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An optimal 3-point quadrature formula of closed type and error bounds

Una fórmula de cuadratura óptima de 3 puntos de tipo cerrado y error de frontera

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Abstract. An optimal 3-point quadrature formula of closed type is derived. The obtained optimal quadrature formula has better estimations of error than the well- known Simpson's formula. A few error inequalities for this formula are established.

Key words and phrases. Optimal quadrature formula, error inequalities, Ostrowski-like inequalities.

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Resumen. Se establece una fórmula de cuadratura óptima de 3 puntos de tipo cerrado. Dicha fórmula mejora la estimación de error de la bien conocida fórmula de Simpson. Se establecen algunas desigualdades de error para esta fórmula.

Palabras y frases clave. Fórmula de cuadratura óptima, desigualdades de error, desigualdades de tipo de Ostrowski.

1. Introduction and preliminary results

In recent years a number of authors have considered error analyses for quadrature rules of Newton-Cotes type. In particular, the mid-point, trapezoid and Simpson rules have been investigated more recently ([2], [3], [4], [5], [6], [8]) with the view of obtaining bounds on the quadrature rule in terms of a variety of norms involving, at most, the first derivative. In the mentioned papers explicit error bounds for the quadrature rules are given. These results are obtained from an inequalities point of view. The authors use Peano type kernels for obtaining a specific quadrature rule. We say that $\int_a^b f(t)dt = \sum_{k=o}^{n} w_k f(x_k) + R$, where $a \leq x_0 < x_1 < \cdots < x_n \leq b$, $w_k \geq 0$, $k = 0, 1, \ldots, n$, is a quadrature

formula of closed type if the end points are included, i.e., $a = x_0$, $b = x_n$ are nodal points.

Quadrature formulas can be formed in many different ways. For example, we can integrate a Lagrange interpolating polynomial of a function f to obtain a corresponding quadrature formula (Newton-Cotes formulas). We can also seek a quadrature formula such that it is exact for polynomials of maximal degree (Gaussian formulas). Gauss-like quadrature formulas are considered in [9].

Here we present a new approach to this topic. Namely, we give a type of quadrature formula. We also give a way of estimating its error and all parameters which appear in the estimation. Then we seek a quadrature formula of the given type such that the estimation of its error is best possible. Let us consider the above described procedure with more details.

If we define

$$K_2(\alpha,\beta,\gamma,\delta,t) = \begin{cases} \frac{1}{2}(t-\alpha)(t-\beta), & t \in \left[a, \frac{a+b}{2}\right] \\ \frac{1}{2}(t-\gamma)(t-\delta), & t \in \left(\frac{a+b}{2}, b\right] \end{cases},$$

then, integrating by parts, we obtain

$$\begin{aligned} \int_a^b K_2(\alpha,\beta,\gamma,\delta,t) f''(t)dt = \frac{1}{2}\Bigg\{ & f'(b)\,(b-\gamma)\,(b-\delta) \\ & + f'\left(\frac{a+b}{2}\right)\left[\left(\frac{a+b}{2}-\alpha\right)\left(\frac{a+b}{2}-\beta\right)\right. \\ & \left. -\left(\frac{a+b}{2}-\gamma\right)\left(\frac{a+b}{2}-\delta\right)\right] \\ & - f'(a)(a-\alpha)(a-\beta)\Bigg\} + \left(a - \frac{\alpha+\beta}{2}\right) f(a) \\ & - \left(\frac{\gamma+\delta}{2} - \frac{\alpha+\beta}{2}\right) f\left(\frac{a+b}{2}\right) \\ & - \left(b - \frac{\gamma+\delta}{2}\right) f(b) + \int_a^b f(t)dt. \end{aligned}$$

If we choose $\alpha = \beta = a$ and $\gamma = \delta = b$ then we get the mid-point quadrature rule. If we choose $\alpha = \gamma = a$ and $\beta = \delta = b$ then we get the trapezoid rule. If we choose $\alpha = 0$, $\beta = \frac{a+2b}{3}$ and $\gamma = \frac{2a+b}{3}$, $\delta = 1$ then we get Simpson's rule.

