$$

Therefore

$$\left\|u_n^+\right\|_*^2 \leq c + c\left\|u_n^-\right\|_* + \varepsilon_n\left\|u_n^+\right\|_* ,$$

since $\lim_{n\to\infty}\varepsilon_n = 0$ this inequality yields (3.42).

Fourth step. We consider the following exhaustive cases:

i) There exists a constant c such that $\|u_n^-\|_* \leq c$, or
ii) $lim_{n\to\infty}\|u_n^-\|_* = \infty$, passing to a subsequence if it would be necessary.

In case i), using (3.42) we have $\|u_n^+\|_* \leq$ c, $\forall n \in \mathbb{N}$ and, from the equality $\|u\|_*^2 = \|u^+\|_*^2 + \|u^-\|_*^2$ $\forall u \in H^1(\Omega)$, we conclude that $(u_n)_{n=1}^\infty$ is bounded. Next let us prove that case ii) can not occur. First, from (3.28) and (3.41) we get

$$\left| \ \left\|u_n^-\right\|_*^2 - \lambda\int_\Omega \left(u_n^-\right)^2\right| \leq c + c\left\|u_n^-\right\|_* . \tag{3.43}$$

If $w_n = \frac{u_n^-}{\left\|u_n^-\right\|_*}$ then there exists $w_0 \in H^1(\Omega)$ and a subsequence from $\{w_n\}_{n=1}^\infty$ that we denote in the same way, such that it converges to w_o weakly in $H^1(\Omega)$

and strongly in $L^2(\Omega)$. Let us see that $w_0 \neq 0$. Dividing (3.43) by $\|u_n^-\|_*^2$ we obtain

$$\left|1-\lambda\int_\Omega w_n^2\right| \leq \frac{c}{\|u_n^-\|_*^2}+\frac{c}{\|u_n^-\|_*}\,.$$

Taking limit when $n\to\infty$ we get

$$\int_\Omega w_0^2=\frac{1}{\lambda}\,,$$

therefore $w_0\neq 0$. Let us see that λ is an eigenvalue and w_0 its eigenfunction. First we prove

$$\left|\left(u_n^-,v\right)_*-\lambda\int_\Omega u_n^- v\right| \leq \left(c+\varepsilon_n+\|u_n^+\|_*+c\,\|u_n^+\|_{L^{p\sigma}}^{\sigma}\right)\|v\|_*\,. \tag{3.44}$$

From (3.33) we get

$$\begin{aligned}\left|\left(u_n^-,v\right)_*-\lambda\int_\Omega u_n^- v\right|-\left|\left(u_n^+,v\right)_*-\lambda\int_\Omega u_n^- v-\int_\Omega f\left(x,u_n\right)v\right| \leq\\ \leq\left|\left(u_n^-,v\right)_*-\lambda\int_\Omega u_n^- v-\left(u_n^+,v\right)_*+\lambda\int_\Omega u_n^- v+\int_\Omega f\left(x,u_n\right)v\right|\\ =\left|\left(u_n,v\right)_*-\int_\Omega f\left(x,u_n\right)v\right|\leq\varepsilon_n\|v\|_*\,.\end{aligned}$$

Then

$$\begin{aligned}\left|\left(u_n^-,v\right)_*-\lambda\int_\Omega u_n^- v\right| &\leq \varepsilon_n\|v\|_*+\left|(u_n^+,v)_*-\lambda\int_\Omega u_n^- v-\int_\Omega f(x,u_n)v\right|\\ &\leq\varepsilon_n\|v\|_*+\|u_n^+\|_*\|v\|_*+\left|\lambda\int_\Omega u_n^- v+\int_\Omega f(x,u_n)v\right|\,.\end{aligned} \tag{3.45}$$

Next we estimate $\left|\lambda\int_\Omega u_n^- v+\int_\Omega f(x,u_n)v\right|$. Conditions (f_0) and (f_2) imply

$$|f(x,s)-\lambda s|\leq c\,,\quad \forall s\leq 0\,,\quad \forall x\in\overline{\Omega}\,. \tag{3.46}$$

using (3.46), we obtain

$$\begin{aligned}\left|\lambda\int_\Omega u_n^- v+\int_\Omega f(x,u_n)v\right| &\le \left|\int_{u_n<0}\{f(x,u_n)-\lambda u_n\}v\right| \\ &\quad+\left|\int_{u_n\ge 0} f(x,u_n)v\right| \\ &\le \int_\Omega \chi_{u_n}|f(x,u_n)-\lambda u_n|\ |v|+\int_\Omega \left|f\left(x,u_n^+\right)\right|\ |v| \\ &\le c\int_\Omega \chi_{u_n}|v|+\int_\Omega \left|f\left(x,u_n^+\right)\right|\ |v| \\ &\le c\int_\Omega |v|+\int_\Omega \left|f\left(x,u_n^+\right)\right|\ |v| \\ &\le c\int_\Omega |v|+c\int_\Omega |v|+c\int_\Omega \left|u_n^+\right|^\sigma |v| \\ &\le c\|v\|_*+c\left\|u_n^+\right\|_{L^{p\sigma}}^\sigma \|v\|_{L^q},\end{aligned}$$

