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Towards a new interpretation of Milnor's number

Hacia una nueva interpretación del número de Milnor

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Abstract. The Milnor number is a fundamental invariant of the biholomorphism type of the singularity of the germ of a holomorphic function f defined on an open neighborhood W of $0 \in \mathbb{C}^n$, and such that 0 is the only critical point of f in W. The present article describes a conjecture that would provide an interpretation of this invariant, in the case $n = 2$, as a sharp lower bound for the number of factors in *any* factorization in terms of right-handed Dehn twists of the monodromy around the singular fiber of f. Also, towards the end of the paper, an analogue conjecture for proper holomorphic maps $f : E \to D_r^0$ where E is a complex surface with boundary, D_r^0 is $\{z \in \mathbb{C} : |z| < r\}$, and f has $f^{-1}(0)$ as its unique singular fiber and all other fibers are closed and connected 2-manifolds of (necessarily the same) genus $g \geq 0$, is briefly described. The latter conjecture has been proved recently by the authors in the case when the regular fiber of f has genus 1 ([3]), and in ([5]), that author provides for each $g \geq 2$ an $f_g : E_g \to D_1^0$ having genus g regular fiber and violating this conjecture.

Key words and phrases. Milnor number, monodromy, right handed Dehn twist, morsification.

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Resumen. El número de Milnor es un invariante fundamental del tipo de biholomorfismo de un germen de una función holomorfa f definida en una vecindad abierta W de $0 \in \mathbb{C}^n$, tal que 0 es el único punto crítico de f en W. En este artículo presentamos una conjetura que daría una interpretación de este invariante en el caso $n = 2$, como una cota inferior exacta para el número de factores de *cualquier* factorización en términos de giros de Dehn derechos de la monodromía alrededor de la fibra singular de f. Además, hacia el final del artículo, se describe brevemente una conjetura análoga para el caso en que tenemos una función holomorfa propia $f : E \to D_r^0$ donde E es una superficie compleja con frontera, D_r^0 es $\{z \in \mathbb{C} : |z| < r\}$, f tiene a $f^{-1}(0)$ como su única fibra singular y todas las otras fibras son 2-variedades cerradas conexas de género, necesariamente constante, $g \geq 0$. Esta última conjetura ha sido demostrada recientemente por los autores en el caso en que el género de la fibra regular es 1 ([3]), y en ([5]), ese autor construye, para cada $g \geq 2$, una fibración $f_g : E_g \to D_1^0$ cuya fibra regular tiene género g y que viola esta conjetura.

Palabras y frases clave. Número de Milnor, monodromía, giro de Dehn derecho, morsificación.

1. **Introduction**

Let $f : W \to \mathbb{C}$ be a holomorphic function defined on an open neighborhood of $0 \in \mathbb{C}^n$, and let us assume that $f(0) = 0$, and that 0 is the only critical point of f in W. Then f determines a singular germ at the origin, denoted by f_0. The Milnor number is a very fundamental invariant of the biholomorphism type of the singularity f_0 and has been intensely studied since its introduction by Milnor in [7]. Several interpretations of the Milnor number have been discovered (see, for instance, [7] and [11]). The present article describes a conjecture that would provide an interpretation of Milnor's number, in the case $n = 2$, as a sharp lower bound for the number of factors in *any* factorization in terms of right-handed Dehn twists of the monodromy around the singular fiber of f. Also, towards the end of the paper, an analogue conjecture for proper holomorphic maps $f : E \to D_r^0$ where E is a complex surface with boundary, D_r^0 is $\{z \in \mathbb{C} : |z| < r\}$, and f has $f^{-1}(0)$ as its unique singular fiber and all other fibers are *closed* (i.e. compact without boundary) connected 2-manifolds of (necessarily the same) genus $g \geq 0$, is briefly described. The latter conjecture has been proved recently by the authors in the case when the regular fiber of f has genus 1 ([3]), and in ([5]), that author provides for each $g \geq 2$ an $f_g : E_g \to D_1^0$ having genus g regular fiber and violating this conjecture.

The conjectural interpretation of Milnor's number we formulate here seems at first sight to be another manifestation of the Topological Economy Principle in Algebraic Geometry, proposed by Arnold and his school. In what follows we will quote comments and facts from the beautiful article [1]. The principle says that *"if you have a geometrical or topological phenomenon, which you can realize by algebraic objects, then the simplest algebraic realizations are topologically as simple as possible"*. This principle has been used to formulate a number of conjectures, many of which have become theorems. Let us mention two examples of this that seem closer in spirit to our prediction.