If we require that

$$\begin{aligned}(b-\gamma)(b-\delta)&=0\\ \left(\frac{a+b}{2}-\alpha\right)\left(\frac{a+b}{2}-\beta\right)-\left(\frac{a+b}{2}-\gamma\right)\left(\frac{a+b}{2}-\delta\right)&=0\\ (a-\alpha)(a-\beta)&=0,\end{aligned}$$

then we get a quadrature formula of the form

$$\int_a^b K_2(\alpha,\beta,\gamma,\delta,t)f''(t)dt=\left(a-\frac{\alpha+\beta}{2}\right)f(a)-\left(\frac{\gamma+\delta}{2}-\frac{\alpha+\beta}{2}\right)f\left(\frac{a+b}{2}\right)$$
$$-\left(b-\frac{\gamma+\delta}{2}\right)f(b)+\int_a^b f(t)dt. \tag{1}$$

In practice we cannot find an exact value of the remainder term (error) $\int_a^b K_2(\alpha,\beta,\gamma,\delta,t)f''(t)dt$. All we can do is to estimate the error. It can be done in different ways. For example,

$$\left|\int_a^b K_2(\alpha,\beta,\gamma,\delta,t)f''(t)dt\right|\leq \max_{t\in[a,b]}|f''(t)|\int_a^b |K_2(\alpha,\beta,\gamma,\delta,t)|\,dt. \tag{2}$$

It is a natural question which formula of the type (1) is optimal, with respect to a given way of estimation of the error. The main aim of this paper is to give an answer to this question and to consider the formula from an inequalities point of view. In fact, we seek a quadrature formula of the given type such that its error bound is minimal. Note that we can minimize only the factor $\int_a^b |K_2(\alpha,\beta,\gamma,\delta,t)|\,dt$ in (2). A general approach is: we first consider the minimization problem and then we formulate final results. A few error inequalities for the obtained optimal formula are established. Let us mention that the obtained optimal quadrature formula has better estimations of error than the Simpson's formula (see Remark 2).

Finally, we also mention that similar optimal quadrature rules are considered in [10]-[11].

2. **Optimal quadrature formula**

We consider the problem, described in Section 1, on the interval $[0,1]$. Let $\alpha,\beta,\gamma,\delta\in R$. We define the mapping

$$K_2(\alpha,\beta,\gamma,\delta,t)=\begin{cases}\frac{1}{2}(t-\alpha)(t-\beta), & t\in\left[0,\frac{1}{2}\right]\\ \frac{1}{2}(t-\gamma)(t-\delta), & t\in\left(\frac{1}{2},1\right]\end{cases}. \tag{3}$$

Let $I \subset R$ be an open interval such that $[0,1] \subset I$ and let $f : I \to R$ be a twice differentiable function such that f'' is bounded and integrable. We denote

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)| . \tag{4}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_0^1 K_2(\alpha,\beta,\gamma,\delta,t)f''(t)dt &= \frac{1}{2}\int_0^{\frac{1}{2}}(t-\alpha)(t-\beta)f''(t) + \frac{1}{2}\int_{\frac{1}{2}}^1 (t-\gamma)(t-\delta)f''(t)dt \\ &= -\frac{1}{2}\alpha\beta f'(0) + \frac{1}{2}(1-\gamma)(1-\delta)f'(1) \\ &\quad + \frac{1}{2}\left[\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}-\beta\right) - \left(\frac{1}{2}-\gamma\right)\left(\frac{1}{2}-\delta\right)\right] f'\left(\frac{1}{2}\right) \\ &\quad - \int_0^1 K_1(\alpha,\beta,\gamma,\delta,t)f'(t)dt,\end{aligned} \tag{5}$$

where

$$K_1(\alpha,\beta,\gamma,\delta,t) = \begin{cases} t-\frac{\alpha+\beta}{2}, & t\in\left[0,\frac{1}{2}\right] \\ t-\frac{\gamma+\delta}{2}, & t\in\left(\frac{1}{2},1\right]\end{cases} .$$

