Chandrasekhar ansatz and the generalized total angular momentum operator for the Dirac equation in the Kerr-Newman metric

El ansatz de Chandrasekhar y el operador generalizado del momento angular para la ecuación de Dirac en la métrica de Kerr-Newman

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Abstract. In this paper we compute the square root of the generalized squared total angular momentum operator J for a Dirac particle in the Kerr-Newman metric. The separation constant λ arising from the Chandrasekahr separation ansatz turns out to be the eigenvalue of J. After proving that J is a symmetry operator, we show the completeness of Chandrasekhar ansatz for the Dirac equation in oblate spheroidal coordinates and derive an explicit formula for the time evolution operator e^{-itH}.

Key words and phrases. Dirac equation, Kerr-Newman metric, general relativity.

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Resumen. En este trabajo derivamos la raíz cuadrada del operador generalizado del momento angular para una partícula de Dirac en la métrica de Kerr-Newman. La constante de separación λ introducida por el ansatz de Chandrasekhar resulta ser el valor propio de J. Después de haber mostrado que J es un operador de simetría, probamos la completitud del ansatz de Chandrasekhar para la ecuación de Dirac en coordenadas esferoidales oblongas y derivamos una expresión analítica para el operador de evolución temporal e^{-itH}.

Palabras y frases clave. Ecuación de Dirac, métrica de Kerr-Newman, relatividad general.

1. **Introduction**

In the Newman-Penrose formalism [16] in two-component spinor notation, the Dirac equation coupled to a general gravitational field and a 4-vector field $\mathbf{V}$ is given in Planck units $\hbar = c = G = 1$ by [18]

$$(\nabla_{AA'} + ieV_{AA'})P^A + i\mu_* Q_{A'}^* = 0 \tag{1}$$

$$(\nabla_{AA'} - ieV_{AA'})Q^A + i\mu_* P_{A'}^* = 0, \tag{2}$$

where $\nabla_{AA'}$ is the symbol for covariant differentiation, e is the charge or coupling constant of the Dirac particle to the vector field $\mathbf{V}$, μ_* is the particle mass m_e divided by $\sqrt{2}$ and P^A and Q^A are the two-component spinors representing the wave function. Here, the asterisk used as a superscript denotes complex conjugation. Notice that the factor $2^{-\frac{1}{2}}$ in the definition of the mass μ_* is due to the fact, that the Pauli matrices as defined in the Newman-Penrose formalism differ from their usual definitions by the factor $\sqrt{2}$. Moreover, the vector potential enters with opposite signs in (1) and (2) in order that gauge invariance be preserved, since the spinors in the above equations are related by complex conjugation.

According to [16], we denote with ξ_a^A a basis for the spinor space and with $\xi_{a'}^{*\ A'}$ a basis for the conjugate spinor space. To the spinor basis we can associate at each point of the space-time a null tetrad $(\mathbf{l}, \mathbf{n}, \mathbf{m}, \mathbf{m}^*)$ obeying the orthogonality relations $\mathbf{l} \cdot \mathbf{n} = 1$, $\mathbf{m} \cdot \mathbf{m}^* = -1$ and $\mathbf{l} \cdot \mathbf{m} = \mathbf{l} \cdot \mathbf{m}^* = \mathbf{n} \cdot \mathbf{m} = \mathbf{n} \cdot \mathbf{m}^* = 0$. Furthermore, the covariant derivative of a spinor ξ^A can be expressed in terms of its components along the spinor basis ξ_a^A as follows [16], [19]

$$\xi_a^A \xi_{a'}^{A'} \xi_b^B \nabla_{AA'} \xi^B = \partial_{aa'} \xi^b + \Gamma_{caa'}^b \xi^c,$$

where $\Gamma_{caa'}^b$ are the spin coefficients and

$$\partial_{00'} := D = l^b \frac{\partial}{\partial x^b}, \qquad \partial_{11'} := \widetilde{\Delta} := n^b \frac{\partial}{\partial x^b}, \tag{3}$$

$$\partial_{01'} := \delta = m^b \frac{\partial}{\partial x^b}, \qquad \partial_{10'} := \delta^* := m^{*b} \frac{\partial}{\partial x^b}, \tag{4}$$

are the directional derivatives along $\mathbf{l}$, $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{m}^*$. Following [18] and letting

$$P^0 =: F_1, \quad P^1 =: F_2, \quad Q^{*1'} =: G_1, \quad Q^{*0'} =: -G_2,$$

equations (1) and (2) can be brought into the form

$$\mathcal{O}_D \Psi = 0, \tag{5}$$

where $\mathcal{O}_D$ is given by

$$\mathcal{O}_D = \begin{pmatrix} I_{m_e} & B_1 \\ B_2 & I_{m_e} \end{pmatrix}, \tag{6}$$

where

$$I_{m_e} = \begin{pmatrix} -im_e & 0 \\ 0 & -im_e \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \widetilde{\Delta} + \mu^* - \gamma^* + ieV_b n^b & -\left(\delta^* + \beta^* - \tau^* + ieV_b m^b\right) \\ -\left(\delta + \pi^* - \alpha^* + ieV_b m^b\right) & D + \epsilon^* - \rho^* + ieV_b l^b \end{pmatrix},$$

$$B_2 = \begin{pmatrix} D + \epsilon - \rho + ieV_b l^b & \delta^* + \pi - \alpha + ieV_b m^{*b} \\ \delta + \beta - \tau + ieV_b m^{*b} & \widetilde{\Delta} + \mu - \gamma + ieV_b n^b \end{pmatrix},$$

and $\Psi = (F_1, F_2, G_1, G_2)^T$. In what follows, we consider the Dirac equation in the Kerr-Newman metric, i.e. in the presence of a rotating charged black hole. In Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$ the Kerr-Newman metric is given by [4]

$$ds^2 = \frac{\Delta}{\Sigma} \left(dt - a\sin^2\vartheta d\varphi\right)^2 - \Sigma\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) - \frac{\sin^2\vartheta}{\Sigma}\left[adt - \left(r^2 + a^2\right)d\varphi\right]^2 \tag{7}$$

where

$$\Sigma := \Sigma(r, \theta) = r^2 + a^2\cos^2\theta, \quad \Delta := \Delta(r) = r^2 - 2Mr + a^2 + Q^2,$$

where M, a and Q are the mass, the angular momentum per unit mass and the charge of the black hole, respectively. In the non-extreme case $M^2 > a^2 + Q^2$ the function Δ has two distinct zeros, namely,

$$r_0 = M - \sqrt{M^2 - a^2 - Q^2} \quad \text{and} \quad r_1 = M + \sqrt{M^2 - a^2 - Q^2},$$

the first one corresponding to the Cauchy horizon and the second to the event horizon, while in the extreme case $M^2 = a^2 + Q^2$ Cauchy horizon and event horizon coincide, since Δ has two double roots at $r_1^* = M$. Throughout our work we will consider the case $M^2 > a^2 + Q^2$ and restrict our attention to the region $r > r_1$ outside the event horizon. Hence, Δ will be always positive. Notice that by setting $M = Q = 0$ in (7) the Kerr-Newman metric goes over into the Minkowski metric in oblate spheroidal coordinates (OSC), namely

$$ds^2 = dt^2 - \frac{\Sigma}{\hat{\Delta}}dr^2 - \hat{\Delta}\left(\frac{\Sigma}{\hat{\Delta}}d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{8}$$

with $\hat{\Delta} = r^2 + a^2$. In fact, by means of the coordinate transformation

$$x = \sqrt{r^2 + a^2}\sin\vartheta\cos\varphi, \quad y = \sqrt{r^2 + a^2}\sin\vartheta\sin\varphi, \quad z = r\cos\vartheta. \tag{9}$$

(8) can be reduced to the Minkowski metric in cartesian coordinates. Moreover, in the OSC the surfaces $r =$const are confocal ellipsoids, while the surfaces $\vartheta =$const are represented by hyperbolids of one sheet, confocal to the ellipsoids. Since sgn$(z) =$sgn$(\cos\vartheta)$ with $\vartheta \in [0, \pi]$, the surface $\vartheta =$const is actually

given by a half-hyperboloid, truncated at its waist and laying in the half-space $z \lesseqgtr 0$ according as $\vartheta \lesseqgtr \frac{\pi}{2}$. At this point it is interesting to observe that at $r = 0$ the ellipsoids degenerate to the disk $x^2 + y^2 = a^2 \sin^2 \vartheta$, whose boundary $\{(r, \vartheta, \varphi)|r = 0, \vartheta = \frac{\pi}{2}\}$ (and hence $\Sigma = 0$) corresponds to the set of points at which the Kerr-Newman metric becomes truly singular. Furthermore, the surfaces $\varphi =$const look like bent planes, which are approximately vertical for large r, but flatten out and become horizontal at the edge of the disk. For further details we refer to [9] and [3].