where the function χ_{u_n} is defined by

$$\chi_{u_n}(x)=\begin{cases}1 & \text{if} \quad u_n(x)<0\,,\\ 0 & \text{if} \quad u_n(x)\ge 0\,,\end{cases}$$

and $p=\frac{2n}{n+2}$, $q=\frac{2n}{n-2}$ for $n\ge 3$, and we take $1<p<1/\sigma\theta$ as long as $\sigma\theta<1$ for $n=2$. Then we obtain (3.44). Now, dividing (3.44) by $\|u_n^-\|_*$, we get

$$\left|(w_n,v)_*-\lambda\int_\Omega w_n v\right| \le \left(\frac{c+\varepsilon_n}{\left\|u_n^-\right\|_*}+\frac{\|u_n^+\|_*}{\left\|u_n^-\right\|_*}+c\frac{\|u_n^+\|_{L^{p\sigma}}^\sigma}{\left\|u_n^-\right\|_*}\right)\|v\|_*\,. \tag{3.47}$$

It is evident that $\lim_{n\to\infty}\frac{c+\varepsilon_n}{\left\|u_n^-\right\|_*}=0$. From (3.42) we have $\lim_{n\to\infty}\frac{\left\|u_n^+\right\|_*}{\|u_n^-\|_*}=0$. Let us prove that

$$\lim_{n\to\infty}\frac{\|u_n^+\|_{L^{p\sigma}}^\sigma}{\left\|u_n^-\right\|_*}=0\,. \tag{3.48}$$

Conditions (f_0) and (f_3) imply the existence of positive constants K and c_2 such that

$$F(x,s)\ge \theta K s^{1/\theta}-c_2\,,\quad \text{for} \quad s>0\,. \tag{3.49}$$

Then (3.34) and (3.49) give

$$\int_\Omega \left|u_n^+\right|^{1/\theta}\le c+\varepsilon_n\left\|u_n^+\right\|_*+c\left\|u_n^-\right\|_*\,. \tag{3.50}$$

Dividing (3.50) by $\|u_n^-\|_*^{1/\sigma\theta}$ we have

$$\frac{1}{\left\|u_n^-\right\|_*^{1/\sigma\theta}}\int_\Omega \left|u_n^+\right|^{1/\theta}\le \frac{c}{\left\|u_n^-\right\|_*^{1/\sigma\theta}}+\varepsilon_n\frac{\|u_n^+\|_*}{\left\|u_n^-\right\|_*^{1/\sigma\theta}}+\frac{c}{\left\|u_n^-\right\|_*^{\frac{1}{\sigma\theta}-1}}\,. \tag{3.51}$$

From (S_2) we have $\frac{1}{\sigma\theta} = 1 + \delta$ for some $\delta > 0$. Hence

$$\lim_{n\to\infty} \int_\Omega \left(\frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{1/\theta} = 0\,. \tag{3.52}$$

From (S_2) and the choice of p in the case $n = 2$ we have that $1 < p\sigma \leq \frac{1}{\theta}$, then

$$\begin{aligned} 0 \leq \lim_{n\to\infty} &\left(\int_\Omega \left(\frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{p\sigma} \right)^{1/p\sigma} \\ \leq \lim_{n\to\infty} &\left(\int_\Omega \left(\frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{1/\theta} \right)^{\theta} = 0\,. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \left(\int_\Omega \left(\frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{p\sigma} \right)^{1/p} = 0\,.$$

Then, the limit in (3.47) yields

$$\lim_{n\to\infty} \left| (w_n, v)_* - \lambda \int_\Omega w_n v \right| = \left| (w_0, v)_* - \lambda \int_\Omega w_0 v \right| = 0\,.$$

Hence

$$(w_0, v)_* = \lambda \int_\Omega w_0 v\,, \quad \forall v \in H^1(\Omega)\,,$$

so, λ is an eigenvalue of $-\Delta$, with boundary condition $\gamma_1 u + \alpha\gamma_0 u = 0$. But this contradicts hypothesis (S_1). Hence, $\|u_n^-\|_*$ cannot tend to $+\infty$ when $n \to \infty$. ☑

4. Results of existence

In this section, we establish the existence of solutions of Problem $(\mathbb{P})$.