We require that the coefficients $-\frac{1}{2}\alpha\beta$, $\frac{1}{2}\left[\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}-\beta\right) - \left(\frac{1}{2}-\gamma\right)\left(\frac{1}{2}-\delta\right)\right]$ and $\frac{1}{2}(1-\gamma)(1-\delta)$ be equal to zero. Hence, we require that $\alpha = 0$ or $\beta = 0$ and $\gamma = 1$ or $\delta = 1$. If we choose $\alpha = 0$ and $\delta = 1$ then we get $\beta+\gamma = 1$. If we now substitute $\alpha = 0$, $\gamma = 1-\beta$ and $\delta = 1$ in (5) then we have

$$\begin{aligned}\int_0^1 K_2(0,\beta,1-\beta,1,t)f''(t)dt &= -\int_0^1 K_1(0,\beta,1-\beta,1,t)f'(t)dt \\ &= -\int_0^{\frac{1}{2}}\left(t-\frac{\beta}{2}\right)f'(t)dt - \int_{\frac{1}{2}}^1\left(t-\frac{2-\beta}{2}\right)f'(t)dt \\ &= -\frac{\beta}{2}f(0) - (1-\beta)f\left(\frac{1}{2}\right) - \frac{\beta}{2}f(1) + \int_0^1 f(t)dt.\end{aligned} \tag{6}$$

We also have

$$\left|\int_0^1 K_2(0,\beta,1-\beta,1,t)f''(t)dt\right| \le \|f''\|_\infty \int_0^1 |K_2(0,\beta,1-\beta,1,t)|\,dt\,, \tag{7}$$

and

$$\int_0^1 |K_2(0,\beta,1-\beta,1,t)|\,dt = \frac{1}{2}\int_0^{\frac{1}{2}} t\,|t-\beta|\,dt + \frac{1}{2}\int_{\frac{1}{2}}^1 |t-1+\beta|\,(1-t)dt. \quad (8)$$

We now define

$$g(\beta) = \frac{1}{2}\int_0^{\frac{1}{2}} t\,|t-\beta|\,dt + \frac{1}{2}\int_{\frac{1}{2}}^1 |t-1+\beta|\,(1-t)dt\,, \quad (9)$$

and consider the problem

$$\text{minimize } g(\beta), \qquad \beta \in R. \quad (10)$$

Hence, we should like to find a global minimizer of g. We consider the following cases:

(i) $\beta \le 0$,
(ii) $0 \le \beta \le \frac{1}{2}$,
(iii) $\beta \ge \frac{1}{2}$.

Case (i). If $\beta \le 0$ then $t\,|t-\beta| = t(t-\beta)$, for $t \in \left[0, \frac{1}{2}\right]$ and $|t-1+\beta| \times |t-1| = (t-1+\beta)(t-1)$, for $t \in \left(\frac{1}{2}, 1\right]$. Thus,

$$\begin{aligned} g(\beta) &= \frac{1}{2}\int_0^{\frac{1}{2}} t(t-\beta)dt + \frac{1}{2}\int_{\frac{1}{2}}^1 (t-1+\beta)(t-1)dt \quad (11)\\ &= \frac{1}{24} - \frac{\beta}{8} \ge \frac{1}{24}. \end{aligned}$$

Case (iii). If $\beta \ge \frac{1}{2}$ then $t\,|t-\beta| = t(\beta-t)$, for $t \in \left[0, \frac{1}{2}\right]$ and $|t-1+\beta| \times |t-1| = (t-1+\beta)(1-t)$, for $t \in \left(\frac{1}{2}, 1\right]$. Thus,

$$\begin{aligned} g(\beta) &= \frac{1}{2}\int_0^{\frac{1}{2}} t(\beta-t)dt + \frac{1}{2}\int_{\frac{1}{2}}^1 (t-1+\beta)(1-t)dt \quad (12)\\ &= \frac{\beta}{8} - \frac{1}{24} \ge \frac{1}{48}. \end{aligned}$$