- **Thom's Conjecture**. Let C be a smooth algebraic curve in $\mathbb{C}P^2$, and let us denote by $[C]$ the homology class of $H_2\left(\mathbb{C}P^2, \mathbb{Z}\right)$ it represents. Then Thom's Conjecture says that if Σ is a closed oriented smoothly embedded surface in $\mathbb{C}P^2$ so that $[\Sigma] = [C]$, then the genus of Σ is at least that of C. The efforts of several authors, including

Kronheimer, Mrowka, Taubes, Morgan, Fintushel, Stern, Szabó and Ozsváth (see the Introduction of [9]) culminated in the proof of a vast generalization of Thom's Conjecture known as the Symplectic Thom Conjecture: *an embedded symplectic surface in a closed, symplectic 4-manifold is genus-minimizing in its homology class.* As a corollary *an embedded holomorphic curve in a Kaehler surface is genus-minimizing in its homology class* (see [9]).

- **Milnor's Conjecture.** Let $f(z,w) \in \mathbb{C}[z,w]$ be an irreducible polynomial sending the origin $\mathbf{0} = (0,0)$ to 0. Let $V = f^{-1}(0)$ be the curve defined by f. Suppose that $\mathbf{0}$ is one of the (necessarily isolated) singular points of V and that exactly r branches of V pass through $\mathbf{0}$. Choose a small ball $B_\epsilon(\mathbf{0})$ centered at the origin and having radius ϵ with the property that for each $0 < \epsilon' \leq \epsilon$ the sphere $S_{\epsilon'}(\mathbf{0})$ intersects V transversely. The intersection $K = V \cap S_\epsilon(\mathbf{0})$ is a link in the sphere $S_\epsilon(\mathbf{0})$. There is an invariant number, associated to the singularity of V at $\mathbf{0}$ and usually denoted by $\delta_{\mathbf{0}}$, which measures the number of ordinary double points of V concentrated at $\mathbf{0}$. This means that if one perturbs a local parametrization $(x_i(t), y_i(t))$ with $i = 1, \ldots, r$ of V near $\mathbf{0}$, one generically obtains a curve $\tilde{V}$ with irreducible components $(\tilde{x}_i(t), \tilde{y}_i(t))$, $i = 1, \ldots, r$, having exactly $\delta_{\mathbf{0}}$ ordinary double points and no other singularities. The intersection K' between these curves and $S_\epsilon(\mathbf{0})$ is a link having the same type as the original link K. Let us choose one such perturbation with the extra property that there is an ϵ_0 so that no ordinary double point of $\tilde{V}$ is contained in the closed ball $\overline{B_{\epsilon_0}(\mathbf{0})}$. If we consider the intersections $\tilde{V} \cap S_{\epsilon'}$ with $0 < \epsilon_0 \leq \epsilon' \leq \epsilon$, as a movie starting at time ϵ and ending at time ϵ_0, we see the link K' passing through itself $\delta_{\mathbf{0}}$ times, and becoming the link formed by r unlinked copies of the unknot. In [7] Milnor conjectured that any other way to transform K' into the link formed by r unlinked copies of the unknot, by allowing strands of K' to pass through each other, would have at least $\delta_{\mathbf{0}}$ crossings. This conjecture has already been proven as an almost direct consequence of Thom's Conjecture.

On the other hand, since the Topological Economy Principle is just a *principle*, i.e., there is no known precise recipe to decide whether a particular phenomenon fits the principle or not, it is not clear to us whether the conjecture formulated in this article is a genuine instance of the Topological Economy Principle, and consequently, whether Ishizaka's counterexample undermines the principle or not.

In our opinion, this state of affairs prompts a number of interesting questions. For instance, is it possible to unify the Topological Economy Principle, at least partially, e.g. to formulate and prove a general theorem, so that a subset of the known manifestations of the principle were particular cases of it? Also, since all known instances of the principle seem to take place in a fixed ambient space, it would be interesting to discover a manifestation which is ambient free (our conjecture seems to have this character).

This article is organized as follows. In Section 2, the situation where the Milnor number originated is described and its classical definition is given. Section 3 provides an algebro-geometric formulation of the Milnor number. It relates the Milnor number with the number of critical points in a deformation of f. Section 4 sketches the notion of monodromy representation. Section 5 defines right-handed Dehn twists. Section 6 makes precise the notions of deformation, morsification and simple morsification of a map f. Section 7 generalizes the Milnor number and proposes a conjecture conducing to a new interpretation of this notion.

The article is a summary of the ideas presented in the "XV Congreso de Matemáticas" (2005). It is intended only as a survey of some results and conjectures by the authors. Proofs are omitted and only a sketch or an indication of how any particular argument would go is given.

The authors wish to express their sincere thanks to the Sociedad Colombiana de Matemáticas, and in particular to the organizers of this event where these ideas were first presented.