Theorem 4.1. *Suppose $n \geq 2$, $\alpha < 0$, $(f_0), (f_1), (f_2), (f_3)$, and let μ_1, μ_2 be the first and the second eigenvalues of $-\Delta$ with the boundary condition of the problem*

$$(P_1) \qquad \begin{cases} -\Delta u = f(x, u(x))\,, & \text{in} \quad \Omega\,, \\ \gamma_1 u + \alpha\gamma_0 u = 0\,, & \text{on} \quad \partial\Omega\,, \end{cases}$$

such that

- $f_4)$ *$\frac{f(x,s)}{s} \geq \mu_1$, $\forall s \in \mathbb{R} - \{0\}$, $\forall x \in \overline{\Omega}$.*
- $f_5)$ *There exist $\varepsilon_0 > 0$ and $p > 0$ such that $\mu_1 < \mu_2 - p < \mu_2$, and*

$$\frac{f(x,s)}{s} \leq \mu_2 - p\,, \quad \forall s \in (-\varepsilon_0, \varepsilon_0) - \{0\}\,, \quad \forall x \in \overline{\Omega}\,.$$

Then Problem (P_1) has at least one nontrivial solution.

Proof. We prove the conditions of Theorem 2.1. The functional Φ associated to the problem (P_1) is defined by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha}{2}\int_{\partial\Omega}(\gamma_0 u)^2 ds - \int_\Omega F(x,u)\,,$$

which satisfies the Palais-Smale condition by Lemma (3.2).

Using decomposition (2.4), $H^1(\Omega) = X_1 \oplus X_2$, we have

I) $\Phi(u) \leq 0$, $\forall u \in X_1$. Indeed, condition (f_4) implies that $F(x,s) \geq \mu_1 \frac{s^2}{2}$, $\forall s \in \mathbb{R}$, $\forall x \in \overline{\Omega}$. Then for each $u \in X_1$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha}{2}\int_{\partial\Omega}(\gamma_0 u)^2 ds - \int_\Omega F(x,u)\\ &= \frac{1}{2}\mu_1 \int_\Omega u^2 - \int_\Omega F(x,u) \quad \text{(by (2.5))}\\ &\leq \frac{1}{2}\mu_1 \int_\Omega u^2 - \frac{1}{2}\mu_1\int_\Omega u^2 = 0\,.\end{aligned}$$

II) There exists $\rho_0 > 0$ such that $\Phi(u) \geq 0$, $\forall u \in \partial B\rho_0(0) \cap X_2$. Condition (f_5) implies

$$F(x,s) \leq (\mu_2 - p)\frac{s^2}{2}\,, \quad \text{for} \quad |s| < \varepsilon_0, \quad \text{and} \quad \forall x \in \overline{\Omega}\,. \tag{4.1}$$

On the other hand, for $|s| \geq \varepsilon_0$, condition (f_1) implies the existence of a positive constant m_0 such that

$$|f(x,s)| \leq m_0 |s|^\sigma\,, \quad \text{for} \quad |s| \geq \varepsilon_0\,, \quad \text{and} \quad \forall x \in \overline{\Omega}\,. \tag{4.2}$$

Now, (4.1) and (4.2) implies

$$F(x,s) \leq \begin{cases} (\mu_2 - p)\frac{s^2}{2}, & \text{if} \quad |s| < \varepsilon_0\,,\\ m|s|^{\sigma+1}, & \text{if} \quad |s| \geq \varepsilon_0\,,\end{cases} \tag{4.3}$$

for any constant m and $x \in \overline{\Omega}$.

Using (4.3), the variational characterization of μ_2, the Sobolev Imbedding Theorem, and the norm $\|u\|_k = \sqrt{(u,u)_k}$, where the inner product $(u,v)_k$ is defined in (2.3), we get for $u \in X_2$,

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|_k^2 - \frac{k}{2}\int_\Omega u^2 - \int_\Omega F(x,u) \\
&\geq \frac{1}{2}\|u\|_k^2 - \frac{k}{2}\int_{|u|<\varepsilon_0} u^2 - \frac{k}{2}\int_{|u|\geq\varepsilon_0} u^2 - \frac{1}{2}(\mu_2 - p) \\
&\quad \int_{|u|<\varepsilon_0} u^2 - m\int_{|u|\geq\varepsilon_0} |u|^{\sigma+1} \\
&= \frac{1}{2}\|u\|_k^2 - \frac{1}{2}(\mu_2 + k - p)\int_{|u|<\varepsilon_0} u^2 - \frac{k}{2}\int_{|u|\geq\varepsilon_0} u^2 - m\int_{|u|\geq\varepsilon_0} |u|^{\sigma+1} \\
&\geq \frac{1}{2}\|u\|_k^2 - \frac{1}{2}(\mu_2 + k - p)\int_\Omega u^2 - \widetilde{c}\int_{|u|\geq\varepsilon_0} |u|^{\sigma+1} - m\int_{|u|\geq\varepsilon_0} |u|^{\sigma+1} \\
&\quad (\text{where } \widetilde{c} = k/2\varepsilon_0^{\sigma-1}) \\
&= \frac{1}{2}\|u\|_k^2 - \frac{1}{2}(\mu_2 + k - p)\int_\Omega u^2 - m_1\int_{|u|\geq\varepsilon_0} |u|^{\sigma+1} \\
&\quad (m_1 = \widetilde{c} + m) \\
&\geq \frac{1}{2}\|u\|_k^2 - \frac{(\mu_2 + k - p)}{2(\mu_2 + k)}\|u\|_k^2 - m_1\int_\Omega |u|^{\sigma+1} \\
&\geq \frac{1}{2}\left(\frac{p\delta}{\mu_2 + k}\right)\|u\|^2 - m_1\int_\Omega |u|^{\sigma+1} \\
&= m_4\|u\|^2 - m_3\|u\|^{\sigma+1}\,.
\end{aligned}$$