Case (ii). If $0 \le \beta \le \frac{1}{2}$ then

$$t\,|t-\beta| = \begin{cases} t(\beta-t) & t \in [0,\beta] \\ t(t-\beta) & t \in \left(\beta, \frac{1}{2}\right] \end{cases},$$

and

$$|t-1+\beta|\,|t-1| = \begin{cases} (t-1+\beta)(t-1) & t \in \left[\frac{1}{2}, 1-\beta\right] \\ (t-1+\beta)(1-t) & t \in (1-\beta, 1] \end{cases}.$$

Thus,

$$\begin{aligned} g(\beta) &= \frac{1}{2}\int_0^{\beta} t(\beta-t)dt + \frac{1}{2}\int_{\beta}^{\frac{1}{2}} t(t-\beta)dt \qquad (13) \\ &\quad + \frac{1}{2}\int_{\frac{1}{2}}^{1-\beta} (t-1+\beta)(t-1)dt + \frac{1}{2}\int_{1-\beta}^{1} (t-1+\beta)(1-t)dt \\ &= \frac{\beta^3}{3} - \frac{\beta}{8} + \frac{1}{24}. \end{aligned}$$

We have

$$g'(\beta) = \beta^2 - \frac{1}{8} \quad \text{and} \quad g''(\beta) = 2\beta. \qquad (14)$$

We now solve the equation $g'(\beta) = 0$. The solutions of this equation are $\beta_{1,2} = \pm\frac{\sqrt{2}}{4}$. Since $g''\left(\frac{\sqrt{2}}{4}\right) > 0$ we conclude that $\beta = \frac{\sqrt{2}}{4}$ is, at least, a local minimizer. We have

$$g\left(\frac{\sqrt{2}}{4}\right) = \frac{2-\sqrt{2}}{48}. \qquad (15)$$

From (11), (12) and (15) we conclude that $\beta = \frac{\sqrt{2}}{4}$ is the global minimizer. If we now substitute $\beta = \frac{\sqrt{2}}{4}$ in (6) then we get

$$\begin{aligned} &\int_0^1 K_2\left(0, \frac{\sqrt{2}}{4}, 1-\frac{\sqrt{2}}{4}, 1, t\right) f''(t)dt \qquad (16) \\ = &\int_0^1 f(t)dt - \frac{\sqrt{2}}{8}f(0) - \left(1-\frac{\sqrt{2}}{4}\right) f\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{8}f(1). \end{aligned}$$

The above quadrature formula is optimal in the sense described in Section 1.

From the previous considerations we can formulate the following result.

Theorem 1. *Let $I \subset R$ be an open interval such that $[0,1] \subset I$ and let $f : I \to R$ be a twice differentiable function such that f'' is bounded and integrable. Then we have*

$$\left|\int_0^1 f(t)dt - \frac{\sqrt{2}}{8}f(0) - \left(1-\frac{\sqrt{2}}{4}\right) f\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{8}f(1)\right| \leq \frac{2-\sqrt{2}}{48}\,\|f''\|_\infty. \quad (17)$$

Remark 2. *If we set* $\beta = \frac{1}{3}$ *in* (6) *then we get Simpson's rule:*

$$\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] = \int_0^1 K_2\left(0, \frac{1}{3}, \frac{2}{3}, 1, t\right) f''(t)dt. \quad (18)$$

We have

$$\left|\int_0^1 f(t)dt - \frac{1}{6}f(0) - \frac{2}{3}f\left(\frac{1}{2}\right) - \frac{1}{6}f(1)\right| \leq \frac{\|f''\|_\infty}{81}. \quad (19)$$

It is obvious that (17) *is a better estimate than* (19). *Note that* (16) *and* (18) *are 3-point quadrature rules of the same (closed) type.*