2. The classical definition of the Milnor number

Let $f : W \to \mathbb{C}$ be a holomorphic function defined in an open neighborhood $W \subset \mathbb{C}^n$ of the origin, with $f(0) = 0$, and having a singularity only at 0, i.e., all the partial derivatives $\partial f/\partial z_i$, $i = 1, \dots, n$ vanish simultaneously only at 0. Let

$$B_\rho = \{(z_1, \dots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq \rho^2\} \subset \mathbb{C}^n$$

be the closed ball in $\mathbb{C}^n$ of radius ρ and centered at the origin. B_ρ is a smooth manifold with boundary of (real) dimension $2n$. Let

$$S_\rho = \left\{(z_1, \dots, z_n) : |z_1|^2 + \cdots + |z_n|^2 = \rho^2\right\}$$

be the sphere of radius ρ and centered at the origin. It is the boundary of B_ρ and it is a smooth manifold of (real) dimension $2n - 1$. Let us denote by $D_r = \{z \in \mathbb{C} : |z| \leq r\}$ the closed disk of radius r centered at the origin in the complex plane. With this notation we have the following theorem [7]. See Figure 1.

Theorem 1 (Milnor)**.** *There exists $\rho_0 > 0$ and $r > 0$ such that*

(1) *For each $0 < \rho \leq \rho_0$, the smooth $(2n-2)$-manifold $f^{-1}(0) - \{0\} \subset W - \{0\}$ is transversal to S_ρ.*
(2) *If $z \neq 0$ and $z \in D_r$, then $X_z = f^{-1}(z) \cap B_{\rho_0}$ is a smooth $(2n-2)$-manifold with boundary.*
(3) *If ∂X_z denotes the boundary of X_z, then $\partial X_z = X_z \cap \partial B_{\rho_0} = X_z \cap S_{\rho_0}$.*
(4) *If $E = f^{-1}(D_r) \cap B_{\rho_0}$ then $f : E \to D_r$ is surjective.*

(5) *Let* $E^* = E - f^{-1}(0)$, *and* $D_r^* = D_r - \{0\}$. *The restriction* $f|_{E^*} : E^* \to D_r^*$ *is, by the Ehresmann Fibration Theorem, a fiber bundle. Consequently,* X_z *is diffeomorphic to* $X_{z'}$ *for* $z, z' \neq 0$.

(6) *For each* $z \in D_r^*$, X_z *is homotopically equivalent to a bouquet of a finite number of spheres of (real) dimension* $(n-1)$. *This number is independent of* z.

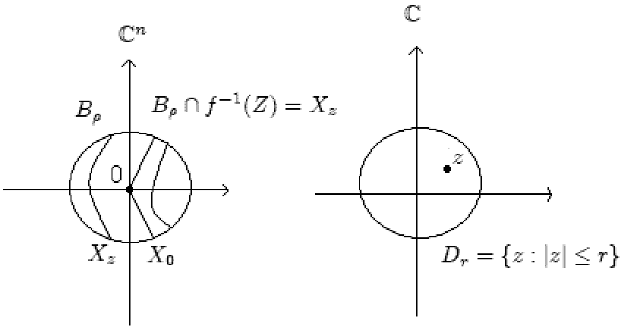

FIGURE 1. The situation giving rise to Milnor's number.

This theorem makes it possible to formulate one of the most important invariants of a singularity: its Milnor number.

Definition 2. *Let* $f : W \to \mathbb{C}$ *be as described above, and let* f_0 *be the holomorphic germ determined by* f *at the origin. Then the* Milnor number of *the isolated singular germ* f_0 *is the number of spheres in the bouquet homotopically equivalent to all of the* X_z *with* $z \in D_r^*$. *This number will be denoted by* $k(f_0)$.

The Milnor number clearly coincides with the torsion free rank of the $n-1$ homology $H_{n-1}(X_z, \mathbb{Z})$.

3. Algebraic interpretation of the Milnor number

Let $X = V(\{f_\alpha(z_1, \ldots, z_n)\}_{\alpha=1,\ldots,m}) \subset \mathbb{C}^n$ be an irreducible affine algebraic variety of dimension d, and let $p \in X$ be a point in X. By $\mathcal{O}_{X,p}$ we will denote the (local) ring of germs of regular functions at p. It is known that $\mathcal{O}_{X,p}$ is isomorphic to R_{m_p}, the localization of the coordinate ring $R = \mathbb{C}[X]$ of regular functions on X at the maximal ideal $m_p = \{f \in R : f(p) = 0\}$. A system of parameters $g_1, \ldots, g_d$ for R_{m_p} are functions on X such that the intersection of X with the variety defined by the g_i's is a finite set of points, one of them

p. Algebraically, this condition is equivalent to the fact that the radical of the ideal generated by the g_i's is the maximal ideal of the local ring (R, m_p). The Serre *multiplicity of the intersection of* $(\underline{g}) = (g_1, \ldots, g_d)$ at p [10] is defined as

$$\mu\left(\underline{g}, R_{m_p}\right) = \sum_{i=0}^{d} (-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i \left(R_{m_p}/(\underline{g}), R_{m_p}\right).$$