So,

$$\Phi(u) \geq \|u\|\left(m_4\|u\| - m_3\|u\|^\sigma\right)\,, \quad u \in X_2\,. \tag{4.4}$$

Recalling that $\sigma > 1$ by condition f_1, the function $d : [0,+\infty) \to \mathbb{R}$ defined by $d(\rho) = m_4\rho - m_3\rho^\sigma$ achieves its global maximum in $\rho_0 = \left(\frac{m_4}{m_3\sigma}\right)^{1/(\sigma-1)}$. Then

$$\Phi(u) \geq \rho_0 d(\rho_0) = \rho_0^2\left(1 - \frac{1}{\sigma}\right)m_4 > 0\,, \quad \forall u \in \partial B_{\rho_0}(0) \cap X_2\,.$$

III) There exists $e \in X_2 - \{0\}$ and a constant M such that

$$\Phi(v + te) \leq M\,, \quad \forall v \in X_1 \quad \text{and} \quad \forall t > 0\,.$$

If $n \geq 2$ the space $H^1(\Omega)$ is not contained in $L^\infty(\Omega)$. Let $e \in X_2$ be a function which is unbounded from above, and λ_* the number given by

$$\lambda_* = \frac{\int_\Omega |\nabla e|^2 + \alpha\int_{\partial\Omega}(\gamma_0 e)^2 ds}{\int_\Omega e^2}\,. \tag{4.5}$$

Then, $\mu_2 \leq \lambda_*$ and $\mu_1 < \lambda_*$. The value of λ_* does not change by substituting e by te, then we suppose that e satisfies the condition

$$2(\lambda_* - \lambda)\int_\Omega e^2 < -\delta^*(\mu_1 - \lambda)\,, \tag{4.6}$$

where λ is the positive constant of condition (f_2), $\delta^* = \frac{\delta_1}{\mu_1+k}$ and δ_1 is a positive constant such that $\delta_1\|v\|^2_{L^\infty} \leq \|v\|^2_k$, $\forall v \in X_1$ where $\|v\|_{L^\infty} = \sup_{x\in\overline{\Omega}} |v(x)|$. Moreover δ^* satisfies

$$\delta^*\|v\|^2_{L^\infty} \leq \int_\Omega v^2\,, \quad \forall v \in X_1\,. \tag{4.7}$$

If $v \in X_1$ we get

$$\begin{aligned} 0 = (v,e)_k &= \int_\Omega \nabla v\cdot\nabla e + k\int_\Omega ve + \alpha\int_{\partial\Omega}(\gamma_0 v)(\gamma_0 e)ds \\ &= (\mu_1+k)\int_\Omega ve\,, \end{aligned}$$

where $\mu_1 + k > 0$, then $\int_\Omega ve = 0$ and we obtain

$$\int_\Omega \nabla v\cdot\nabla e + \alpha\int_{\partial\Omega}(\gamma_0 v)(\gamma_0 e)ds = 0\,. \tag{4.8}$$

From (f_3) there exist $m_5 > 0$ and $s_1 \geq s_0$ such that

$$F(x,s) \geq \frac{\lambda}{2}s^2 + m_5 s^{1/\theta}\,, \quad \forall s \geq s_1 \quad \text{and} \quad \forall x \in \overline{\Omega}\,. \tag{4.9}$$

Conditions (f_0) and (f_2) imply the existence of a positive constant $m_6 > 0$ such that

$$F(x,s) \geq \frac{\lambda}{2}s^2 - m_6|s|\,, \quad \forall s \leq s_1 \quad \text{and} \quad \forall x \in \overline{\Omega}\,. \tag{4.10}$$

