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Existence of global weak solutions to a symmetrically hyperbolic system with a source

Existencia de soluciones débiles globales para un sistema hiperbólico simétrico con una fuente

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Abstract. In this paper the existence of global bounded weak solutions is obtained for the Cauchy problem of a symmetrically hyperbolic system with a source by using the theory of compensated compactness. This system arises in such areas as elasticity theory, magnetohydrodynamics, and enhanced oil recovery.

Key words and phrases. Symmetrically hyperbolic system, source terms, weak solution, compensated compactness method.

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Resumen. En este artículo se obtiene la existencia de soluciones débiles acotadas globalmente para el problema de Cauchy de un sistema simétricamente hiperbólico con una fuente, usando la teoría de la compacidad compensada. Este sistema surge en areas como la teoría de la elasticidad, la magneto-hidrodinámica y el mejoramiento en la recuperación de petróleo.

Palabras y frases clave. Sistema simétrico hiperbólico, términos fuente, solución débil, método de compacidad compensada.

1. Introduction

In this paper, we are concerned with a symmetrically hyperbolic system of two equations with source terms

$$\begin{cases} u_t + (u\phi(r))_x + g_1(u,v) &= 0 \\ v_t + (v\phi(r))_x + g_2(u,v) &= 0 \end{cases}, \tag{1.1}$$

and bounded measurable initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \tag{1.2}$$

where $\phi(r)$ is a nonlinear symmetric function of u, v with $r = u^2 + v^2$. In the paper [9], Lu studied the homogeneous system of Cauchy problem (1.1)-(1.2) with $g_1(u,v) = g_2(u,v) = 0$, this homogeneous system is interesting because it arises from such areas as elasticity theory, magnetohydrodynamics, and enhanced oil recovery (cf. [5, 6]). In this article we study a symmetrically hyperbolic system with source terms, which is also interesting in mathematic and fluid mechanics.

Let F be the mapping from R^2 into R^2 defined by

$$F : (u,v) \to (u\phi(r), v\phi(r)),$$

then two eigenvalues of dF are

$$\lambda_1 = \phi(r), \quad \lambda_2 = \phi(r) + 2r\phi'(r), \tag{1.3}$$

with corresponding right eigenvectors

$$r_1 = (-v, u)^T, \quad r_2 = (u, v)^T. \tag{1.4}$$

By simple calculations,

$$\nabla\lambda_1 \cdot r_1 = 0, \quad \nabla\lambda_2 \cdot r_2 = 6r\phi^{'}(r) + 4r^2\phi^{''}(r). \tag{1.5}$$

Therefore, from (1.3) the strict hyperbolicity of system (1.1) fails at the points where $r\phi'(r) = 0$, and from (1.5) the first characteristic field is always linearly degenerate and the second characteristic field is either genuinely nonlinear or linearly degenerate, depending on the behavior of ϕ.

In this article, we suppose that

$$\phi \in C^2\left(R^+\right), \qquad \text{meas } \left\{r : 3r\phi'(r) + 2r^2\phi''(r) = 0\right\} = 0. \tag{1.6}$$

Therefore the second characteristic field could be linearly degenerate on a set of Lebesgue measure zero.

The study of the Cauchy problem (1.1)-(1.2) with $g_1(u,v)$=$g_2(u,v)$=0 by using the compensated compactness method started from [1], where Chen first considered the propagation and cancelation of oscillations for the weak solution. Along the second genuinely nonlinear characteristic field, the initial oscillations cannot propagate and are killed instantaneously as time evolves, but along the first linearly degenerate field, the initial oscillations can propagate. Lu [9] studied the homogeneous system of Cauchy problem (1.1)-(1.2) with $g_1(u,v) = g_2(u,v) = 0$ and obtained the existence of bounded weak solutions for the Cauchy problem of a symmetrically homogeneous hyperbolic system.

In this work, we study the system of Cauchy problem (1.1)-(1.2) and have the same conclusion for some source terms.

2. The main theorem

For studying the Cauchy problem (1.1)-(1.2), we consider the Cauchy problem for the related parabolic system.

$$\begin{cases} u_t + (u\phi(r))_x + g_1(u,v) = \varepsilon u_{xx} \\ v_t + (v\phi(r))_x + g_2(u,v) = \varepsilon v_{xx} \end{cases}, \tag{2.1}$$

with the initial data (1.2).

We suppose that the functions $g_1(u,v)$ and $g_2(u,v)$ satisfy the following conditions:

(H1) Both $g_1(u,v)$ and $g_2(u,v)$ are local Lipchitz continuous functions,

(H2) $ug_1(u,v) + vg_2(u,v) \geq Cr + \widetilde{C}$, where $C, \widetilde{C}$ are constants.

(H3) $g_2(u,v) = vh(u,v)$, $h(u,v) \in C\left(R^2\right)$.

(H4) $\left|\frac{vg_1(u,v)-ug_2(u,v)}{v^2}\right| \leq C_1\left|\frac{u}{v}\right| + \widetilde{C_1}$. where $C_1, \widetilde{C_1} > 0$ are constants.