It is well known that if R_{m_p} is a Cohen-Macaulay ring, then all the Tor_i vanish for $i > 0$ and in this case $\mu\left(\left(\underline{g}\right), R_{m_p}\right) = \dim_{\mathbb{C}} R_{m_p}/\left(\underline{g}\right)$. It is a theorem [4] that this number can be computed as the number of different solutions of a system of equations $\mathcal{E}$

$$\mathcal{E} : \{f_\alpha(z_1, \ldots, z_n),\ \alpha = 1, \ldots, m,\ g_i(z_1, \ldots, z_n) + \varepsilon_i = 0,\ i = 1, \ldots, d\},$$

determined by the equations that define X together with a set of equations $g_i(x) + \varepsilon_i = 0$, obtained by perturbing the g_i's in a neighborhood of the origin U_0, and where the perturbation $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ can chosen arbitrarily in a sufficiently small disk D_δ, and outside a proper Zariski subset of D_δ.

These two formulations of the notion of intersection multiplicity are all equivalent in the case where R is a complete local ring, in particular, if R is the ring of formal power series in several variables over $\mathbb{C}$. They are also equivalent for its holomorphic counterpart, the ring $S = \mathbb{C}\{z_1, \ldots, z_n\}$ of power series at 0 with positive radius of convergence.

If we take $X = \mathbb{C}^n$, the (local) ring of germs of holomorphic functions at the origin, then $\mathcal{O}_{X,0}$ can be identified with S, the ring of power series at 0 with positive radius of convergence. If f has an isolated singularity at the origin the quotient ring $S/(g_1, \ldots, g_n)$, where $g_i = \partial f/\partial z_i$, is zero dimensional. Since S is a regular ring it is in particular a Cohen Macaulay ring and in this case we have

$$\mu(f_0) = \mu\left(\partial f/\partial z_1, \ldots, \partial f/\partial z_n, S\right) = \dim_{\mathbb{C}} \left(S/\left(\partial f/\partial z_1, \ldots, \partial f/\partial z_n\right)\right).$$

We want to see that $\mu(f_0) = k(f_0)$. On the one hand, we first note that the perturbation ε that leads to system $\mathcal{E}$ *can also be interpreted as a parameter deformation of* f, i.e., if $\widetilde{f}$ is the function defined by $\widetilde{f} = f + \varepsilon_1 z_1 + \cdots + \varepsilon_n z_n$, then clearly the system of equations

$$\mathcal{A} : \{z \in U_0 : \partial f/\partial z_i(z) + \varepsilon_i = 0,\ i = 1, \ldots, n\},$$

is the same system as

$$\mathcal{B} : \{z \in U_0 : \partial \widetilde{f}/\partial z_i = 0,\ i = 1, \ldots, n\},$$

where U_0 denotes a sufficiently small neighborhood of 0. Hence, $\mu(f_0)$ coincides with the number of critical values of $\widetilde{f}$ in a sufficiently small neighborhood of the origin. On the other hand, it can be seen that the critical value of f breaks into $k(f_0)$ critical values of a "more simple type" of the *morsified* function $\widetilde{f}$. More precisely, with notation as above, we have the following theorem (see [11] and Chapter 1 of [6]). See Figure 2.

Theorem 3. *There exist $\rho, r, \varepsilon > 0$ and Z, a Zariski closed proper subset of $B_\varepsilon \subset \mathbb{C}^n$ such that for all $(\varepsilon_1, \ldots, \varepsilon_n)$ in $B_\varepsilon - Z$ the function $\widetilde{f}$ has $k(f_0)$ critical values in B_r which correspond to exactly the same number of critical points in B_ρ, each one of Morse type, i.e., there exist coordinates around each critical point p and around $\widetilde{f}(p)$ such that in these coordinates $\widetilde{f} = z_1^2 + \cdots + z_n^2 + c$.*

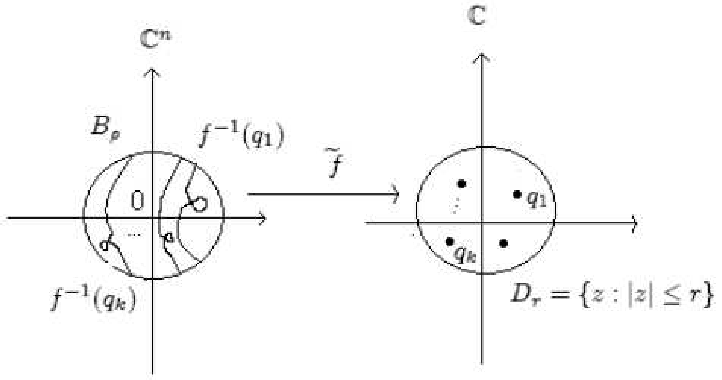

FIGURE 2. The structure of $\tilde{f}$.

From Theorems 1 and 3 it immediately follows that $\mu(f_0) = k(f_0)$.