Now, if $v \in X_1$ and $t > 0$ then (4.5), (4.8), (4.9) and (4.10) yield

$$\begin{aligned} \Phi(v+te) &= \frac{1}{2}\int_\Omega |\nabla(v+te)|^2 + \frac{\alpha}{2}\int_{\partial\Omega}(\gamma_0(v+te))^2 - \int_\Omega F(x,v+te) \\ &= \frac{1}{2}\mu_1\int_\Omega v^2 + \frac{t^2}{2}\lambda_*\int_\Omega e^2 - \int_\Omega F(x,v+te) \\ &= \frac{1}{2}\mu_1\int_\Omega v^2 + \frac{t^2}{2}\lambda_*\int_\Omega e^2 - \int_{v(x)+te(x)\leq s_1} F(x,v+te) \\ &\quad - \int_{v(x)+te(x)>s_1} F(x,v+te) \\ &\leq \frac{1}{2}\mu_1\int_\Omega v^2 + \frac{t^2}{2}\lambda_*\int_\Omega e^2 - \frac{\lambda}{2}\int_\Omega (v+te)^2 \\ &\quad + m_6\int_{v(x)+te(x)\leq s_1}|v+te| - m_5\int_{v(x)+te(x)>s_1}(v+te)^{1/\theta} \\ &\leq \frac{1}{2}(\mu_1-\lambda)\int_\Omega v^2 + \frac{t^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 + m_6\int_\Omega |v| \\ &\quad + m_6 t\int_\Omega |e| - m_5\int_{v(x)+te(x)>s_1}(v+te)^{1/\theta}. \end{aligned}$$

From (4.7) we have

$$\Phi(v+te) \leq \frac{\delta^*}{2}(\mu_1-\lambda)\|v\|^2_{L^\infty} + m_6|\Omega|\ \|v\|_{L^\infty} + \frac{t^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 + m_6 t\int_\Omega |e| - m_5\int_{v(x)+te(x)>s_1}(v+te)^{1/\theta}\,. \tag{4.11}$$

Observing (4.11) we have the following cases:

Case 1. If $\lambda > \lambda_*$, then

$$\Phi(v+te) \leq \frac{\delta^*}{2}(\mu_1-\lambda)\|v\|^2_{L^\infty} + m_6|\Omega|\ \|v\|_{L^\infty} + \frac{t^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 + m_6 t\int_\Omega |e|\,,$$

where the coefficients of $\|v\|^2_{L^\infty}$ and t^2 are negative, therefore there exists a constant $M_1 > 0$ such that $\Phi(v+te) \leq M_1$, $\forall v \in X_1$ and $\forall t > 0$.

Case 2. If $0 < \lambda \leq \lambda_*$, and $v_0 = \min\left\{v(x) : x \in \overline{\Omega}\right\}$ then $v_0 + t \leq s_1$ or $v_0 + t > s_1$. Let $t \leq s_1 - v_0$.

- If $v_0 = 0$, from (4.11) we have

$$\Phi(v+te) \leq \frac{\delta^*}{2}(\mu_1-\lambda)\|v\|^2_{L^\infty} + m_6|\Omega|\ \|v\|_{L^\infty} + \frac{s_1^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 + m_6 s_1\int_\Omega |e|.$$

Since the coefficient of $\|v\|^2_{L^\infty}$ is negative, there is $M_2 > 0$ such that $\Phi(v+te) \leq M_2$.

- If $v_0 \neq 0$ then $|v_0| \leq \|v\|_{L^\infty}$ and from (4.11) we have,

$$\Phi(v+te) \leq \frac{\delta^*}{2}(\mu_1-\lambda)\|v\|^2_{L^\infty} + m_6|\Omega|\ \|v\|_{L^\infty} + \frac{1}{2}(s_1-v_0)^2(\lambda_*-\lambda)\int_\Omega e^2 + (s_1-v_0)m_6\int_\Omega |e|\,.$$

Using the inequality

$$(s_1-v_0)^2 \leq 2\left(s_1^2 + |v_0|^2\right)\,, \tag{4.12}$$

and calling

$$c = m_6\left(|\Omega| + \int_\Omega |e|\right)\,, \tag{4.13}$$

we obtain

$$\Phi(v+te) \leq \left[\frac{\delta^*}{2}(\mu_1-\lambda) + (\lambda_*-\lambda)\int_\Omega e^2\right]\|v\|^2_{L^\infty} + c\|v\|_{L^\infty} + s_1^2(\lambda_*-\lambda)\int_\Omega e^2 + s_1 m_6\int_\Omega |e|.$$

The coefficient of $\|v\|^2_{L^\infty}$ is negative, therefore there exists $M_3 > 0$ such that $\Phi(v + te) \leq M_3$.