3. Error inequalities

On the space of square integrable functions, $L_2(a, b)$, we introduce the standard inner product

$$(f, g) = \int_a^b f(t)g(t)dt, \quad (20)$$

with the corresponding norm written $\|f\|_2$. The resulting space is a Hilbert space. We also define the Chebyshev functional

$$T(f, g) = \langle f, g\rangle - \langle f, e\rangle \langle g, e\rangle, \quad (21)$$

where $f, g \in L_2(a, b)$ and $e = 1$. This functional satisfies the pre-Grüss inequality

$$T(f, g)^2 \leq T(f, f)\, T(g, g). \quad (22)$$

We now define

$$\sigma(f) = \sigma(f; a, b) = \sqrt{(b-a)T(f, f)}. \quad (23)$$

More about the above mentioned quantities can be found, for example, in [7] and [1].

Finally, we define the functional

$$\begin{aligned} Q(f) &= Q(f; a, b) \quad (24)\\ &= \int_a^b f(t)dt - \left[\frac{\sqrt{2}}{8}f(a) + \left(1 - \frac{\sqrt{2}}{4}\right) f\left(\frac{a+b}{2}\right) + \frac{\sqrt{2}}{8}f(b)\right](b-a). \end{aligned}$$

We need the following lemma.

Lemma 3. *Let*

$$f(t) = \begin{cases} f_1(t), & t \in [a, x_0] \\ f_2(t), & t \in (x_0, b] \end{cases}, \quad (25)$$

where $x_0 \in [a, b]$, $f_1 \in C^1(a, x_0)$, $f_2 \in C^1(x_0, b)$. *If* $f_1(x_0) = f_2(x_0)$ *then* f *is an absolutely continuous function.*

A proof of this lemma can be found in [12]. We now define

$$P(f;a,b)=\frac{(b-a)^2}{96}\left(4-3\sqrt{2}\right)\left[f'(b)-f'(a)\right]. \tag{26}$$

Note that

$$Q(f;a,b)-P(f;a,b)=R(f;a,b)$$

is a corrected quadrature formula (with the remainder $R(f;a,b)$) which is similar to the corrected trapezoid and corrected mid-point quadrature formulas. It has similar properties as the last two mentioned formulas which are known and can be found in the literature. Here we only mention that the corrected formula improves the original formula.

For the simplicity, in this paper we choose $[a,b]=[0,1]$. How we can obtain corresponding results in the arbitrary interval $[a,b]$ it is described, for example, in [13]. In this book we can also find how to write corresponding compound formulas.

Theorem 4. *Let $f':[0,1]\to R$ be an absolutely continuous function such that $f''\in L_1(0,1)$ and there exist real numbers γ_2,Γ_2 such that $\gamma_2\le f''(t)\le\Gamma_2$, $t\in[0,1]$. Then*

$$|Q(f;0,1)-P(f;0,1)|\le\frac{\Gamma_2-\gamma_2}{2}\left(\frac{5}{96}\sqrt{6}-\frac{29}{432}\sqrt{3}\right), \tag{27}$$

where $Q(f;0,1)$ and $P(f;0,1)$ are defined by (24) and (26), respectively.

If there exists a real number γ_2 such that $\gamma_2\le f''(t)$, $t\in[0,1]$ then

$$|Q(f;0,1)-P(f;0,1)|\le\left(\frac{1}{12}-\frac{\sqrt{2}}{32}\right)(S_1-\gamma_2), \tag{28}$$

where $S_1=f'(1)-f'(0)$.