In order to write out explicitely (5) for the Kerr-Newman metric in Boyer-Lindquist coordinates, we make use of the Kinnersley tetrad [13], namely

$$l^b = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right) \tag{10}$$

$$n^b = \frac{1}{2\Sigma} \left(r^2 + a^2, -\Delta, 0, a \right) \tag{11}$$

$$m^b = \frac{1}{\sqrt{2}(r + ia \cos \vartheta)} \left(ia \sin \vartheta, 0, 1, i \csc \vartheta \right). \tag{12}$$

Then, the non vanishing spin coefficients in (6) are [2]

$$\rho = -\frac{1}{r - ia\cos\vartheta}, \quad \beta = \frac{\rho^* \cot \vartheta}{2\sqrt{2}}, \quad \mu = \frac{\Delta\rho}{2\Sigma},$$

$$\pi = \frac{ia}{\sqrt{2}}\rho^2 \sin\vartheta, \quad \tau = -\frac{ia\sin\vartheta}{\sqrt{2}\Sigma}, \quad \gamma = \mu + \frac{r-M}{2\Sigma},$$

and

$$\alpha = \pi - \beta^* = \frac{\rho}{\sqrt{2}} \left(ia\rho \sin\vartheta + \frac{1}{2}\cot\vartheta \right),$$

where ρ^* denotes the complex conjugate of ρ. Notice that while the vanishing of ϵ is due to the particular choice of the tetrad (10), (11), (12), the other spin coefficients κ, σ, λ and ν are zero according to the Goldberg-Sachs theorem [11]. By means of (3) and (4) the directional derivatives along the tetrad are computed to be

$$D = D_0, \quad D_0 := \frac{\partial}{\partial r} + \frac{1}{\Delta}\left[(r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \varphi} \right],$$

$$\widetilde{\Delta} = -\frac{\Delta\hat{D}_0}{2\Sigma}, \quad \hat{D}_0 := \frac{\partial}{\partial r} - \frac{1}{\Delta}\left[(r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \varphi} \right],$$

$$\delta = -\frac{\rho^* \hat{\mathcal{L}}_0}{\sqrt{2}}, \quad \mathcal{L}_0 := \frac{\partial}{\partial \vartheta} - i\left(a\sin\vartheta\frac{\partial}{\partial t} + \csc\vartheta\frac{\partial}{\partial \varphi} \right),$$

$$\delta^* = -\frac{\rho\mathcal{L}_0}{\sqrt{2}}, \quad \hat{\mathcal{L}}_0 := \frac{\partial}{\partial \vartheta} + i\left(a\sin\vartheta\frac{\partial}{\partial t} + \csc\vartheta\frac{\partial}{\partial \varphi} \right).$$

From [15] we know that the vector potential for a rotating charged black hole is

$$V_b = \frac{1}{\Sigma}(-Qr, 0, 0, aQr\sin^2\vartheta),$$

and therefore

$$V_b l^b = -\frac{rQ}{\Delta}, \quad V_b n^b = -\frac{rQ}{2\Sigma}, \quad V_b m^b = V_b m^{*b} = 0.$$

Notice that the vector potential $\mathbf{V}$ is stationary and axialsymmetric. Regarding the elements of the matrix operator $\mathcal{O}_D$, we find that

$$\widetilde{\Delta} + \mu^* - \gamma^* + ieV_b n^b = -\frac{\Delta}{\sqrt{2}\Sigma}\left(\hat{D}_0 + \frac{r-M}{\Delta} + i\frac{eQr}{\Delta}\right)$$

$$-\left(\delta^* + \beta^* - \tau^* + ieV_b m^b\right) = \rho\left(\mathcal{L}_0 + \frac{1}{2}\cot\vartheta + ia\rho^*\sin\vartheta\right)$$

$$-\left(\delta + \pi^* - \alpha^* + ieV_b m^b\right) = \rho^*\left(\hat{\mathcal{L}}_0 + \frac{1}{2}\cot\vartheta\right)$$

$$D + \epsilon^* - \rho^* + ieV_b l^b = \sqrt{2}\left(D_0 - \rho^* - i\frac{eQr}{\Delta}\right)$$

$$D + \epsilon - \rho + ieV_b l^b = \sqrt{2}\left(D_0 - \rho - i\frac{eQr}{\Delta}\right)$$

$$\delta^* + \pi - \alpha + ieV_b m^{*b} = -\rho\left(\mathcal{L}_0 + \frac{1}{2}\cot\vartheta\right)$$

$$\delta + \beta - \tau + ieV_b m^{*b} = -\rho^*\left(\hat{\mathcal{L}}_0 + \frac{1}{2}\cot\vartheta - ia\rho\sin\vartheta\right)$$

$$\widetilde{\Delta} + \mu - \gamma + ieV_b n^b = -\frac{\Delta}{\sqrt{2}\Sigma}\left(\hat{D}_0 + \frac{r-M}{\Delta} + i\frac{eQr}{\Delta}\right).$$

According to [5] we may replace the Dirac equation $\mathcal{O}_D\Psi(t,r,\vartheta,\varphi) = 0$ by a modified but equivalent equation

$$\mathcal{W}\hat{\psi}(t,r,\vartheta,\varphi) = 0 \quad \text{with} \quad \mathcal{W} = \Gamma S^{-1}\mathcal{O}_D S, \tag{13}$$

where $\hat{\psi} = S^{-1}\Psi = (\hat{F}_1\ \hat{F}_2\ \hat{G}_1\ \hat{G}_2)^T$ and Γ and S are non singular 4×4 matrices, whose elements may depend on the variables r and ϑ.

2. **The decomposition of the operator $\mathcal{W}$**

In this section we show that it is possible to find non singular matrices S and Γ such that the operator $\mathcal{W}$ decomposes into the sum of an operator containing only derivatives respect to the variables t, r and φ and of an operator involving only derivatives respect to t, ϑ and φ. There are several choices of S and Γ, which accomplish this. Here, we look for matrices S and Γ such that theysimplify later on the transformation of equation (13) into its Schrödinger form.

Lemma 2.1. *Let r be positive with $r > r_1$. There exist a non singular 4×4 matrix*

$$S = c\,\text{diag}\left(-\rho, 2^{\frac{1}{2}}\Delta^{-\frac{1}{2}}, 2^{\frac{1}{2}}\Delta^{-\frac{1}{2}}, -\rho^*\right), \quad \det(S) = \frac{c^4}{\Sigma\Delta},$$

with $0 \neq c \in \mathbb{C}$ and a non singular 4×4 matrix

$$\Gamma = \text{diag}\left(\rho^{*-1}, -\rho^{*-1}, -\rho^{-1}, \rho^{-1}\right), \quad \det(\Gamma) = \Sigma,$$

such that

$$\mathcal{W} = \mathcal{W}_{(t,r,\varphi)} + \mathcal{W}_{(t,\vartheta,\varphi)} \tag{14}$$

with

$$
rcl\mathcal{W}_{(t,r,\varphi)} = \begin{pmatrix} im_e r & 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 \\ 0 & -im_e r & 0 & \sqrt{\Delta}\mathcal{D}_- \\ \sqrt{\Delta}\mathcal{D}_- & 0 & -im_e r & 0 \\ 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 & im_e r \end{pmatrix}, \tag{15}
$$

$$
\mathcal{W}_{(t,\vartheta,\varphi)} = \begin{pmatrix} -am_e\cos\vartheta & 0 & 0 & \mathcal{L}_+ \\ 0 & am_e\cos\vartheta & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & -am_e\cos\vartheta & 0 \\ -\mathcal{L}_- & 0 & 0 & am_e\cos\vartheta \end{pmatrix} \tag{16}
$$

where

$$\mathcal{D}_\pm := \frac{\partial}{\partial r} \mp \frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi} - ieQr\right],$$

$$\mathcal{L}_\pm := \frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta \mp i\left(a\sin\vartheta\frac{\partial}{\partial t} + \csc\vartheta\frac{\partial}{\partial\varphi}\right).$$

Proof. Let us define the matrix

$$S := \text{diag}(h(r,\vartheta), \Lambda(r,\vartheta), \sigma(r,\vartheta), \gamma(r,\vartheta)),$$

such that $\det(S) \neq 0$ and h, Λ, σ, γ at least $\mathcal{C}^1((r_1,+\infty)\times[0,\pi])$. The equation $\mathcal{O}_D S\hat{\psi} = 0$ gives rise to the system

$$0 = -im_e h\hat{F}_1 + \left(\widetilde{\Delta} + \mu^* - \gamma^* + ieV_b n^b\right)\sigma\hat{G}_1 - \left(\delta^* + \beta^* - \tau^* + ieV_b m^b\right)\gamma\hat{G}_2, \tag{17}$$

$$0 = -im_e\Lambda\hat{F}_2 - \left(\delta + \pi^* - \alpha^* + ieV_b m^b\right)\sigma\hat{G}_1 + \left(D + \epsilon^* - \rho^* + ieV_b l^b\right)\gamma\hat{G}_2, \tag{18}$$

$$0 = -im_e\sigma\hat{G}_1 + \left(D + \epsilon - \rho + ieV_b l^b\right)h\hat{F}_1 + \left(\delta^* + \pi - \alpha + ieV_b m^{*b}\right)\Lambda\hat{F}_2, \tag{19}$$

$$0 = -im_e\gamma\hat{G}_2 + \left(\delta + \beta - \tau + ieV_b m^{*b}\right)h\hat{F}_1 + \left(\widetilde{\Delta} + \mu - \gamma + ieV_b n^b\right)\Lambda\hat{F}_2. \tag{20}$$

Starting with (17), we find that

$$\left(\widetilde{\Delta} + \mu^* - \gamma^* + ieV_b n^b\right) \sigma \hat{G}_1 = -\frac{\Delta}{\sqrt{2}\Sigma}\left(\sigma\mathcal{D}_+ + \frac{\partial\sigma}{\partial r} + \frac{r-M}{\Delta}\sigma\right)\hat{G}_1, \quad (21)$$

$$-\left(\delta^* + \beta^* - \tau^* + ieV_b m^b\right)\gamma\hat{G}_2 = \rho\left(\gamma\mathcal{L}_+ + \frac{\partial\gamma}{\partial\vartheta} + ia\rho^*\gamma\sin\vartheta\right)\hat{G}_2, \quad (22)$$

with

$$\mathcal{D}_+ := \frac{\partial}{\partial r} - \frac{1}{\Delta}\left[(r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi} - ieQr\right],$$

$$\mathcal{L}_+ := \frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta - i\left(a\sin\vartheta\frac{\partial}{\partial t} + \csc\vartheta\frac{\partial}{\partial\varphi}\right). \quad (23)$$

In order to obtain $\mathcal{W}$ as in the statement of this Lemma, we impose in (21) that

$$\frac{\partial\sigma}{\partial r} + \frac{r-M}{\Delta}\sigma = 0,$$

whose solution is given by $\sigma(r,\vartheta) = \widetilde{\sigma}(\vartheta)\Delta^{-\frac{1}{2}}$ with $\widetilde{\sigma}(\vartheta) \neq 0$ for every $\vartheta \in [0,\pi]$, while in (22) we require that

$$\frac{\partial\gamma}{\partial\vartheta} + ia\rho^*\sin\vartheta\,\gamma = 0,$$

whose solution is $\gamma(r,\vartheta) = -\widetilde{\gamma}(r)\rho^*$ with $\widetilde{\gamma}(r) \neq 0$ for every $r \in (r_1, +\infty)$. Hence, (21) and (22) simplify to

$$\left(\widetilde{\Delta} + \mu^* - \gamma^* + ieV_b n^b\right)\sigma\hat{G}_1 = -\frac{\sqrt{2}\Delta^{\frac{1}{2}}\widetilde{\sigma}(r)}{2\Sigma}\mathcal{D}_+\hat{G}_1 \quad (24)$$

$$-\left(\delta^* + \beta^* - \tau^* + ieV_b m^b\right)\gamma\hat{G}_2 = -\frac{\widetilde{\gamma}(r)}{\Sigma}\mathcal{L}_+\hat{G}_2. \quad (25)$$