4. Monodromy representation of a fiber bundle

For an oriented smooth manifold with boundary X, there are several topological groups, which are relevant to the definition of the monodromy representation of a fiber bundle. First, the group $\mathrm{Diff}^+(X)$ formed by the orientation preserving diffeomorphisms of X, under composition. Second, the subgroup $\mathrm{Isot}(X)$ of $\mathrm{Diff}^+(X)$ formed by those diffeomorphisms which are isotopic to the identity diffeomorphism though elements of $\mathrm{Diff}^+(X)$. It can be seen that $\mathrm{Isot}(M)$ is a normal subgroup of $\mathrm{Diff}^+(X)$. The third relevant group is the quotient $\mathcal{M}(X) := \mathrm{Diff}^+(X)/\mathrm{Isot}(X)$, called the *mapping class group of* X. The fourth one is the subgroup $\mathrm{Diff}^+(X, \partial X)$ of $\mathrm{Diff}^+(X)$ formed by those elements whose restriction to the boundary ∂X of X is equal to the identity map. Finally, it can be seen that $\mathrm{Diff}^+(X, \partial X) \cap \mathrm{Isot}(X)$ is a normal subgroup of $\mathrm{Diff}^+(X, \partial X)$, and their quotient, which we shall denote by $\mathcal{M}(X, \partial X)$, is called the *mapping class group of* X *relative to* ∂X. An elementary group theory argument shows that $\mathcal{M}(X, \partial X)$ injects canonically into $\mathcal{M}(X)$ so it can be regarded as a subgroup of $\mathcal{M}(X)$.

Now, let $f : E \longrightarrow B$ be an oriented smooth fiber bundle with fiber F, a smooth oriented manifold with (possibly empty) boundary. Choose an *orientation preserving good* trivialization $\mathcal{T} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ for f. (This means that the collection $\{U_\alpha\}_{\alpha\in A}$ is a *good* open covering of B, that is one such that the intersection of any finite subcollection of it is diffeomorphic to $\mathbb{R}^m$, $m = \dim B$, and each ϕ_α is a fiber preserving diffeomorphism from $f^{-1}(U_\alpha)$ to $U_\alpha \times F$, which also preserves the orientation fiberwise.) Let $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Diff}^+(F)$ be the cocycle determined by the trivialization $\mathcal{T}$. If we fix a base point $x_0 \in B$ for the fundamental group $\pi_1(x_0, B)$ and a pair $U_{\alpha_0}, \phi_{\alpha_0}$ in $\mathcal{T}$ such that $x_0 \in U_{\alpha_0}$, the *monodromy representation of* $f : E \to B$ is the anti-homomorphism $\lambda : \pi_1(x_0, B) \to \mathcal{M}(F)$ defined in the following way. Take any loop γ based at x_0 and divide it into arcs $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that γ_0 has initial point x_0, γ_i ends where γ_{i+1} begins, for $i = 0, \ldots, n-1$, and γ_n ends at x_0, and for which there exist open sets $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ in $\{U_\alpha\}_{\alpha\in A}$ with $\gamma_i \subset U_{\alpha_i}$ for each $i = 0, \ldots, n$. See Figure 3. Let us denote the starting point of each γ_i by x_i.

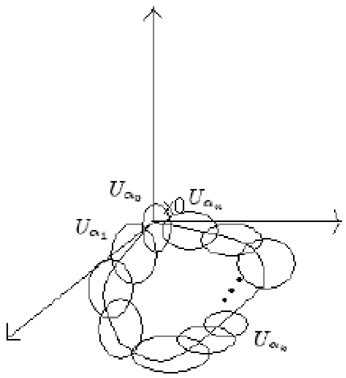

FIGURE 3. Trivialization of the bundle along γ.

Now, if $g_{i+1,i}$ denotes the transition function on $U_{\alpha_i} \cap U_{\alpha_{i+1}}$ for $i = 0, \ldots, n-1$, and $g_{0,n}$ denotes the transition function on $U_{\alpha_n} \cap U_{\alpha_0}$, then the image of the class $[\gamma]$ in $\pi_1(x_0, B)$ is defined as the element in $\mathcal{M}(F)$ determined by the diffeomorphism

$$\lambda([\gamma]) = g_{0,n}(x_0) g_{n,n-1}(x_n) \cdots g_{2,1}(x_2) g_{1,0}(x_1)$$

in $\mathrm{Diff}^+(F)$. It can be proved that λ is a well defined anti-homomorphism.

According to Theorem 1, the map $f : E^* \to D_r^*$ is a smooth oriented fiber bundle whose fiber is a smooth compact oriented $(2n-2)$-manifold with boundary. We can take the model fiber to be $f^{-1}(z_0) = X_{z_0}$, the fiber over an arbitrarily chosen point $z_0 \in D_r^*$. It can be seen that in this case it is possible to choose an orientation preserving good trivialization of the bundle such that all transition functions have range in the subgroup $\mathcal{M}(X_{z_0}, \partial X_{z_0})$ of $\mathcal{M}(X_{z_0})$.