If there exists a real number Γ_2 such that $f''(t)\le\Gamma_2$, $t\in[0,1]$ then

$$|Q(f;0,1)-P(f;0,1)|\le\left(\frac{1}{12}-\frac{\sqrt{2}}{32}\right)(\Gamma_2-S_1). \tag{29}$$

Proof. We define the function

$$\tilde{p}_2(t)=\begin{cases}\frac{1}{2}t\left(t-\frac{\sqrt{2}}{4}\right)+\frac{\sqrt{2}}{32}-\frac{1}{24}, & t\in\left[0,\frac{1}{2}\right]\\ \frac{1}{2}(t-1)\left(t-1+\frac{\sqrt{2}}{4}\right)+\frac{\sqrt{2}}{32}-\frac{1}{24}, & t\in\left(\frac{1}{2},1\right]\end{cases}. \tag{30}$$

Let p_1 be defined by

$$p_1(t)=\begin{cases}t-\frac{\sqrt{2}}{8}, & t\in\left[0,\frac{1}{2}\right]\\ t-1+\frac{\sqrt{2}}{8}, & t\in\left(\frac{1}{2},1\right]\end{cases}. \tag{31}$$

Then we have

$$(\tilde{p}_2,f'')=-(p_1,f')-P(f;0,1)=Q(f;0,1)-P(f;0,1) \tag{32}$$

since

$$(p_1, f') = -Q(f; 0, 1)\,, \tag{33}$$

holds.

On the other hand, we have

$$\left(f'' - \frac{\Gamma_2 + \gamma_2}{2}, \tilde{p}_2\right) = (f'', \tilde{p}_2), \tag{34}$$

since $(\tilde{p}_2, e) = 0$. From (22) we get

$$\begin{aligned} \left|\left(f'' - \frac{\Gamma_2 + \gamma_2}{2}, \tilde{p}_2\right)\right| &\leq \left\|f'' - \frac{\Gamma_2 + \gamma_2}{2}\right\|_\infty \|\tilde{p}_2\|_1 \\ &\leq \left(\frac{5}{96}\sqrt{6} - \frac{29}{432}\sqrt{3}\right)\frac{\Gamma_2 - \gamma_2}{2}, \end{aligned} \tag{35}$$

since

$$\left\|f'' - \frac{\Gamma_2 + \gamma_2}{2}\right\|_\infty \leq \frac{\Gamma_2 - \gamma_2}{2}\,,$$

and

$$\|\tilde{p}_2\|_1 = \frac{5}{96}\sqrt{6} - \frac{29}{432}\sqrt{3}\,.$$

From (32)-(35) we see that (27) holds.

We now prove that (28) holds. We have

$$|(f'' - \gamma_2, \tilde{p}_2)| \leq \|f'' - \gamma_2\|_1 \|\tilde{p}_2\|_\infty = \left(\frac{1}{12} - \frac{\sqrt{2}}{32}\right)(S_1 - \gamma_2),$$

since

$$\|f'' - \gamma_2\|_1 = \int_0^1 (f''(t) - \gamma_2)\, dt = f'(1) - f'(0) - \gamma_2\,,$$

and

$$\|\tilde{p}_2\|_\infty = \frac{1}{12} - \frac{\sqrt{2}}{32}\,.$$

In a similar way we can prove that (29) holds. ☑

Remark 5. *Note that we can apply the estimate (27) only if the second derivative f'' is bounded. It means that we cannot use (27) to estimate directly the error when approximating the integral of such a well-behaved function as $f(t) = \sqrt{t^3}$ on $[0, 1]$, (since $f''(t) = 3/(4\sqrt{t})$ is unbounded on $[0, 1]$). On the other hand, we can use the estimation (28), (since $\gamma = 3/4$ on $[0, 1]$ for the given function).*

Theorem 6. *Let $f' : [0, 1] \to R$ be an absolutely continuous function such that $f'' \in L_2(0, 1)$. Then*

$$|Q(f; 0, 1) - P(f; 0, 1)| \leq \sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}}\,\sigma\,(f''; 0, 1)\,, \tag{36}$$

where $\sigma(f;0,1)$ is defined by (23). The inequality (36) is sharp in the sense that the constant $\sqrt{\frac{47}{23040}-\frac{\sqrt{2}}{768}}$ cannot be replaced by a smaller one.