Regarding (18), we have

$$-\left(\delta + \pi^* - \alpha^* + ieV_b m^b\right)\sigma\hat{G}_1 = \frac{\rho^*}{\Delta^{\frac{1}{2}}}\left(\widetilde{\sigma}\mathcal{L}_- + \frac{d\widetilde{\sigma}}{d\vartheta}\right)\hat{G}_1, \quad (26)$$

$$\left(D + \epsilon^* - \rho^* + ieV_b l^b\right)\gamma\hat{G}_2 = \sqrt{2}\left(\gamma\mathcal{D}_- + \frac{\partial\gamma}{\partial r} - \rho^*\gamma\right)\hat{G}_2, \quad (27)$$

where

$$\mathcal{D}_- := \frac{\partial}{\partial r} + \frac{1}{\Delta}\left[(r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi} - ieQr\right],$$

$$\mathcal{L}_- := \frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta + i\left(a\sin\vartheta\frac{\partial}{\partial t} + \csc\vartheta\frac{\partial}{\partial\varphi}\right).$$

By imposing $\widetilde{\sigma}(\vartheta) = c_3$ with some constant $0 \neq c_3 \in \mathbb{C}$ and observing that $\gamma(r,\vartheta) = -c_4\rho^*$ with $0 \neq c_4 \in \mathbb{C}$ sets the last two terms in the parenthesis on

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the r.h.s. of (27) equal to zero, (24), (25), (26) and (27) become

$$\left(\widetilde{\Delta} + \mu^* - \gamma^* + ieV_b n^b\right)\sigma\hat{G}_1 = -\frac{c_3\sqrt{2}\Delta^{\frac{1}{2}}}{2\Sigma}\mathcal{D}_+\hat{G}_1$$

$$-\left(\delta^* + \beta^* - \tau^* + ieV_b m^b\right)\gamma\hat{G}_2 = -\frac{c_4}{\Sigma}\mathcal{L}_+\hat{G}_2$$

$$-\left(\delta + \pi^* - \alpha^* + ieV_b m^b\right)\sigma\hat{G}_1 = \frac{c_3\rho^*}{\Delta^{\frac{1}{2}}}\mathcal{L}_-\hat{G}_1$$

$$\left(D + \epsilon^* - \rho^* + ieV_b l^b\right)\gamma\hat{G}_2 = -c_4\sqrt{2}\rho^*\mathcal{D}_-\hat{G}_2.$$

Concerning (19), a short computation gives

$$\left(D + \epsilon - \rho + ieV_b l^b\right)h\hat{F}_1 = \sqrt{2}\left(h\mathcal{D}_- + \frac{\partial h}{\partial r} - \rho h\right)\hat{F}_1 \qquad (28)$$

$$\left(\delta^* + \pi - \alpha + ieV_b m^{*b}\right)\Lambda\hat{F}_2 = -\rho\left(\Lambda\mathcal{L}_+ + \frac{\partial\Lambda}{\partial\vartheta}\right)\hat{F}_2\,, \qquad (29)$$

and by similar reasonings as we did for (26) and (27), we obtain $h(r,\vartheta) = -\rho\widetilde{h}(\vartheta)$ and $\Lambda(r,\vartheta) = c_2\widetilde{\lambda}(r)$ with $0 \neq c_2 \in \mathbb{C}$. Finally, regarding (20), we get

$$\left(\delta + \beta - \tau + ieV_b m^{*b}\right)h\hat{F}_1 = -\rho^*\left(h\mathcal{L}_- + \frac{\partial h}{\partial\vartheta} - ia\rho\sin\vartheta\, h\right)\hat{F}_1 \qquad (30)$$

$$\left(\widetilde{\Delta} + \mu - \gamma + ieV_b n^b\right)\Lambda\hat{F}_2 = -\frac{\sqrt{2}\Delta}{2\Sigma}\left(\Lambda\mathcal{D}_+ + \frac{\partial\Lambda}{\partial r} + \frac{r - M}{\Delta}\Lambda\right)\hat{F}_2. \qquad (31)$$

By proceeding similarly as we did for (26) and (27), we find that $\Lambda(r,\vartheta) = c_2\Delta^{-\frac{1}{2}}$ and $h(r,\vartheta) = -c_1\rho$ with $0 \neq c_1 \in \mathbb{C}$. Hence, (28), (29), (30) and (31) become

$$\left(D + \epsilon - \rho + ieV_b l^b\right)h\hat{F}_1 = -c_1\sqrt{2}\rho\mathcal{D}_-\hat{F}_1$$

$$\left(\delta^* + \pi - \alpha + ieV_b m^{*b}\right)\Lambda\hat{F}_2 = -c_2\rho\Delta^{-\frac{1}{2}}\mathcal{L}_+\hat{F}_2$$

$$\left(\delta + \beta - \tau + ieV_b m^{*b}\right)h\hat{F}_1 = \frac{c_1}{\Sigma}\mathcal{L}_-\hat{F}_1$$

$$\left(\widetilde{\Delta} + \mu - \gamma + ieV_b n^b\right)\Lambda\hat{F}_2 = -\frac{c_2\sqrt{2}\Delta^{\frac{1}{2}}}{2\Sigma}\mathcal{D}_+\hat{F}_2.$$

Then, the operator $S^{-1}\mathcal{O}S$ is computed to be

$$S^{-1}\mathcal{O}S = \begin{pmatrix} -im_e & 0 & \frac{\sqrt{2}c_3\rho^*\Delta^{\frac{1}{2}}}{2c_1}\mathcal{D}_+ & \frac{c_4\rho^*}{c_1}\mathcal{L}_+ \\ 0 & -im_e & \frac{c_3\rho^*}{c_2}\mathcal{L}_- & -\frac{\sqrt{2}c_4\rho^*\Delta^{\frac{1}{2}}}{c_2}\mathcal{D}_- \\ -\frac{\sqrt{2}c_1\rho\Delta^{\frac{1}{2}}}{c_3}\mathcal{D}_- & -\frac{c_2\rho}{c_3}\mathcal{L}_+ & -im_e & 0 \\ -\frac{c_1\rho}{c_4}\mathcal{L}_- & \frac{\sqrt{2}c_2\rho\Delta^{\frac{1}{2}}}{2c_4}\mathcal{D}_+ & 0 & -im_e \end{pmatrix}.$$

By choosing a non singular 4×4 matrix Γ as follows

$$\Gamma = \mathrm{diag}\left(\rho^{*-1}, -\rho^{*-1}, -\rho^{-1}, \rho^{-1}\right),$$

it can be checked that $\mathcal{W} = \Gamma S^{-1} \mathcal{O} S$ is given by

$$\begin{pmatrix} im_e r - am_e \cos\vartheta & 0 & \frac{\sqrt{2}c_3}{2c_1}\sqrt{\Delta}\mathcal{D}_+ & \frac{c_4}{c_1}\mathcal{L}_+ \\ 0 & -im_e r + am_e \cos\vartheta & -\frac{c_3}{c_2}\mathcal{L}_- & -\frac{\sqrt{2}c_4}{c_2}\sqrt{\Delta}\mathcal{D}_- \\ \frac{\sqrt{2}c_1}{c_3}\sqrt{\Delta}\mathcal{D}_- & \frac{c_2}{c_3}\mathcal{L}_+ & -im_e r - am_e \cos\vartheta & 0 \\ -\frac{c_1}{c_4}\mathcal{L}_- & \frac{\sqrt{2}c_2}{2c_4}\sqrt{\Delta}\mathcal{D}_+ & 0 & im_e r + am_e \cos\vartheta \end{pmatrix}.$$

If we impose the following conditions

$$c_4 = \frac{c_3}{\sqrt{2}} = \frac{c_2}{\sqrt{2}} = c_1 =: c,$$

we finally have

$$\mathcal{W} = \begin{pmatrix} im_e r - am_e \cos\vartheta & 0 & \sqrt{\Delta}\mathcal{D}_+ & \mathcal{L}_+ \\ 0 & -im_e r + am_e \cos\vartheta & -\mathcal{L}_- & \sqrt{\Delta}\mathcal{D}_- \\ \sqrt{\Delta}\mathcal{D}_- & \mathcal{L}_+ & -im_e r - am_e \cos\vartheta & 0 \\ -\mathcal{L}_- & \sqrt{\Delta}\mathcal{D}_+ & 0 & im_e r + am_e \cos\vartheta \end{pmatrix}.$$

This completes the proof. ☑

Clearly, the result obtained in Lemma 2.1 is also valid in the extreme case, where Δ is simply $(r-M)^2$. Without loss of generality we set $c = 1$. The next result, which holds for the Dirac equation in oblate spheroidal coordinates, follows directly from Lemma 2.2 by setting $M = Q = 0$.