• In the case $t > s_1 - v_0$, let $\Omega_1 = \{x \in \Omega : e(x) > 1\}$ then $|\Omega_1| > 0$. Since the function e is not bounded from above, and $\Omega_1 \subset \{x \in \Omega : v(x)+te(x) > s_1\}$ then (4.11) yields

$$\Phi(v+te) \leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|^2_{L^\infty} + m_6|\Omega|\ \|v\|_{L^\infty} + \frac{t^2}{2}(\lambda_* - \lambda)\int_\Omega e^2 + m_6 t\int_\Omega |e| - m_5|\Omega_1|(v_0+t)^{1/\theta}\,. \tag{4.14}$$

Setting $v_0 + t = s$ we have

$$\Phi(v+te) \leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|^2_{L^\infty} + m_6|\Omega|\ \|v\|_{L^\infty} + \frac{(s-v_0)^2}{2}(\lambda_* - \lambda)\int_\Omega e^2 + m_6(s-v_0)\int_\Omega |e| - m_5|\Omega_1|s^{1/\theta}\,.$$

From $(s - v_0)^2 \leq 2s^2 + 2\|v\|^2_{L^\infty}$ and (4.13) we obtain

$$\Phi(v+te) \leq \left(\frac{\delta^*}{2}(\mu_1 - \lambda) + (\lambda_* - \lambda)\int_\Omega e^2\right)\|v\|^2_{L^\infty} + c\|v\|_{L^\infty} + m_6 s\int_\Omega |e| + s^2(\lambda_* - \lambda)\int_\Omega e^2 - m_5|\Omega_1|s^{1/\theta}\,.$$

Since the coefficients of $\|v\|^2_{L^\infty}$ and $s^{1/\theta}$ are negative, then there exists $M_4 > 0$ such that $\Phi(v + te) \leq M_4$. If $M = \max\{M_2, M_3, M_4\}$, then $\Phi(v + te) \leq M$ $\forall v \in X_1$, and $t > 0$. ☑

In the following theorem we consider the case $\alpha > 0$, and we use the following condition (f_2^*): the number λ of condition (f_2) is such that $\lambda > \beta_1$, and $\lambda \neq \beta_j$, for $j = 2, 3, \cdots$, (λ is not an eigenvalue).

Theorem 4.2. *Suppose: $n \geq 2$, $\alpha > 0$, (f_0), (f_1), (f_3), (f_2^*), (S_2), and the conditions:*

(f_4^*) $\frac{f(x,s)}{s} \geq \beta_1$, $\forall s \in \mathbb{R} - \{0\}$, $\forall x \in \overline{\Omega}$,

(f_5^*) *there exist $\varepsilon_0 > 0$ and $\beta \in (\beta_1, \beta_2)$ such that*

$$\frac{f(x,s)}{s} \leq \beta \quad \forall s \in (-\varepsilon_0, \varepsilon_0) - \{0\} \quad \forall x \in \overline{\Omega}\,.$$

Then the problem

$$(\mathbb{P}_2)\begin{cases} -\Delta u = f(x,u(x))\,, & \text{in} \quad \Omega\,, \\ \gamma_1 u + \alpha\gamma_0 u = 0\,, & \text{on} \quad \partial\Omega\,, \end{cases}$$

has at least a nontrivial solution.

Proof. To prove the conditions of Theorem 2.1 we use the decomposition (2.7), $H^1(\Omega) = Y_1 \oplus Y_2$. The functional Φ associated to problem $(\mathbb{P}_2)$ is

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha}{2}\int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_\Omega F(x,u)\,,$$

which satisfies the Palais-Smale condition by Lemma (3.2).

I) $\Phi(u) \leq 0, \forall u \in Y_1$. Let $u \in Y_1$, condition (f_4^*) and the equality (2.8), yields

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha}{2}\int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_\Omega F(x,u)\\ &\leq \frac{1}{2}\beta_1 \int_\Omega u^2 - \frac{\beta_1}{2}\int_\Omega u^2 = 0\,.\end{aligned}$$

II) There exists $\rho_0 > 0$ such that $\Phi(u) \geq 0 \ \ \forall u \in \partial B_{\rho_0}(0) \cap Y_2$. Condition (f_5^*) implies

$$F(x,s) \leq \frac{\beta}{2}s^2\,, \quad |s| \leq \varepsilon_0\,, \quad \forall x \in \overline{\Omega}\,. \tag{4.15}$$

On the other hand, for $|s| \geq \varepsilon_0$ and $x \in \overline{\Omega}$ condition (f_1) implies the existence of a positive constant m_0 such that $|f(x,s)| \leq m_0|s|^\sigma$ and its integrals yield

$$|F(x,s)| \leq m|s|^{\sigma+1}\,, \quad \forall |s| \geq \varepsilon_0 \quad \text{and} \quad \forall x \in \overline{\Omega}\,, \tag{4.16}$$

for any constant $m > 0$. From (4.15) and (4.16), we have

$$F(x,s) \leq \frac{\beta}{2}s^2 + m|s|^{\sigma+1}\,, \quad \forall s \in \mathbb{R}\,, \quad \forall x \in \overline{\Omega}\,. \tag{4.17}$$