For the rest of the article the holomorphic function f will have domain W, an open set in $\mathbb{C}^2$, and the fiber bundles considered will have a real four-dimensional manifold as total space and a real 2-dimensional manifold as base.

5. Dehn twists

Let $A = \{z : 1 \leq |z| \leq 2\}$ be an annulus in the complex plane with the standard orientation. The *right-handed Dehn twist in A* is the element D of $\mathrm{Diff}^+(A)$ defined as $D(re^{i\theta}) = re^{i(\theta - 2\pi\psi(r))}$, where $\psi : [1,2] \to \mathbb{R}$ is a fixed smooth function which is constantly zero on the interval $[1, 1+1/3]$, monotone increasing in the interval $[1+1/3, 2-1/3]$, and constantly 1 on the interval $[2-1/3, 2]$. Let now F be any oriented smooth 2-manifold, possibly with boundary, and α a simple closed curve in $int(F)$, that is, a smoothly embedded circle. If we take the closure $\overline{T}_\alpha$ of a tubular neighborhood of the 1-submanifold α and an orientation preserving diffeomorphism $g : A \to \overline{T}_\alpha$, then *a right-handed Dehn twist around α* is the element D_α of $\mathrm{Diff}^+(F)$ defined as

$$D_\alpha(p) = \begin{cases} gDg^{-1}(p) & \text{if } p \text{ is in } \overline{T}_\alpha \\ p & \text{if } p \text{ is in } F - \overline{T}_\alpha \end{cases} .$$

See Figure 4.

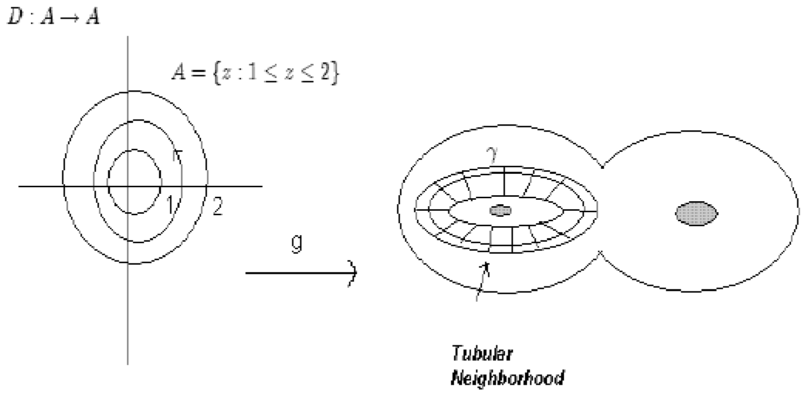

Figure 4. A right-handed Dehn twist on an oriented surface.

It can be proven that the isotopy class of D_α in $\mathrm{Diff}^+(F)$ does not depend on either the chosen neighborhood or the diffeomorphism g. Moreover, if α is isotopic to β (as embeddings of S^1) then D_α and D_β determine the same elements of $\mathcal{M}(F)$. All of this is true if the D_α are considered as elements of $\mathrm{Diff}^+(F, \partial F)$ [2].

6. Morsification

Let E be a connected complex surface and let $D_r^\circ = \{z \in \mathbb{C} : |z| < r\}$ be the open disk of radius $r > 0$ in the complex plane. Let $f : E \to D_r^\circ$ be a proper holomorphic map such that $f^{-1}(D_r^\circ - \{0\})$ does not contain any critical point of f.

Definition 4. *By a* deformation *of $f : E \to D_r^\circ$ we shall mean a surjective proper holomorphic map $F : \mathcal{S} \to D_r^\circ \times \Delta_\epsilon$, where $\mathcal{S}$ is a three-dimensional complex manifold and $\Delta_\epsilon = \{z \in \mathbb{C} : |z| < \epsilon\}$, and such that*

(1) *The composition $\mathcal{S} \xrightarrow{F} D_r^\circ \times \Delta_\epsilon \xrightarrow{pr_2} \Delta_\epsilon$ does not have critical points.*
(2) *If $D_t^\circ := D_r^\circ \times \{t\}$, $\mathcal{S}_t := F^{-1}(D_t^\circ)$ and $f_t := F|_{\mathcal{S}_t} : \mathcal{S}_t \to D_t^\circ$ then the maps $f : E \to D_r^\circ$ and $f_0 : \mathcal{S}_0 \to D_0^\circ$ are topologically equivalent.*

Furthermore, the deformation $F : \mathcal{S} \to D_r^\circ \times \Delta_\epsilon$ is called a morsification *of the map $f : E \to D_r^\circ$ if for any $t \neq 0$, each singular fiber of the map $f_t : \mathcal{S}_t \to D_t^\circ$ is of simple Lefschetz type, that is, it contains a single nodal singularity or smooth multiple (see* [8]*). The morsification is called a* simple morsification *if for each $t \neq 0$, all the singular fibers of f_t are of simple Lefschetz type.*

An Euler characteristic invariance argument shows that the number of singular fibers of any map f_t with $t \neq 0$ of any simple morsification F of f is independent of both the simple morsification F and the map f_t chosen.