Corollary 2.2. *Let $r > 0$. There exist a non singular 4×4 matrix*

$$S^{(OSC)} = \mathrm{diag}\left(-\rho, 2^{\frac{1}{2}}\hat{\Delta}^{-\frac{1}{2}}, 2^{\frac{1}{2}}\hat{\Delta}^{-\frac{1}{2}}, -\rho^*\right), \quad \det\left(S^{(OSC)}\right) = \frac{1}{\Sigma\hat{\Delta}}$$

and a non singular 4×4 matrix

$$\Gamma = \mathrm{diag}\left(\rho^{*-1}, -\rho^{*-1}, -\rho^{-1}, \rho^{-1}\right), \quad \det(\Gamma) = \Sigma$$

such that

$$\mathcal{W}^{(OSC)} = \mathcal{W}^{(OSC)}_{(t,r,\varphi)} + \mathcal{W}^{(OSC)}_{(t,\vartheta,\varphi)} \tag{32}$$

with

$$
\mathcal{W}^{(OSC)}_{(t,r,\varphi)} = \begin{pmatrix} im_e r & 0 & \sqrt{\hat{\Delta}}\widetilde{\mathcal{D}}_+ & 0 \\ 0 & -im_e r & 0 & \sqrt{\hat{\Delta}}\widetilde{\mathcal{D}}_- \\ \sqrt{\hat{\Delta}}\widetilde{\mathcal{D}}_- & 0 & -im_e r & 0 \\ 0 & \sqrt{\hat{\Delta}}\widetilde{\mathcal{D}}_+ & 0 & im_e r \end{pmatrix},
$$

$$
\mathcal{W}^{(OSC)}_{(t,\vartheta,\varphi)} = \mathcal{W}_{(t,\vartheta,\varphi)}, \tag{33}
$$

where $\mathcal{W}_{(t,\vartheta,\varphi)}$ is given by (16), $\hat{\Delta} = r^2 + a^2$ and

$$
\widetilde{\mathcal{D}}_\pm := \frac{\partial}{\partial r} \mp \left(\frac{\partial}{\partial t} + \frac{a}{\hat{\Delta}} \frac{\partial}{\partial \varphi} \right).
$$

Notice that the angular part of the operator $\mathcal{W}_{(t,\vartheta,\varphi)}$ remains unchanged, when we treat the Dirac equation in the Minkowski space-time with oblate spheroidal coordinates. Turning back to the Kerr-Newman metric, it can easily be checked that

$$
\left[\mathcal{W}_{(t,r,\varphi)}, \mathcal{W}_{(t,\vartheta,\varphi)} \right] = 0, \tag{34}
$$

which also implies

$$
\left[\mathcal{W}_{(t,r,\varphi)}, \mathcal{W} \right] = \left[\mathcal{W}_{(t,\vartheta,\varphi)}, \mathcal{W} \right] = 0.
$$

An analogous commutator holds for the Dirac equation in OSC. Going back to Lemma 2.1, we observe that the matrix Γ splits into the sum

$$
\Gamma = \Gamma_{(r)} + \Gamma_{(\vartheta)}, \tag{35}
$$

where $\Gamma_{(r)} = \mathrm{diag}(-r, r, r, -r)$ and $\Gamma_{(\vartheta)} = ia\,\mathrm{diag}(-\cos\vartheta, \cos\vartheta, -\cos\vartheta, \cos\vartheta)$ satisfy the commutation relations

$$
\left[\Gamma_{(r)}, \Gamma_{(\vartheta)} \right] = 0, \tag{36}
$$

$$
\left[\Gamma_{(r)}, \mathcal{W}_{(t,\vartheta,\varphi)} \right] = \left[\Gamma_{(\vartheta)}, \mathcal{W}_{(t,r,\varphi)} \right] = 0. \tag{37}
$$

3. Construction of the symmetry operator J

In this section we will make use of Chandrasekhar separation Ansatz in order to construct a new operator J for the Dirac equation in the Kerr-Newman metric. We show that J commutes with the Dirac operator $\mathcal{O}_D$ given by (6), thus being a symmetry operator for $\mathcal{O}_D$. Moreover, the separation constant turns to be eigenvalue of J, whose physical interpretation is that one of the square root of the squared total angular momentum for a Dirac particle in the presence of a charged rotating black hole.

Taking into account that the geometry of our problem is axial symmetric and considering stationary waves with energy ω, the (t, φ) dependence of the spinors $\hat{\psi}$ entering in (13) is given by

$$\hat{\psi}(t, r, \vartheta, \varphi) = e^{i\omega t} e^{i\left(k + \frac{1}{2}\right)\varphi} \widetilde{\psi}(r, \vartheta) \tag{38}$$

where $k \in \mathbb{Z}$ is the azimuthal quantum number of the particle and $\widetilde{\psi}(r, \vartheta) \in \mathbb{C}^4$. Inserting (38) in (13), it is straightforward to verify that $\widetilde{\psi}(r, \vartheta)$ satisfies the following equation

$$\left(\mathcal{W}_{(r)} + \mathcal{W}_{(\vartheta)}\right) \widetilde{\psi} = 0, \tag{39}$$

where

$$\mathcal{W}_{(r)} = \begin{pmatrix} im_e r & 0 & \sqrt{\Delta}\hat{\mathcal{D}}_+ & 0 \\ 0 & -im_e r & 0 & \sqrt{\Delta}\hat{\mathcal{D}}_- \\ \sqrt{\Delta}\hat{\mathcal{D}}_- & 0 & -im_e r & 0 \\ 0 & \sqrt{\Delta}\hat{\mathcal{D}}_+ & 0 & im_e r \end{pmatrix}, \tag{40}$$

and

$$\mathcal{W}_{(\vartheta)} = \begin{pmatrix} -am_e \cos\vartheta & 0 & 0 & \hat{\mathcal{L}}_+ \\ 0 & am_e \cos\vartheta & -\hat{\mathcal{L}}_- & 0 \\ 0 & \hat{\mathcal{L}}_+ & -am_e \cos\vartheta & 0 \\ -\hat{\mathcal{L}}_- & 0 & 0 & am_e \cos\vartheta \end{pmatrix}, \tag{41}$$

with

$$\hat{\mathcal{D}}_\pm \ := \ \frac{\partial}{\partial r} \mp i\frac{K(r)}{\Delta}, \qquad\qquad K(r) \ = \ \omega(r^2 + a^2) - eQr + \left(k + \tfrac{1}{2}\right)a,$$

$$\hat{\mathcal{L}}_\pm \ := \ \frac{\partial}{\partial \vartheta} + \tfrac{1}{2}\cot\vartheta \pm Q(\vartheta), \quad Q(\vartheta) \ = \ a\omega \sin\vartheta + \left(k + \tfrac{1}{2}\right)\csc\vartheta.$$

By defining $\widetilde{\psi}(r, \vartheta) = (f_1(r, \vartheta), f_2(r, \vartheta), g_1(r, \vartheta), g_2(r, \vartheta))^T$, equation (39) gives rise to the following systems of first order linear partial differential equations

$$\sqrt{\Delta}\hat{\mathcal{D}}_+ g_1 + im_e r f_1 + \left(\hat{\mathcal{L}}_+ g_2 - am_e f_1 \cos\vartheta\right) = 0$$

$$\sqrt{\Delta}\hat{\mathcal{D}}_- g_2 - im_e r f_2 - \left(\hat{\mathcal{L}}_- g_1 - am_e f_2 \cos\vartheta\right) = 0$$

$$\sqrt{\Delta}\hat{\mathcal{D}}_- f_1 - im_e r g_1 + \left(\hat{\mathcal{L}}_+ f_2 - am_e g_1 \cos\vartheta\right) = 0$$

$$\sqrt{\Delta}\hat{\mathcal{D}}_+ f_2 + im_e r g_2 - \left(\hat{\mathcal{L}}_- f_1 - am_e g_2 \cos\vartheta\right) = 0.$$

Let us now define

$$f_1(r, \vartheta) = \gamma_1(r)\delta_1(\vartheta), \quad f_2(r, \vartheta) = \sigma_2(r)\tau_2(\vartheta), \tag{42}$$

$$g_1(r, \vartheta) = \alpha_1(r)\beta_1(\vartheta), \quad g_2(r, \vartheta) = \epsilon_2(r)\mu_2(\vartheta). \tag{43}$$

If we substitute (42) and (43) into the above system, it can be easily seen that the requirement of separability implies

$$\beta_1(\vartheta) = \delta_1(\vartheta), \quad \epsilon_2(r) = \gamma_1(r), \quad \mu_2(\vartheta) = \tau_2(\vartheta), \quad \alpha_1(r) = \sigma_2(r).$$

According to the notation used in [6], we set

$$\gamma_1(r) = R_-(r), \quad \delta_1(\vartheta) = S_-(\vartheta), \quad \sigma_2(r) = R_+(r), \quad \tau_2(\vartheta) = S_+(\vartheta).$$

Hence, we get

$$f_1(r, \vartheta) = R_-(r)S_-(\vartheta), \quad f_2(r, \vartheta) = R_+(r)S_+(\vartheta),$$

$$g_1(r, \vartheta) = R_+(r)S_-(\vartheta), \quad g_2(r, \vartheta) = R_-(r)S_+(\vartheta),$$

and the system of partial differential equations become

$$\left(\sqrt{\Delta}\hat{\mathcal{D}}_+ R_+ + im_e r R_-\right) S_- + \left(\hat{\mathcal{L}}_+ S_+ - am_e \cos\vartheta S_-\right) R_- = 0 \qquad (44)$$

$$\left(\sqrt{\Delta}\hat{\mathcal{D}}_- R_- - im_e r R_+\right) S_+ - \left(\hat{\mathcal{L}}_- S_- - am_e \cos\vartheta S_+\right) R_+ = 0 \qquad (45)$$

$$\left(\sqrt{\Delta}\hat{\mathcal{D}}_- R_- - im_e r R_+\right) S_- + \left(\hat{\mathcal{L}}_+ S_+ - am_e \cos\vartheta S_-\right) R_+ = 0 \qquad (46)$$

$$\left(\sqrt{\Delta}\hat{\mathcal{D}}_+ R_+ + im_e r R_-\right) S_+ - \left(\hat{\mathcal{L}}_- S_- - am_e \cos\vartheta S_+\right) R_- = 0. \qquad (47)$$

Introducing four separation constants $\lambda_1, \ldots, \lambda_4$ as follows

$$\hat{\mathcal{L}}_+ S_+ - am_e \cos\vartheta S_- = -\lambda_1 S_- \qquad (48)$$

$$\hat{\mathcal{L}}_- S_- - am_e \cos\vartheta S_+ = +\lambda_2 S_+ \qquad (49)$$

$$\hat{\mathcal{L}}_+ S_+ - am_e \cos\vartheta S_- = -\lambda_3 S_- \qquad (50)$$

$$\hat{\mathcal{L}}_- S_- - am_e \cos\vartheta S_+ = +\lambda_4 S_+, \qquad (51)$$

we obtain from the set of equations (44), (45), (46) and (47)

$$\sqrt{\Delta}\hat{\mathcal{D}}_+ R_+ + im_e r R_- = \lambda_1 R_- \qquad (52)$$

$$\sqrt{\Delta}\hat{\mathcal{D}}_+ R_+ + im_e r R_- = \lambda_4 R_- \qquad (53)$$

$$\sqrt{\Delta}\hat{\mathcal{D}}_- R_- - im_e r R_+ = \lambda_2 R_+ \qquad (54)$$

$$\sqrt{\Delta}\hat{\mathcal{D}}_- R_- - im_e r R_+ = \lambda_3 R_+. \qquad (55)$$

Clearly, the systems of equations (48),$\cdots$,(51) and (52),$\cdots$,(55) will be consistent if

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 =: \lambda.$$

Notice that the decoupled equations give rise to the following two systems of linear first order differential equations, namely

$$
\begin{pmatrix} \sqrt{\Delta}\hat{\mathcal{D}}_- & -im_e r - \lambda \\ im_e r - \lambda & \sqrt{\Delta}\hat{\mathcal{D}}_+ \end{pmatrix} \begin{pmatrix} R_- \\ R_+ \end{pmatrix} = 0 \,, \tag{56}
$$

$$
\begin{pmatrix} -\hat{\mathcal{L}}_- & \lambda + am_e \cos\theta \\ \lambda - am_e \cos\theta & \hat{\mathcal{L}}_+ \end{pmatrix} \begin{pmatrix} S_- \\ S_+ \end{pmatrix} = 0 \,. \tag{57}
$$

Starting from the systems of equations $(48),\cdots,(51)$ and $(52),\cdots,(55)$, we are now able to construct a new operator J. Moreover, we will show that such operator commutes with the Dirac operator $\mathcal{O}_D$.