If $u \in Y_2$ then

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha}{2}\int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_\Omega F(x,u)\\ &= \frac{1}{2}\|u\|_*^2 - \int_\Omega F(x,u)\\ &\geq \frac{1}{2}\|u\|_*^2 - \frac{\beta}{2}\int_\Omega u^2 - m\int_\Omega |u|^{\sigma+1}\\ &\geq \frac{1}{2}\|u\|_*^2 - \frac{\beta}{2\beta_2}\|u\|_*^2 - mc\|u\|^{\sigma+1}\\ &\geq \frac{1}{2}\left(1 - \frac{\beta}{\beta_2}\right) d_1^{-1}\|u\|^2 - mc\|u\|^{\sigma+1}\,.\end{aligned}$$

Since $\sigma + 1 > 2$, there exist $\rho_0 > 0$ and $a > 0$ such that

$$\Phi(u) \geq a > 0\,, \quad \forall u \in \partial B\rho_0(0) \cap Y_2\,.$$

III) There exist a function $e \in Y_2 - \{0\}$ and a constant $M > 0$ such that

$$\Phi(v + te) \leq M\,, \quad \forall v \in Y_1 \quad \text{and} \quad \forall t > 0\,.$$

Let $e \in Y_2$ be a function which is unbounded from above and λ_* the number defined by

$$\lambda_* = \frac{\int_\Omega |\nabla e|^2 + \alpha \int_{\partial\Omega} (\gamma_0 e)^2 ds}{\int_\Omega e^2}\,, \tag{4.18}$$

then $\beta_2 \leq \lambda_*$ and $\lambda > \lambda_*$ or $\lambda \leq \lambda_*$. We suppose that e satisfies the condition

$$2(\lambda_* - \lambda) \int_\Omega e^2 < -\delta^* \left(1 - \frac{\lambda}{\beta_1}\right)\,, \tag{4.19}$$

where $\delta^* > 0$ is such that

$$\delta^* \|v\|^2_{L^\infty} \leq \|v\|^2_*\,, \quad \forall v \in Y_1\,. \tag{4.20}$$

For $v \in Y_1$ and $k > 0$ we have,

$$(v,u)_k = (v,u)_* + k \int_\Omega vu = (\beta_1 + k) \int_\Omega vu\,, \quad \forall u \in H^1(\Omega)\,.$$

Making $u = e$ we get

$$0 = (v,e)_k = (v,e)_* + k \int_\Omega ve = (\beta_1 + k) \int_\Omega ve\,,$$

then

$$\int_\Omega ve = 0\,, \tag{4.21}$$

and

$$(v,e)_* = 0\,. \tag{4.22}$$

We also use

$$F(x,s) \geq \frac{\lambda}{2} s^2 + m_5 s^{1/\theta}\,, \quad \forall s \geq s_1\,, \quad \forall x \in \overline{\Omega} \quad \text{and} \tag{4.23}$$

$$F(x,s) \geq \frac{\lambda}{2} s^2 - m_6 |s|\,, \quad \forall s \leq s_1 \quad \text{and} \quad \forall x \in \overline{\Omega}\,. \tag{4.24}$$

If $v \in Y_1$ and $t > 0$ then using (4.18), (4.21), (4.22), (4.23) and (4.24), we obtain

$$\begin{aligned}
\Phi(v+te) &= \frac{1}{2} \int_\Omega |\nabla(v+te)|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0(v+te))^2 - \int_\Omega F(x, v+te) \\
&= \frac{1}{2}\beta_1 \int_\Omega v^2 + \frac{t^2}{2}\lambda_* \int_\Omega e^2 - \int_{v(x)+te(x)\leq s_1} F(x, v+te) \\
&\quad - \int_{v(x)+te(x)>s_1} F(x, v+te)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\beta_1 \int_\Omega v^2 + \frac{t^2}{2}\lambda_* \int_\Omega e^2 - \frac{\lambda}{2}\int_{v+te\le s_1}(v+te)^2 \\
&\quad + m_6 \int_{v+te\le s_1} |v+te| \\
&- \frac{\lambda}{2}\int_{v+te>s_1}(v+te)^2 - m_5 \int_{v+te>s_1}(v+te)^{1/\theta} \\
&\le \frac{1}{2}(\beta_1-\lambda)\int_\Omega v^2 + \frac{t^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 + m_6\int_\Omega |v+te| \\
&\quad - m_5 \int_{v+te>s_1}(v+te)^{1/\theta} \\
&\le \frac{1}{2}(\beta_1-\lambda)\int_\Omega v^2 + m_6\int_\Omega |v| + \frac{t^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 \\
&\quad + tm_6\int_\Omega |e| - m_5\int_{v+te>s_1}(v+te)^{1/\theta}\,.
\end{aligned}$$