For instance, let us consider the map $f : E = f^{-1}(D_r) \cap B_{\rho_0} \to D_r$ treated in the first section. It is possible to obtain a deformation of this map in the following way. Fix any nonzero complex linear function $\lambda : \mathbb{C}^2 \to \mathbb{C}$ and consider the function $F : W \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ defined as $F(z_1, z_2, t) = (f(z_1, z_2) + t\lambda(z_1, z_2), t)$. It is possible to choose an $\epsilon > 0$ so that its restriction $F' : B_{\rho_0} \times \Delta_\epsilon \to \mathbb{C} \times \Delta_\epsilon$ satisfies that $F'(\cdot, t) : B_{\rho_0} \to D_r$ is surjective for each $t \in \Delta_\epsilon$. Let $\mathcal{E}$ be the three-dimensional complex manifold $(F')^{-1}(D_r) \subset B_{\rho_0} \times \Delta_\epsilon$. Then the map $F' : \mathcal{E} \to D_r \times \Delta_\epsilon$ is a deformation of $f : E \to D_r \times \Delta_\epsilon$. It can also be proved that for a generic choice of λ, this deformation is a morsification such that for each $t \neq 0$, the map $(f')_t$ has only simple Lefschetz type singular fibers.

It is important to point out that simple morsifications are known to exist in some cases. Such is the case where the regular fiber of f is a closed oriented surface of genus 1 in which case a theorem of Moishezon [8] guarantees that a simple morsification exists.

7. A new interpretation of the Milnor number

In this section we will propose a new interpretation of the Milnor number. We will work in dimension $n = 2$, and regard f as a holomorphic function from $E = f^{-1}(D_r) \cap B_\rho$ to D_r as in Theorem 1. Let E^* denote the complement in E of the central fiber $X_0 = f^{-1}(0)$, and C_r denote the boundary circle of the closed disk D_r. We fix a base point $z_0 \in C_r$. As stated in Theorem 1, $f : E^* \to D_r^*$ is a smooth oriented fiber bundle over the punctured disk. Since $n = 2$, each fiber X_z, $z \neq 0$ is an oriented surface with boundary. Let us denote by T the union of the boundaries of all the fibers X_z, $T = \cup_{z \in D_r} \partial X_z$, and let P be $f^{-1}(C_r)$. See Figure 5.

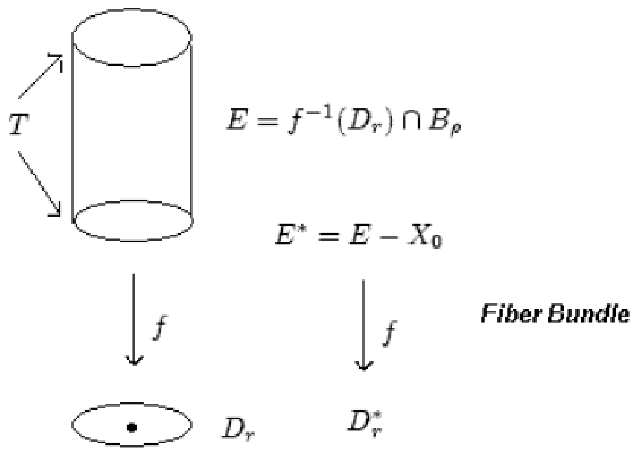

FIGURE 5. Milnor's picture seen abstractly.

Then it is easy to verify that P and T are smooth manifolds with boundary such that $P \cap T = \partial P = \partial T$, $P \cup T = \partial E$, and f restricted to P induces a (usually nontrivial) fiber bundle $f|_P : P \to C_r$. As remarked at the end of section 6 the monodromy representation of $f : E^* \to D_r^*$ can be taken to have image in the subgroup $\mathcal{M}(X_{z_0}, \partial X_{z_0})$ of $\mathcal{M}(X_{z_0})$.

Now, let $F : \mathcal{S} \to D_r \times \Delta_\epsilon$ be a simple morsification of f and let us fix a $t_0 \neq 0$ in Δ_ϵ. If $Q = \{z_1, \ldots, z_k\}$ is the set of critical values of f_{t_0}, then the restriction $f_{t_0} : \mathcal{S}_{t_0} - f_{t_0}^{-1}(Q) \to D_{t_0} - Q$ is a fiber bundle. Let us denote $f_{t_0}^{-1}(C_r)$ by P_{t_0}. This is a smooth 3-manifold and the restriction $f_{t_0} : P_{t_0} \to C_r$ fibers it over a circle. It can be proved that $f : P \to C_r$ and $f_{t_0} : P_{t_0} \to C_r$ are equivalent as smooth maps, i.e. that there exist orientation preserving diffeomorphisms φ_P and φ_{C_r} such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi_P} & P_{t_0} \\ f \downarrow & & f_{t_0} \downarrow \\ C_r & \xrightarrow{\varphi_{C_r}} & C_r \end{array}$$