By setting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 =: \lambda$ in $(48),\cdots,(51)$ and $(52),\cdots,(55)$, multiplying (48) by $-rR_-$ and (52) by $ia\cos\vartheta S_-$, respectively, and summing together the resulting equations, we find that

$$
-ia\rho^* \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_+ g_1 + r\rho^* \hat{\mathcal{L}}_+ g_2 = \lambda f_1 \,, \tag{58}
$$

with $\rho^* = -(r + ia\cos\vartheta)^{-1}$. By means of the same method we obtain from the couples of equations $((49),(53))$ and $((51),(55))$

$$
-ia\rho^* \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_- g_2 - r\rho^* \hat{\mathcal{L}}_- g_1 = \lambda f_2 \tag{59}
$$

$$
+ia\rho \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_- f_1 + r\rho \hat{\mathcal{L}}_+ f_2 = \lambda g_1 \tag{60}
$$

$$
+ia\rho \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_+ f_2 - r\rho \hat{\mathcal{L}}_- f_1 = \lambda g_2 \,, \tag{61}
$$

with $\rho = -(r - ia\cos\vartheta)^{-1}$. Equations $(58),\cdots,(61)$ give the entries of the matrix operator $\hat{J}$ we are looking for, namely

$$
\begin{pmatrix} 0 & 0 & -ia\rho^* \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_+ & r\rho^* \hat{\mathcal{L}}_+ \\ 0 & 0 & -r\rho^* \hat{\mathcal{L}}_- & -ia\rho^* \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_- \\ ia\rho \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_- & r\rho \hat{\mathcal{L}}_+ & 0 & 0 \\ -r\rho \hat{\mathcal{L}}_- & ia\rho \cos\vartheta \sqrt{\Delta}\hat{\mathcal{D}}_+ & 0 & 0 \end{pmatrix} .
$$

The operator $\hat{J}$ can be written in a more compact form as follows

$$
\hat{J} = \Gamma^{-1} \left(\Gamma_{(\vartheta)} \mathcal{W}_{(r)} - \Gamma_{(r)} \mathcal{W}_{(\vartheta)} \right) . \tag{62}
$$

Clearly, a similar expression holds for the Dirac equation in OSC.

Lemma 3.1. $J = S\hat{J}S^{-1}$ *is a symmetry matrix operator for the Dirac operator* $\mathcal{O}_D$ *in the Kerr-Newman metric, i.e.*

$$
[\mathcal{O}_D, J] = 0.
$$

Proof. From (13) we have $\mathcal{O}_D = S\Gamma^{-1}\mathcal{W}S^{-1}$, which acts on the spinors Ψ and we consider the operator $J = S\hat{J}S^{-1}$ with $\hat{J}$ given by (62). It holds

$$\mathcal{O}_D J = S\Gamma^{-1}\mathcal{W}\Gamma^{-1}(\Gamma_{(\vartheta)}\mathcal{W}_{(r)} - \Gamma_{(r)}\mathcal{W}_{(\vartheta)})S^{-1}$$

$$J\mathcal{O}_D = S\Gamma^{-1}(\Gamma_{(\vartheta)}\mathcal{W}_{(r)} - \Gamma_{(r)}\mathcal{W}_{(\vartheta)})\Gamma^{-1}\mathcal{W}S^{-1}$$

and therefore we obtain

$$[\mathcal{O}_D, J] = S\Gamma^{-1}PS^{-1}$$

with

$$P := \mathcal{W}\Gamma^{-1}(\Gamma_{(\vartheta)}\mathcal{W}_{(r)} - \Gamma_{(r)}\mathcal{W}_{(\vartheta)}) - (\Gamma_{(\vartheta)}\mathcal{W}_{(r)} - \Gamma_{(r)}\mathcal{W}_{(\vartheta)})\Gamma^{-1}\mathcal{W}.$$

Making use of the decomposition $\mathcal{W} = \mathcal{W}_{(r)} + \mathcal{W}_{(\vartheta)}$ and of the commutation relations (37), we obtain

$$P = \mathcal{W}_{(\vartheta)}(\Gamma_{(r)}\Gamma^{-1} + \Gamma^{-1}\Gamma_{(\vartheta)})\mathcal{W}_{(r)} - \mathcal{W}_{(r)}(\Gamma^{-1}\Gamma_{(r)} + \Gamma_{(\vartheta)}\Gamma^{-1})\mathcal{W}_{(\vartheta)}$$

$$= \mathcal{W}_{(\vartheta)}\mathcal{W}_{(r)} - \mathcal{W}_{(r)}\mathcal{W}_{(\vartheta)} = 0$$

where in the last line we employed (34). ☑

Concerning the physical meaning of J, it is interesting to observe that, when $a = 0$ the angular eigenfunctions $S_{\pm}(\vartheta)$ can be expressed in terms of the spin-weighted spherical harmonics $Y_{\pm\frac{1}{2}}^{jk}$ (see for instance [17, 10]). According to [5], we will call J the square root of the total squared angular momentum operator. In [5] it was proved that the separation constant λ is the eigenvalue of the square root of the total squared angular momentum for the Dirac equation in any type-D vacuum space-time. Moreover, the occurrence of this operator and therefore the separability of the Dirac equation in the Kerr-Newman metric by means of Chandrasekhar ansatz arises from the presence of a Killing spinor field on the space-time under consideration. For a detailed description of the concept of Killing spinors we refer to [22].

4. Schrödinger form of the Dirac equation in OSC

In order to bring the matrix equation

$$\mathcal{W}^{(OSC)}\hat{\psi} = 0, \tag{63}$$

with $\mathcal{W}^{(OSC)}$ given by (32) into the form of a Schrödinger equation

$$i\partial_t\hat{\psi} = H_D\hat{\psi}, \tag{64}$$

we apply a method similar to that one used in [8]. Without risk of confusion we can simplify our notation by omitting to write explicitly the superscript (OSC) attached to the operator $\mathcal{W}$. Starting with (63), we bring the time derivatives on the l.h.s. of the equation and find that

$$-iT\partial_t\hat{\psi} = \left(\mathcal{W}_{(r,\varphi)} + \mathcal{W}_{(\vartheta,\varphi)}\right)\hat{\psi}, \tag{65}$$

with

$$
T = \begin{pmatrix}
0 & 0 & i\sqrt{\Delta} & -a\sin\vartheta \\
0 & 0 & -a\sin\vartheta & -i\sqrt{\Delta} \\
-i\sqrt{\Delta} & -a\sin\vartheta & 0 & 0 \\
-a\sin\vartheta & i\sqrt{\Delta} & 0 & 0
\end{pmatrix}.
$$

and with $\mathcal{W}_{(r,\varphi)}$, $\mathcal{W}_{(\vartheta,\varphi)}$ already defined in Corollary 2.2, where the operators $\hat{\mathcal{D}}_\pm$ and $\hat{\mathcal{L}}_\pm$ have been now replaced by

$$
\hat{\mathcal{D}}_\pm = \frac{\partial}{\partial r} \mp \frac{a}{\hat{\Delta}} \frac{\partial}{\partial \varphi}
$$

$$
\hat{\mathcal{L}}_\pm = \frac{\partial}{\partial \vartheta} + \frac{1}{2}\cot\vartheta \mp i\csc\vartheta \frac{\partial}{\partial \varphi}.
$$

Since $\det(T) = \Sigma^2 \neq 0$ for $r > 0$ and $\vartheta \in [0, \pi]$, T is non singular and we can multiply (65) on both sides by T^{-1} to obtain (64) with Hamiltonian operator

$$
H_D = \frac{\Delta}{\Sigma}\mathcal{S} \cdot \left(\hat{\mathcal{W}}_{(r,\varphi)} + \hat{\mathcal{W}}_{(\vartheta,\varphi)}\right), \tag{66}
$$

where σ_2 is the Pauli matrix and

$$
\mathcal{S} = \mathbb{1}_4 - \frac{a\sin\vartheta}{\sqrt{\Delta}}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix},
$$

$$
\mathcal{W}_{(r,\varphi)} = \frac{m_e r}{\sqrt{\Delta}}\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} + \mathrm{diag}(\mathcal{E}_-, -\mathcal{E}_+, -\mathcal{E}_+, \mathcal{E}_-),
$$

$$
\mathcal{W}_{(\vartheta,\varphi)} = \frac{am_e\cos\vartheta}{\sqrt{\Delta}}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{M}_+ & 0 & 0 \\ \mathcal{M}_- & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{M}_+ \\ 0 & 0 & -\mathcal{M}_- & 0 \end{pmatrix},
$$

with

$$
\mathcal{E}_\pm = i\hat{\mathcal{D}}_\pm \quad \text{and} \quad \mathcal{M}_\pm = \frac{i}{\sqrt{\Delta}}\hat{\mathcal{L}}_\pm
$$

such that

$$
\mathcal{E}_\pm^* = -\mathcal{E}_\pm \quad \text{and} \quad \mathcal{M}_\pm^* = -\mathcal{M}_\mp.
$$