Proof. We define the function

$$p_2(t)=\begin{cases} \frac{1}{2}t\left(t-\frac{\sqrt{2}}{4}\right) & t\in\left[0,\frac{1}{2}\right] \\ \frac{1}{2}(t-1)\left(t-1+\frac{\sqrt{2}}{4}\right) & t\in\left(\frac{1}{2},1\right] \end{cases}. \tag{37}$$

Then we have

$$\langle\tilde{p}_2,f''\rangle=\langle p_2,f''\rangle-\langle p_2,e\rangle\langle f'',e\rangle\,, \tag{38}$$

since $\tilde{p}_2=p_2-\langle p_2,e\rangle$. From (32) and (38) it follows

$$T(p_2,f'')=Q(f;0,1)-P(f;0,1)\,, \tag{39}$$

since $\langle\tilde{p}_2,f''\rangle=(\tilde{p}_2,f'')$ if $[a,b]=[0,1]$. From (22) we get

$$|T(p_2,f'')|\le\sqrt{T(p_2,p_2)}\sqrt{T(f'',f'')}=\sqrt{\frac{47}{23040}-\frac{\sqrt{2}}{768}}\sigma(f'';0,1)\,, \tag{40}$$

since

$$T(p_2,p_2)=\frac{47}{23040}-\frac{\sqrt{2}}{768}\,.$$

From (39) and (40) we see that (36) holds.

We now prove that (36) is sharp. For that purpose we define the function

$$f(t)=\begin{cases} \frac{t^4}{24}-\frac{\sqrt{2}}{48}t^3, & t\in\left[0,\frac{1}{2}\right] \\ \frac{t^4}{24}-\left(\frac{1}{6}-\frac{\sqrt{2}}{48}\right)t^3+\left(\frac{1}{4}-\frac{\sqrt{2}}{16}\right)t^2-\left(\frac{1}{8}-\frac{\sqrt{2}}{32}\right)t+\frac{1}{48}-\frac{\sqrt{2}}{192}, & t\in\left(\frac{1}{2},1\right] \end{cases}, \tag{41}$$

such that

$$f'(t)=\begin{cases} \frac{t^3}{6}-\frac{\sqrt{2}}{16}t^2, & t\in\left[0,\frac{1}{2}\right] \\ \frac{t^3}{6}-\left(\frac{1}{2}-\frac{\sqrt{2}}{16}\right)t^2+\left(\frac{1}{2}-\frac{\sqrt{2}}{8}\right)t-\left(\frac{1}{8}-\frac{\sqrt{2}}{32}\right), & t\in\left(\frac{1}{2},1\right] \end{cases}, \tag{42}$$

and $f''(t)=p_2(t)$. From Lemma 3 we see that the function f', defined by (42), is an absolutely continuous function. For the function defined by (41) the left-hand side of (36) becomes

$$L.H.S.(36)=\frac{47}{23040}-\frac{\sqrt{2}}{768}\,.$$

The right-hand side of (36) becomes

$$R.H.S.(36)=\frac{47}{23040}-\frac{\sqrt{2}}{768}\,.$$

We see that $L.H.S.(36)=R.H.S.(36)$. Thus, (36) is sharp. ☑

Remark 7. *The estimation (27) is better than the estimation (36). However, note that we can apply the estimate (27) only if the second derivative f'' is bounded. It means that we cannot use (27) to estimate directly the error when approximating the integral of such a well-behaved function as $f(t) = \sqrt[3]{t^5}$ on $[0,1]$, (since $f''(t) = 10/\left(9\sqrt[3]{t}\right)$ is unbounded on $[0,1]$). On the other hand, we can use the estimation (36) (since $\|f''\|_2^2 = \frac{100}{27}$ for the given function).*

Note also that the estimation (36) is expressed by means of the quantity $\sigma(f'';0,1)$. This is a better estimation than an estimation expressed by means of the norm $\|f''\|_2$, since

$$\sigma(f'';0,1) = \sqrt{\|f''\|_2^2 - (f'(1) - f'(0))^2} \leq \|f''\|_2 .$$

Furthermore, the term $(f'(1) - f'(0))^2$ can be easily calculated.

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