commutes. This implies that if we identify $f^{-1}(z_0)$ and $f_{t_0}^{-1}(z_0)$ via an orientation preserving diffeomorphism, and induce in this way an identification between the groups $\mathcal{M}(f^{-1}(z_0), \partial f^{-1}(z_0))$ and $\mathcal{M}(f_{t_0}^{-1}(z_0), \partial f_{t_0}^{-1}(z_0))$, then the images of the element of $\pi_1(z_0, C_r)$ represented by the loop based traversing once and positively C_r, under the monodromy representations of the bundles $f : P \to C_r$ and $f_{t_0} : P_{t_0} \to C_r$, are conjugate as elements of $\mathcal{M}\left(f^{-1}(z_0), \partial f^{-1}(z_0)\right)$. Now, let us denote by $[C_r]$ the element in $\pi_1(z_0, D_{t_0} - Q)$ represented by the loop traversing once and positively the circle C_r. Let us choose mutually disjoint small closed disks $D_1, \dots, D_k$ contained in D_r and such that each D_i is centered at z_i. Now choose Jordan arcs $\gamma_1, \dots, \gamma_k$ contained in $D_r - \cup D_i$ such that each γ_i begins at z_0 and ends at some point z_i' on ∂D_i, and $\gamma_i \cap \gamma_j = \{z_0\}$ for $i \neq j$. Let us denote by α_i the loop that starts at z_0, then traverses γ_i until it reaches z_i', then goes once and positively around ∂D_i, and finally comes back to z_0 again following γ_i. After a renumbering (if necessary) it can be assumed that $[\alpha_1] \dots [\alpha_k] = [C_r]$. See Figure 6.

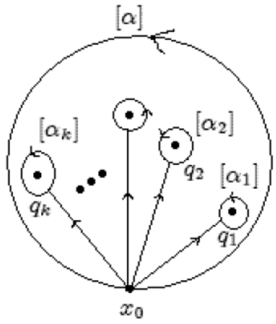

FIGURE 6. The system of curves $\alpha_1, \dots, \alpha_k$ on the disk.

Since

$$\lambda_{t_0} : \pi_1(z_0, D_r - Q) \to \mathcal{M}(f^{-1}(z_0), \partial f^{-1}(z_0)),$$

is an anti-homomorphism, we have that

$$\begin{aligned} \lambda_{t_0}([C_r]) &= \lambda_{t_0}([\alpha_1] \dots [\alpha_k]) && (1) \\ &= \lambda_{t_0}([\alpha_k]) \dots \lambda_{t_0}([\alpha_1]). && (2) \end{aligned}$$

It is known that the local monodromy around a singular fiber of simple Lefschetz type is a right-handed Dehn twist. This implies that each factor $\lambda_{t_0}([\alpha_i])$ in the last member of equation (1) is a right-handed Dehn twist in $\mathcal{M}(f^{-1}(z_0), \partial f^{-1}(z_0))$. Consequently, the Milnor number of the germ f_0 is equal to the number of factors in a factorization of $\lambda([C_r])$ obtained from any simple morsification of f_0 by the process we have just described. The new interpretation of Milnor's number we propose is expressed in the following conjecture.

Conjecture 1. *The number of factors in any factorization of* $\lambda([C_r]) \in \mathcal{M}\left(f^{-1}(z_0), \partial f^{-1}(z_0)\right)$ *in terms of right handed Dehn twists is bigger than or equal to the Milnor number of* f_0.

Milnor's number can be generalized in the following way. Let $f : E \to D_r^\circ$ be as described in section 6. Let us assume that it admits a simple morsification $F : \mathcal{S} \to D_r^\circ \times \Delta_\epsilon$, and let $t_0 \neq 0$ in Δ_ϵ. Then the last discussion is valid in this more general setting. In particular, according to the comment made right after Definition 4 the number of singular fibers in any member of any simple morsification is the same. So it is an invariant of the "fiber germ" of f at the fiber $f^{-1}(0)$, and therefore the number of factors in a factorization of the monodromy around $f^{-1}(0)$ obtained in this way can be reasonably called the Milnor number of the fiber germ.

The authors have also conjectured that this generalized Milnor number is actually a lower bound for the number of factors of *any* factorization of $\lambda([C_r])$ in terms of right-handed Dehn twists in the mapping class group of $f^{-1}(z_0)$ relative to its (possibly empty) boundary $\partial f^{-1}(z_0)$, attained by those factorizations arising from simple morsifications. This conjecture has recently been confirmed by the authors in the case when the regular fiber of f is a closed (i.e. compact and without boundary) connected 2-manifold with genus 1 (see [3]). However, in ([5]), that author provides for each $g \geq 2$ an $f_g : E_g \to D_1^0$ having genus g closed regular fiber and violating this conjecture.

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