Notice that the formal operator H_D acts on the spinors $\hat{\psi}$ on the hypersurfaces $t = $const. Moreover, the matrix $\mathcal{S}$ entering in (66) is hermitian. For simplicity in the notation we will omit the hat of the wave functions. It can be checked

that for every $\psi, \phi \in \mathcal{C}_0^\infty \left([0,\infty) \times S^2\right)^4$ the Hamiltonian H_D is hermitian, i.e. formally self-adjoint with respect to the positive scalar product

$$\langle \psi | \phi \rangle = \int_0^\infty dr \int_{-1}^1 d(\cos\vartheta) \int_0^{2\pi} d\varphi \, (\psi | C\phi) \,, \tag{67}$$

with inner product given by

$$(\psi | C\phi) = \overline{\psi} C\phi \,, \tag{68}$$

where the $\overline{\psi}$ denotes the complex conjugated, transposed spinor and

$$C = \mathbb{1}_4 + \frac{a \sin\vartheta}{\sqrt{\Delta}} \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{69}$$

In order to see the positivity of the scalar product defined by (67) and (68) it suffices to show that the matrix C entering in the inner product is positive definite. Indeed, the eigenvalues of C are given by

$$\Lambda_1 = \Lambda_2 = 1 - \frac{|a| \sin\vartheta}{\sqrt{\Delta}} \quad \text{and} \quad \Lambda_3 = \Lambda_4 = 1 + \frac{|a| \sin\vartheta}{\sqrt{\Delta}}$$

and since the following inequality holds for all $r > 0$

$$\frac{|a| \sin\vartheta}{\sqrt{\Delta}} \le \frac{|a|}{\sqrt{\Delta}} < 1 \,,$$

we can conclude that $\Lambda_i > 0$ for every $i = 1, \cdots, 4$. In what follows we consider the Hilbert space $\mathcal{H} = \{\mathbb{C}^4, \langle \cdot | \cdot \rangle\}$ made of wave functions $\phi : [0,\infty) \times S^2 \longrightarrow \mathbb{C}^4$ together with the scalar product (67).

5. Completeness of Chandrasekhar ansatz

We begin with some preliminary observations. Energy, generalized squared angular momentum and the z-component of the total angular momentum form a set of commuting observables $\{H_D, J^2, J_z\}$. Moreover, the angular system (57) can be brought in the so-called Dirac form [24], namely

$$\mathcal{U}S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dS}{d\vartheta} + \begin{pmatrix} -am_e \cos\vartheta & -\frac{k+\frac{1}{2}}{\sin\vartheta} - a\omega \sin\vartheta \\ -\frac{k+\frac{1}{2}}{\sin\vartheta} - a\omega \sin\vartheta & am_e \cos\vartheta \end{pmatrix} S = \lambda S$$

with $S(\vartheta) = (S_-(\vartheta), S_+(\vartheta))^T$ and $\vartheta \in (0,\pi)$. According to [1], in $L_2((0,\pi))^2$ the angular operator $\mathcal{U}$ defined on $D(\mathcal{U}) = \mathcal{C}_0^\infty((0,\pi))^2$ is essentially self-adjoint, its spectrum is discrete, non degenerate (i.e. simple) and depends smoothly on ω. Therefore, its eigenvalues can be written as $\lambda_j(\omega)$ with $j \in \mathbb{Z}$ and $\lambda_j < \lambda_{j+1}$ for every $j \in \mathbb{Z}$. Moreover, the functions $S_\pm^{kj}(\vartheta)$ satisfy a generalized Heun equation [1] and become the well-known spin-weighted spherical harmonics by setting $a = 0$. Furthermore, the functions $e^{i\left(k+\frac{1}{2}\right)\varphi}$ are eigenfunctions of the z-component of the total angular momentum operator J_z with

eigenvalues $-\left(k + \frac{1}{2}\right)$ with $k \in \mathbb{Z}$. Hence, we will label the states ϕ in the Hilbert space $\mathcal{H}$ by ϕ_ω^{kj}. In what follows, we consider the free Dirac operator H_D given by (66) in $\mathcal{H} = L_2\left([0,\infty) \times S^2\right)^4$. From Theorem 1.1 [21] the operator H_D defined on $\mathcal{C}_0^\infty\left([0,\infty) \times S^2\right)^4$ is essentially self-adjoint and has a unique self-adjoint extension on the Sobolev space $W^{1,2}\left([0,\infty) \times S^2\right)^4$. In addition, the spectrum of H_D, which we will denote by σ_D, is purely absolutely continuous and given by $\sigma_D = (-\infty, m_e] \cup [m_e, +\infty)$. Notice that, since H_D possesses a unique self-adjoint extension, when we will derive an integral representation for the Dirac propagator, there will be no need to impose Dirichlet boundary conditions or other boundary conditions on the spinors, as it was done in [8]. In order to find a suitable decomposition of $\mathcal{H}$, we begin by showing that

$$\left\{Y_\omega^{kj}(\vartheta, \varphi)\right\}, \quad Y_\omega^{kj}(\vartheta, \varphi) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} S_{\omega,-}^{kj}(\vartheta) \\ S_{\omega,+}^{kj}(\vartheta) \end{pmatrix} e^{i\left(k+\frac{1}{2}\right)\varphi}, \qquad (70)$$

with $k, j \in \mathbb{Z}$ is a complete orthonormal basis of $L_2\left(S^2\right)^2$ and that it is possible to define isometric operators $\hat{W}_{k,j} : \mathcal{C}_0^\infty\left([0,\infty)\right)^2 \longrightarrow \mathcal{C}_0^\infty\left([0,\infty) \times S^2\right)^4$. In the next result we prove the orthonormality of the set $\left\{Y_\omega^{kj}(\vartheta, \varphi)\right\}$.

Lemma 5.1. *For every k, k', j, $j' \in \mathbb{Z}$ it holds $\left\langle Y_\omega^{kj} | Y_\omega^{k'j'} \right\rangle_{S^2} = \delta_{kk'}\delta_{jj'}$, where $\langle \cdot | \cdot \rangle_{S^2}$ is the usual scalar product on S^2.*

Proof. Making use of (70), a direct computation gives

$$\left\langle Y^{kj} | Y^{k'j'} \right\rangle_{S^2} = \delta_{kk'} \int_0^\pi d\vartheta \, \sin\vartheta \overline{S}^{kj}(\vartheta) S^{k'j'}(\vartheta),$$

where for simplicity in notation we omitted to write the subscript ω attached to the angular eigenfunctions. In order to investigate the above integral we need to go back to the angular system (57). It is easy to see that $S_+^{k'j'}$, $S_-^{k'j'}$ and $\left(S_+^{kj}\right)^*$, $\left(S_-^{kj}\right)^*$ satisfy the following two systems of first order linear ODEs, namely

$$+\hat{\mathcal{L}}_+' S_+^{k'j'} + \left(\lambda_{j'} - am_e \cos\theta\right) S_-^{k'j'} = 0, \qquad (71)$$

$$-\hat{\mathcal{L}}_-' S_-^{k'j'} + \left(\lambda_{j'} + am_e \cos\theta\right) S_+^{k'j'} = 0, \qquad (72)$$

and

$$+\hat{\mathcal{L}}_+ \left(S_+^{kj}\right)^* + (\lambda_j - am_e \cos\theta)\left(S_-^{kj}\right)^* = 0, \qquad (73)$$

$$-\hat{\mathcal{L}}_- \left(S_-^{kj}\right)^* + (\lambda_j + am_e \cos\theta)\left(S_+^{kj}\right)^* = 0, \qquad (74)$$

where $(\cdot)^*$ denotes complex conjugation and

$$\hat{\mathcal{L}}'_{\pm} = \frac{d}{d\vartheta} + \widetilde{Q}(\vartheta)_{\pm},$$

$$\widetilde{Q}(\vartheta)_{\pm} = \frac{1}{2}\cot\vartheta \pm \widetilde{Q}(\vartheta),$$

$$\widetilde{Q}(\vartheta) = a\omega\sin\vartheta + \left(k' + \frac{1}{2}\right)\csc\vartheta.$$

After multiplication of (71) by $\sin\vartheta\left(S_-^{kj}\right)^*$, of (72) by $\sin\vartheta\left(S_+^{kj}\right)^*$, of (73) by $\sin\vartheta S_-^{k'j'}$ and of (74) by $\sin\vartheta S_+^{k'j'}$, we consider the equations

$$(71)\sin\vartheta\left(S_-^{kj}\right)^* - (73)\sin\vartheta S_-^{k'j'} = 0,$$

and

$$(72)\sin\vartheta\left(S_+^{kj}\right)^* - (74)\sin\vartheta S_+^{k'j'} = 0,$$

build their sum and integrate over ϑ from 0 to π. Hence, we obtain

$$2(\lambda_{j'} - \lambda_j)\int_0^\pi d\vartheta\,\sin\vartheta\overline{S}^{kj}(\vartheta)S^{k'j'}(\vartheta) = \int_0^\pi d\vartheta\,\sin\vartheta$$

$$\left[\left(S_+^{kj}\right)^*\hat{\mathcal{L}}'_-S_-^{k'j'} - S_+^{k'j'}\hat{\mathcal{L}}_-\left(S_-^{kj}\right)^* - \left(S_-^{kj}\right)^*\hat{\mathcal{L}}'_+S_+^{k'j'} + S_-^{k'j'}\hat{\mathcal{L}}_+\left(S_+^{kj}\right)^*\right].$$

Since the angular eigenfunctions $S_\pm(\vartheta)$ even vanish at $\vartheta = 0$ and at $\vartheta = \pi$ (see [1]), it can be checked that the following relations hold

$$+\int_0^\pi d\vartheta\,\sin\vartheta S_-^{k'j'}\hat{\mathcal{L}}_+\left(S_+^{kj}\right)^* = -\int_0^\pi d\vartheta\,\sin\vartheta\left(S_+^{kj}\right)^*\hat{\mathcal{L}}_-S_-^{k'j'}$$

$$-\int_0^\pi d\vartheta\,\sin\vartheta S_+^{k'j'}\hat{\mathcal{L}}_-\left(S_-^{kj}\right)^* = +\int_0^\pi d\vartheta\,\sin\vartheta\left(S_-^{kj}\right)^*\hat{\mathcal{L}}_+S_+^{k'j'}.$$

By means of the above results we obtain that

$$2(\lambda_{j'} - \lambda_j)\int_0^\pi d\vartheta\,\sin\vartheta\overline{S}^{kj}(\vartheta)S^{k'j'}(\vartheta) =$$

$$\int_0^\pi d\vartheta\,\sin\vartheta\left[\left(S_+^{kj}\right)^*\left(\hat{\mathcal{L}}'_- - \hat{\mathcal{L}}_-\right)S_-^{k'j'} + \left(S_-^{kj}\right)^*\left(\hat{\mathcal{L}}_+ - \hat{\mathcal{L}}'_+\right)S_+^{k'j'}\right]$$

and this completes the proof.   ☑

In what follows, we prove that the orthonormal set $\left\{Y_\omega^{kj}(\vartheta,\varphi)\right\}$ is complete and therefore an orthonormal basis for $L_2\left(S^2\right)^2$. The main idea of the proof relies on the Projection Theorem (Theorem II.3 [20]), i.e. we show that in $L_2\left(S^2\right)^2$ the only element orthogonal to our orthonormal basis is the zero vector.