Using $\beta_1 \int_\Omega v^2 = \|v\|_*^2$, $\forall v \in Y_1$ and (4.20) we get

$$\begin{aligned}
\Phi(v+te) \le \frac{\delta^*}{2}\left(1-\frac{\lambda}{\beta_1}\right)\|v\|_{L^\infty}^2 + m_6|\Omega|\|v\|_{L^\infty} \\
+ \frac{t^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 + tm_6\int_\Omega |e| - m_5\int_{v+te>s_1}(v+te)^{1/\theta}\,.
\end{aligned} \tag{4.25}$$

Observing (4.25) we have the following cases:

Case $\lambda > \lambda_*$. In this case, the coefficients of $\|v\|_{L^\infty}^2$ and t^2 in (4.25) are negative, therefore, there exists a constant $M_1^* > 0$ such that

$$\Phi(v+te) \le M_1^* \quad \forall v \in Y_1 \quad \text{and} \quad \forall t > 0\,.$$

Case $\mathbf{0} < \lambda \le \lambda_*$. If $v_0 = \min\{v(x) : x \in \overline{\Omega}\}$ then $v_0 + t \le s_1$ or $v_0 + t > s_1$. Let $t \le s_1 - v_0$.

If $v_0 = 0$ then from (4.25) we have,

$$\begin{aligned}
\Phi(v+te) \le \frac{\delta^*}{2}\left(1-\frac{\lambda}{\beta_1}\right)\|v\|_{L^\infty}^2 + m_6|\Omega|\ \|v\|_{L^\infty} \\
+ \frac{s_1^2}{2}(\lambda_*-\lambda)\int_\Omega e^2 + s_1 m_6\int_\Omega |e|\,,
\end{aligned} \tag{4.26}$$

then there exists $M_2^* > 0$ such that $\Phi(v+te) \le M_2^*$.

If $v_0 \neq 0$ then $|v_0| \le \|v\|_{L^\infty} \quad \forall v \in Y_1$. From (4.25) we have

$$\begin{aligned}
\Phi(v+te) \le \frac{\delta^*}{2}\left(1-\frac{\lambda}{\beta_1}\right)\|v\|_{L^\infty}^2 + m_6|\Omega|\ \|v\|_{L^\infty} \\
+ \frac{1}{2}(s_1-v_0)^2(\lambda_*-\lambda)\int_\Omega e^2 + (s_1-v_0)m_6\int_\Omega |e|\,.
\end{aligned}$$

Using (4.12) and (4.13), we obtain

$$\Phi(v+te) \leq \left[\frac{\delta^*}{2}\left(1-\frac{\lambda}{\beta_1}\right)+(\lambda_*-\lambda)\int_\Omega e^2\right]\|v\|^2_{L^\infty}$$
$$+c\|v\|_{L^\infty}+s_1^2(\lambda_*-\lambda)\int_\Omega e^2+s_1m_6\int_\Omega |e|.$$

From (4.19) there exists $M_3^*>0$ such that $\Phi(v+te)\leq M_3^*$.

In the case $v_0+t>s_1$, let $\Omega_1=\{x\in\Omega : e(x)>1\}$. Clearly, $|\Omega_1|>0$. From (4.25) we have

$$\Phi(v+te)\leq \frac{\delta^*}{2}\left(1-\frac{\lambda}{\beta_1}\right)\|v\|^2_{L^\infty}m_6|\Omega|\;\|v\|_{L^\infty}+\frac{t^2}{2}(\lambda_*-\lambda)\int_\Omega e^2$$
$$+tm_6\int_\Omega|e|-m_5|\Omega_1|\;(v_0+t)^{1/\theta}\,. \tag{4.27}$$

Making $v_0+t=s$, using (4.12) and (4.13), we obtain

$$\Phi(v+te)\leq\left[\frac{\delta^*}{2}\left(1-\frac{\lambda}{\beta_1}\right)+(\lambda_*-\lambda)\int_\Omega e^2\right]\|v\|^2_{L^\infty}+c\|v\|_{L^\infty}$$
$$+sm_6\int_\Omega|e|+s^2(\lambda^*-\lambda)\int_\Omega e^2-m_5|\Omega_1|s^{1/\theta}.$$

and there exists $M_4^*>0$ such that $\Phi(v+te)\leq M_4^*$. If $M^*=\max\{M_2^*,M_3^*,M_4^*\}$ then

$$\Phi(v+te)\leq M^*\,,\quad \forall v\in Y_1\quad\text{and}\quad\forall t>0\,.$$

☑

Recalling that for $n=1$ the space $H^1(\Omega)$ is contained in $L^\infty(\Omega)$, and the fact that the above proofs require an unbounded function in $H^1(\Omega)$, we conclude that theorems 4.1 and 4.2 can not be applied to the case $n=1$.

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