(H5) There exists a continuous function $G(w)$, such that:

$$\frac{vg_1(u,v) - ug_2(u,v)}{v^2} = G\left(\frac{u}{v}\right),$$

and

$$G^{'}(w) \geq 0.$$

The main result in this work is given as follows:

Theorem 2.1. *(1) Suppose the initial data $(u_0(x), v_0(x))$ be bounded measurable and the conditions* (H1)-(H2) *are hold. Then for fixed $\varepsilon > 0$, the viscosity solution $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ of the Cauchy problem (2.1) and (1.2) exists and is uniformly bounded with respect to the viscosity parameter ε.*

(2) Moreover, if condition (1.6) holds, then there exists a subsequence of $r^\varepsilon = (u^\varepsilon)^2 + (v^\varepsilon)^2$ (still labeled r^ε) which converges pointwisely to a function $l(x,t)$.

(3) If $v_0(x) \geq c_0 > 0$ for a constant c_0 , the total variation of $\frac{u_0(x)}{v_0(x)}$ is bounded in $(-\infty, +\infty)$ and the conditions (H1)-(H5) *are hold, then there exists a subsequence of $(u^\varepsilon, v^\varepsilon)$ (still denoted by $(u^\varepsilon, v^\varepsilon)$) which converges pointwisely to a pair of functions $(u(x,t), v(x,t))$ satisfying $l(x,t) = u^2(x,t) + v^2(x,t)$, which, combining with* 2.*, implies that the limit (u,v) is a weak solution of the hyperbolic system* (1.1) *with the initial data* (1.2).

Remark 2.1. *Since $(u^\varepsilon, v^\varepsilon)$ is uniformly bounded with respect to ε, its weak-star limit (u,v) always exists. However, the strong limit $l(x,t)$ of $(u^\varepsilon)^2 + (v^\varepsilon)^2$ need not equal $u^2(x,t) + v^2(x,t)$. If this equality is true, then, at least, (u,v) is a weak solution of* (1.1)-(1.2) *without any more conditions, such as which are given in part* 3.

Remark 2.2. *There are many functions $g_1(u,v)$ and $g_2(u,v)$ which are satisfied the conditions* (H1)-(H2)*, but there are some functions $g_1(u,v)$ and $g_2(u,v)$ which satisfy the conditions* (H1)-(H5)*. For example, $g_1(u,v) = au + bv$ and*

$g_2(u,v) = cv$ satisfy the conditions (H1)-(H5)*, where a, b, c is constants and $a \geq c$. For another instance, $g_1(u,v) = \alpha_1\sqrt{u^2+v^2} + \alpha_2 v$ and $g_2(u,v) = \alpha_3 v$ also satisfy the conditions* (H1)-(H5)*, where $\alpha_i \in R$, $i = 1, 2, 3$ and $\alpha_1 \geq \alpha_3$.*

The proof of Theorem 2.1 will be given in Section 4.

3. Some lemmas

By using the theory of compensated compactness, BV compactness and the maximum principle, the existence of global bounded weak solutions is obtained for the Cauchy problem (1.1)-(1.2). To prove this conclusion, at first we introduce some lemmas which are useful later in this paper.

Let us consider the following Cauchy problem for the general parabolic system

$$\begin{cases} u_t + f_1(u,v)_x + k_1(u,v) &= \varepsilon u_{xx} \\ v_t + f_2(u,v)_x + k_2(u,v) &= \varepsilon v_{xx} \end{cases}, \tag{3.1}$$

with the initial data (1.2).

Lemma 3.1. *Suppose that the initial data $(u_0(x), v_0(x))$ be bounded measurable (that is $|u_0(x)| \leq M$, $|v_0(x)| \leq M$), $f_i(u,v) \in C^1\left(R^2\right)$ and $k_i(u,v)$ is locally Lipschitz continuous function, $i = 1, 2$. Then*

(1) The Cauchy problem (3.1) *and* (1.2) *has unique solution $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \in C^\infty(R \times (0,t_0))$ for a small $t_0 > 0$ which depends only on the L^∞ norm of the initial data, and*

$$|u^\varepsilon(x,t)| \leq 2M, \qquad |v^\varepsilon(x,t)| \leq 2M, \qquad \forall\,(x,t) \in R \times [0,t_0).$$

(2) Moreover, if the solution $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ has an a priori estimate for arbitrary fixed $T > 0$

$$|u^\varepsilon(x,t)| \leq M(T), \quad |v^\varepsilon(x,t)| \leq M(T), \quad \forall\,(x,t) \in R \times [0,T],$$

where $M(T)$ is a positive constant, being independent of ε for arbitrary fixed $T > 0$, then the solution $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ exists on $R \times [0,T]$.

Proof. We will give a sketch of the proof; for details see [3, 7, 12].