Lemma 5.2. *Let* $\left\{ Y_\omega^{kj}(\vartheta, \varphi) \right\}$ *with* $Y_\omega^{kj}(\vartheta, \varphi)$ *as given in (70) and* $k, \, j \in \mathbb{Z}$ *be an orthonormal set for* $L_2\left(S^2\right)^2$. *Then for every* $Y_\omega^{kj} \in L_2\left(S^2\right)^2$ *it holds* $\left\langle \widetilde{\varphi} | Y_\omega^{kj} \right\rangle_{S^2} = 0$ *if and only if* $\widetilde{\varphi} = 0$.

Proof. Since there is no risk of confusion, we will omit to write explicitely the subscript ω and the superscripts k, j attached to the angular eigenfunctions. Without loss of generality let us consider

$$\widetilde{\varphi} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \varphi_1(\vartheta) \\ \varphi_2(\vartheta) \end{pmatrix} e^{i\left(k+\frac{1}{2}\right)\varphi} \quad \text{with} \quad \varphi_i \in \mathcal{C}_0^\infty((0,\pi)) \quad \text{for} \quad i = 1, 2\,.$$

Then we have

$$\left\langle \widetilde{\varphi} | Y^{kj} \right\rangle_{S^2} = \int_{-1}^1 dx \, \left(\varphi_1^*(x) S_+(x) + \varphi_2^*(x) S_-(x) \right),$$

where we made use of the transformation $x = \cos \vartheta$. The following estimate holds

$$\left| \left\langle \widetilde{\varphi} | Y^{kj} \right\rangle_{S^2} \right| \; \leq \; \int_{-1}^1 dx \, \left(|\varphi_1^*(x)| \, |S_+(x)| + |\varphi_2^*(x)| \, |S_-(x)| \right)$$

$$\leq \; \|\varphi_1\|_{L_2}^2 \int_{-1}^1 dx \, |S_+(x)|^2 + \|\varphi_2\|_{L_2}^2 \int_{-1}^1 dx \, |S_-(x)|^2$$

$$\leq \; d \int_{-1}^1 dx \, \left(|S_+(x)|^2 + |S_-(x)|^2 \right) = d\,,$$

where in the second line we used Hölder inequality, in the third line the orthonormality condition for $S_\pm(\vartheta)$ and we defined $d := \max \left\{ \|\varphi_1\|_{L_2}^2, \|\varphi_2\|_{L_2}^2 \right\}$. Clearly, $\left| \left\langle \widetilde{\varphi} | Y^{kj} \right\rangle_{S^2} \right| \leq 0$ if and only if $d = 0$. Since $d = 0$ implies $\varphi_1 = \varphi_2 = 0$ and the scalar product $\langle \cdot | \cdot \rangle_{S^2}$ is positive, the proof is completed. ☑

We show now that it is possible to construct isometric operators

$$\hat{W}_{k,j} : \mathcal{C}_0^\infty([0,+\infty))^2 \longrightarrow \mathcal{C}_0^\infty \left([0,+\infty) \times S^2\right)^4,$$

such that

$$R_\omega^{kj} = \begin{pmatrix} R_{\omega,-}^{kj}(r) \\ R_{\omega,+}^{kj}(r) \end{pmatrix} \longmapsto A \begin{pmatrix} R_{\omega,-}^{kj}(r) Y_{\omega,-}^{kj}(\vartheta, \varphi) \\ R_{\omega,+}^{kj}(r) Y_{\omega,+}^{kj}(\vartheta, \varphi) \\ R_{\omega,+}^{kj}(r) Y_{\omega,-}^{kj}(\vartheta, \varphi) \\ R_{\omega,-}^{kj}(r) Y_{\omega,+}^{kj}(\vartheta, \varphi) \end{pmatrix},$$

with A a positive definite hermitian matrix and

$$Y_{\omega,\pm}^{kj}(\vartheta, \varphi) = \frac{1}{\sqrt{2\pi}} S_{\omega,\pm}^{kj}(\vartheta) e^{i\left(k+\frac{1}{2}\right)\varphi}\,.$$

Since the angular eigenfunctions Y_ω^{kj} are normalized, we have

$$\left\|R_\omega^{kj}\right\|^2_{L_2([0,+\infty))^2} = \int_0^\infty dr \int_{S^2} d\Omega \; (\psi_\omega^{kj}|\psi_\omega^{kj}) \quad \text{with} \quad \psi_\omega^{kj} = \begin{pmatrix} R_{\omega,-}^{kj} Y_{\omega,-}^{kj} \\ R_{\omega,+}^{kj} Y_{\omega,+}^{kj} \\ R_{\omega,+}^{kj} Y_{\omega,-}^{kj} \\ R_{\omega,-}^{kj} Y_{\omega,+}^{kj} \end{pmatrix}.$$

Let us define a matrix

$$A^2 := \frac{\Delta}{\Sigma}\left[\mathbb{1}_4 + \frac{a\sin\vartheta}{\sqrt{\Delta}}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}\right].$$

Clearly, it holds $A^2 C = \mathbb{1}_4$ with C given by (69) and A will be a positive definite hermitian matrix. Therefore, taking into account that the matrices C and A^2 commute, we obtain that

$$\left\|R^{kj}\right\|^2_{L_2([0,+\infty))^2} = \int_0^\infty dr \int_{-1}^1 d(\cos\vartheta) \int_0^{2\pi} d\varphi \; (\psi_\omega^{kj}|CA^2\psi_\omega^{kj})$$
$$= \left\langle\psi_\omega^{kj}|A^2\psi_\omega^{kj}\right\rangle = \left\langle A\psi_\omega^{kj}|A\psi_\omega^{kj}\right\rangle = \left\langle\hat{W}_{k,j}\left(R^{kj}\right)|\hat{W}_{k,j}(R^{kj})\right\rangle$$
$$= \|\hat{W}_{k,j}\left(R^{kj}\right)\|^2_{L_2([0,+\infty)\times S^2)^4}.$$

By means of the isometric operators $\hat{W}_{k,j}$ we can now introduce for every $\omega \in \sigma_D$ an auxiliary separable Hilbert space $\mathfrak{h}(\omega)$ as follows

$$\mathfrak{h}(\omega) = \bigoplus_{k,j\in\mathbb{Z}} \mathcal{H}_{k,j} \quad \text{with} \quad \mathcal{H}_{k,j} = \hat{W}_{k,j}\left(\mathcal{C}_0^\infty([0,+\infty))^2\right).$$

Moreover, the expansion theorem (Th.3.7 [23]) implies that

$$\psi_\omega = \sum_{k,j\in\mathbb{Z}} \left\langle\psi_\omega^{kj}|\psi_\omega\right\rangle \psi_\omega^{kj}.$$

Notice that $\mathfrak{h}(\omega)$ is by no means a subspace of $L_2\left([0,+\infty)\times S^2\right)^4$ with respect to the spatial measure, since the solutions $R_{\omega,\pm}^{kj}(r)$ of the radial system (56) oscillate asymptotically for $r \to +\infty$. To circumvent this problem we proceed as follows. First we identify the absolutely continuous part of $H_D^{(A)}$ with the operator H_D, since the spectrum σ_D is purely absolutely continuous. Then, by applying lemma 10 (Ch.1 §3 [25]) we have that H_D is unitary equivalent to the operator of multiplication by ω in $L_2(\sigma_D;d\omega)^4$ and the following isomorphisms holds, namely

$$\mathcal{H} \cong \int_{\sigma_D}^\oplus d\omega\,\mathfrak{h}(\omega) =: \mathfrak{H}. \tag{75}$$

Moreover, since the operator H_D is self-adjoint, the spectral representation theorem (Theorem 7.18, [23]) implies the existence of a unitary map $\mathcal{F}$ from $\mathcal{H}$ onto $\mathfrak{H}$. The vector valued function $\psi_\omega = (\mathcal{F}\psi)(\omega)$ is called the representative element of the element $\psi \in \mathcal{H}$ in the decomposition (75). The scalar product in $\mathfrak{H}$ will be introduced according to [25], as follows

$$\langle \phi | \psi \rangle_{\mathfrak{H}} = \int_{\sigma_D} d\omega \, \langle \phi_\omega | \psi_\omega \rangle ,$$

where $\langle \cdot | \cdot \rangle$ denotes the scalar product given by (67). Furthermore, in $\mathfrak{H}$ we can introduce a norm by means of $\| \cdot \|_{\mathfrak{H}} = \left(\int_{\sigma_D} d\omega \, \| \cdot \|_{\mathfrak{h}}^2 \right)^{\frac{1}{2}}$. The self-adjointness of H_D in $\mathcal{H}$ implies also (see theorem 7.37 [23]) that $\{ U(t) = e^{itH_D} | t \in \mathbb{R} \}$ is a strongly continuous one-parameter unitary group with $U(t)\psi \in \mathcal{C}_\infty^0([0, +\infty) \times S^2)^4$ for every $t \in \mathbb{R}$ and every $\psi \in \mathcal{C}_\infty^0 \left([0, +\infty) \times S^2\right)^4$. By employing the version of the spectral theorem as given in [25] (Ch.1 § 4-5), we have

$$\langle \phi | e^{itH_D} \psi \rangle = \int_{\sigma_D} d\omega \, e^{i\omega t} \langle \phi_\omega | \psi_\omega \rangle , \qquad (76)$$

where $\langle \phi_\omega | \psi_\omega \rangle$ is given in terms of the resolvent of H_D as follows

$$\langle \phi_\omega | \psi_\omega \rangle = \frac{d}{d\omega} \langle \phi | E(\omega)\psi \rangle = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \langle \phi | [R(\omega - i\epsilon) - R(\omega + i\epsilon)]\psi \rangle , \qquad (77)$$

with $E(\omega)$ the spectral family associated to H_D. In addition Theorem 1.7 (Ch.10, §1 [12]) implies that $\langle \phi_\omega | \psi_\omega \rangle$ is absolutely continuous in ω, while the unicity of the spectral family $E(\omega)$ and the existence of a spectral family $F(\omega) = \mathcal{F}E(\omega)\mathcal{F}^{-1}$ on $\mathfrak{H}$ follow directly from theorem 7.15 ([23]). Making use of the relation (3) in (Ch.1, §4 [25]) adapted to our case

$$R(z) = \int_{-\infty}^{+\infty} \frac{1}{\tilde{\omega} - z} dE(\tilde{\omega}) ,$$

we get

$$\frac{1}{2\pi i}[R(\omega - i\epsilon) - R(\omega + i\epsilon)] = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\epsilon}{(\omega - \tilde{\omega})^2 + \epsilon^2} dE(\tilde{\omega}) . \qquad (78)$$

Since the integrand in (78) is bounded and integrable, when we take the limit for $\epsilon \to 0$ in the above expression, we may apply Lebesgue dominated convergence theorem in order to take this limit under the integral sign. Taking into account that

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{(\omega - \tilde{\omega})^2 + \epsilon^2} = \delta(\omega - \tilde{\omega}) ,$$

we find that

$$\frac{1}{2\pi i} \lim_{\epsilon \to 0} [R(\omega - i\epsilon) - R(\omega + i\epsilon)] = \mathrm{Id}.$$

Finally, from (76) it follows that every solution $\hat{\psi}$ of equation (63) can be written as follows

$$\hat{\psi}(t,x) = e^{itH_D}\psi(x) = \int_{\sigma_D} d\omega\, e^{i\omega t} \sum_{k,j \in \mathbb{Z}} \psi_\omega^{kj}(x) \langle \psi_\omega^{kj} | \psi_\omega \rangle \,, \qquad (79)$$

with $x = (r, \vartheta, \varphi)$. In order to construct square integrable wave packets in the coordinate space, we go over in the Minkowski metric in cartesian coordinates by means of the inverse of the coordinate transformations (9) and apply Theorem 1.8 in [21]. Hence, we can choose $\mathcal{H} = L_2 \left([0, +\infty) \times S^2\right)^4$ and normalize the wave packets according to $\langle \psi | \psi \rangle_{\mathcal{H}} = 1$.

The next Lemma describes locally uniformly in ω the asymptotics of the solutions of the radial system

$$\begin{pmatrix} \sqrt{\Delta}\hat{\mathcal{D}}_- & -im_e r - \lambda \\ im_e r - \lambda & \sqrt{\Delta}\hat{\mathcal{D}}_+ \end{pmatrix} \begin{pmatrix} R_- \\ R_+ \end{pmatrix} = 0 \,, \qquad (80)$$

for $r \to +\infty$, where

$$\hat{\mathcal{D}}_\pm = \frac{d}{dr} \mp i \left[\omega + \frac{\left(k + \frac{1}{2}\right) a}{\Delta} \right], \quad \Delta = r^2 + a^2 \,.$$

Lemma 5.3. *Every non trivial solution R of (80) for $|\omega| > m_e$ behave asymptotically for $r \to +\infty$ like*

$$R(r) = \begin{pmatrix} R_-(r) \\ R_+(r) \end{pmatrix} = \begin{pmatrix} \cosh\Theta & \sinh\Theta \\ \sinh\Theta & \cosh\Theta \end{pmatrix} \begin{pmatrix} e^{-i\kappa r}\left[f_-^\infty + \mathcal{O}(r^{-1})\right] \\ e^{+i\kappa r}\left[f_+^\infty + \mathcal{O}(r^{-1})\right] \end{pmatrix},$$

with constants $f_\pm^\infty \neq 0$ and

$$\Theta = \frac{1}{4} \log\left(\frac{\omega + m_e}{\omega - m_e}\right), \quad \kappa = \epsilon(\omega)\sqrt{\omega^2 - m_e^2} \,,$$

where $\epsilon(\omega)$ is a sign function such that $\epsilon(\omega) = +1$ for $\omega > m_e$ and $\epsilon(\omega) = -1$ for $\omega < -m_e$.

Proof. Let us first rewrite (80) as follows

$$R'(r) = V(r)R(r)\,, \quad V(r) = \begin{pmatrix} -i\Omega(r) & \varphi(r) \\ \varphi^*(r) & i\Omega(r) \end{pmatrix}, \qquad (81)$$

where $*$ denotes complex conjugation and

$$\Omega(r) = \omega + \frac{\left(k + \frac{1}{2}\right) a}{\Delta}, \quad \varphi(r) = \frac{\lambda + i m_e r}{\sqrt{\Delta}}.$$

By means of the Ansatz

$$R(r) = \hat{A}\tilde{R}(r), \quad \hat{A} = \begin{pmatrix} C_1 & C_2 \\ C_2 & C_1 \end{pmatrix}, \quad \tilde{R}(r) = \begin{pmatrix} \tilde{R}_-(r) \\ \tilde{R}_+(r) \end{pmatrix},$$

with constants C_1 and C_2 such that $C_1^2 - C_2^2 \neq 0$, (81) becomes

$$\tilde{R}'(r) = T(r)\tilde{R}(r), \quad T(r) = \hat{A}^{-1}V(r)\hat{A} = \begin{pmatrix} -T_{11}(r) & -T_{12}(r) \\ T_{12}(r) & T_{11}(r) \end{pmatrix},$$

where

$$T_{11}(r) = \frac{i\left(C_1^2 + C_2^2\right)\Omega(r) + C_1 C_2 \left(\varphi^*(r) - \varphi(r)\right)}{C_1^2 - C_2^2}$$

$$T_{12}(r) = \frac{2iC_1 C_2 \Omega(r) + C_1^2 \varphi^*(r) - C_2^2 \varphi(r)}{C_1^2 - C_2^2}.$$

We fix now C_1 and C_2 by requiring that

$$\lim_{r \to +\infty} T_{11}(r) = i\kappa, \quad C_1^2 - C_2^2 = 1. \tag{82}$$

Taking into account that asymptotically for $r \to +\infty$, it results that

$$T_{11}(r) = \frac{i\left(C_1^2 + C_2^2\right)\omega - 2i m_e C_1 C_2}{C_1^2 - C_2^2} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

condition (82) becomes

$$\left(C_1^2 + C_2^2\right)\omega - 2m_e C_1 C_2 - \kappa = 0.$$

A simple calculation gives

$$C_1 = \cosh\Theta, \quad C_2 = \sinh\Theta, \quad \Theta = \frac{1}{4}\log\left(\frac{\omega + m_e}{\omega - m_e}\right). \tag{83}$$

With the help of (83) T_{11} and T_{12} can be rewritten as follows

$$T_{11}(r) = \frac{i}{\kappa}\left(\omega\Omega(r) - m_e^2 \frac{r}{\sqrt{\Delta}}\right)$$

$$T_{12}(r) = \frac{\lambda}{\sqrt{\Delta}} + i\frac{m_e}{\kappa}\left(\Omega(r) - \omega\frac{r}{\sqrt{\Delta}}\right).$$

Let us now introduce the decomposition $T(r) = A(r) + B(r)$ with

$$A(r) = \begin{pmatrix} -\frac{i}{\kappa}\left(\omega^2 - m_e^2 \frac{r}{\sqrt{\Delta}}\right) & -\frac{\lambda}{\sqrt{\Delta}} - i m_e \frac{\omega}{\kappa}\left(1 - \frac{r}{\sqrt{\Delta}}\right) \\ \frac{\lambda}{\sqrt{\Delta}} + i m_e \frac{\omega}{\kappa}\left(1 - \frac{r}{\sqrt{\Delta}}\right) & \frac{i}{\kappa}\left(\omega^2 - m_e^2 \frac{r}{\sqrt{\Delta}}\right) \end{pmatrix},$$

and

$$B(r) = \frac{i}{\kappa} \frac{\left(k + \frac{1}{2}\right) a}{\Delta} \begin{pmatrix} -1 & -m_e \\ m_e & 1 \end{pmatrix}.$$

Since $A(r)$ and $B(r)$ are continuously differentiable on $[0, +\infty)$,

$$\int_0^{+\infty} dr \left| A^{'}(r) \right| < \infty \,, \quad \int_0^{+\infty} dr |B(r)| < \infty$$

and the eigenvalues λ_1 and λ_2 of the matrix A_0 defined by

$$A_0 := \lim_{r \to +\infty} A(r) = \begin{pmatrix} -i\kappa & 0 \\ 0 & i\kappa \end{pmatrix}$$

are simple, then Theorem 11 in [7], which is actually a simplified version of a theorem due to Levinson [14], implies that the equation $\tilde{R}^{'}(r) = T(r)\tilde{R}(r)$ has solutions such that for $r \to +\infty$

$$\tilde{R}(r) = \begin{pmatrix} e^{-i\kappa r} \left[f_-^\infty + \mathcal{O}(r^{-1}) \right] \\ e^{+i\kappa r} \left[f_+^\infty + \mathcal{O}(r^{-1}) \right] \end{pmatrix},$$

where f_-^∞ and f_+^∞ are the non zero components of the eigenvectors ξ_- and ξ_+ belonging to the eigenvalues $-i\kappa$ and $+i\kappa$ of the matrix A_0, respectively.   ☑

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