(1) The Cauchy problem (3.1) and (1.2) is equivalent to the following integral equations:

$$\begin{aligned} u(x,t) = &\int_{-\infty}^{+\infty} u_0(y) G^\varepsilon(x-y,t)\,dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} \big[f_1(u(y,\tau), v(y,\tau)) G_y^\varepsilon(x-y,t-\tau) \\ &- k_1(u(y,\tau), v(y,\tau)) G^\varepsilon(x-y,t-\tau)\big]\,dy\,d\tau\,. \end{aligned}$$

$$
\begin{aligned}
v(x,t) = &\int_{-\infty}^{+\infty} v_0(y) G^{\varepsilon}(x-y,t)dy \\
&+ \int_0^t \int_{-\infty}^{+\infty} \big[f_2(u(y,\tau),v(y,\tau)) G^{\varepsilon}_y(x-y,t-\tau) \\
&- k_2(u(y,\tau),v(y,\tau)) G^{\varepsilon}(x-y,t-\tau) \big] dy d\tau \,.
\end{aligned}
$$

The existence of the local solution can be easily obtained by applying the contraction mapping principle to above integral representation for a solution. Following the standard theory of semilinear parabolic systems, we get unique solution $(u^{\varepsilon}(x,t), v^{\varepsilon}(x,t)) \in C^{\infty}(R\times(0,t_0))$ for a small $t_0>0$ which depends only on the L^{∞} norm of the initial data.

(2) Since the solution $(u^{\varepsilon}(x,t), v^{\varepsilon}(x,t))$ has an a priori estimate $M(T)$ for arbitrary fixed $T>0$ and $t_0>0$ depends only on the L^{∞} norm of the initial data, we can use $(u(x,t_0), v(x,t_0))$ as new initial data on the line $t=t_0$ and above a priori estimate $M(T)$, we have a smooth solution on $t_0\le t\le t_0+\tau$ for the Cauchy problem (3.1) and (1.2). So we repeat this process to find a solution on $t_0\le t\le t_0+2\tau$, and eventually after a finite number of steps we obtain a solution on $0\le t\le T$. ☑

Lemma 3.2. *Suppose that $u(x,t)$ is a solution for the Cauchy problem of the parabolic equation*

$$u_t + a(u,x,t)u_x + g(u,x,t) = u_{xx}\,, \tag{3.2}$$

and the initial data

$$u(x,0) = u_0(x)\,. \tag{3.3}$$

Also suppose that the functions $u_0(x)$ and $g(u,x,t)$ satisfy the following conditions: $|u_0(x)|\le M$, $|g(u,x,t)|\le C|u|+\bar{C}$, where $C,\bar{C}>0$ and $a(u,x,t)$ is bounded. Then for any $T>0$, there exists $M(T)>0$ such that $|u(x,t)|\le M(T)$ on $R\times[0,T]$.

Proof. Multiplying equation (3.2) by $2u$, we have

$$
\begin{aligned}
\big(u^2\big)_t + a(u,x,t)\big(u^2\big)_x &= 2uu_{xx} - 2ug(u,x,t) \\
&\le (2uu_x)_x - 2u_x^2 + 2|u|\left(C|u|+\bar{C}\right) \\
&\le \big(u^2\big)_{xx} + (2C+1)u^2 + \bar{C}^2.
\end{aligned}
$$

Let $w = \left(u^2 + \frac{\bar{C}^2}{2C+1}\right) e^{-(2C+1)t}$. Direct calculations show that

$$w_t + a(u,x,t)w_x \le w_{xx}\,. \tag{3.4}$$

Since the initial data $u_0\le M$, so $w(x,0) = (u_0)^2 + \frac{\bar{C}^2}{2C+1} \le M^2 + \frac{\bar{C}^2}{2C+1}$. Using the maximum principle [11] to (3.4), we get $w(x,t)\le M^2+\frac{\bar{C}^2}{2C+1}$, from the relationship between $w(x,t)$ and $u(x,t)$, we obtain the following L^∞ estimates of $u(x,t)$:

$$|u(x,t)| \leq M(T)\,, \quad (x,t) \in R \times [0,T]\,,$$

where $M(T) = \left[\left(M^2 + \frac{\tilde{C}^2}{2C+1}\right) e^{(2C+1)t}\right]^{\frac{1}{2}}$. ☑

From Lemma (3.2), we have

Corollary 3.1. *Suppose that* $u(x,t) \geq (\leq) 0$ *satisfies*

$$u_t + a(u,x,t)u_x + g(u,x,t) \leq (\geq) u_{xx}\,,$$

and $|u(x,0)| \leq M$, $g(u,x,t) \geq (\leq) Cu + \bar{C}$, *where* $C, \bar{C} \in R$ *and* $a(u,x,t)$ *is bounded. Then for any* $T > 0$, *there exists* $M(T) > 0$ *such that* $u(x,t) \leq M(T)(u(x,t) \geq -M(T))$ *on* $R \times [0,T]$.

Lemma 3.3. *Suppose that* $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ *is a solution for Cauchy problem (2.1) and (1.2). Also suppose that the conditions* $v_0(x) \geq c_0 > 0$, (H1) *and* (H3) *are hold. If* $|u^\varepsilon(x,t)| \leq M(\varepsilon, c_0, T)$, $|v^\varepsilon(x,t)| \leq M(\varepsilon, c_0, T)$ *on* $R \times [0,T]$, *then the solution* $v^\varepsilon(x,t) \geq c(t, c_0, \varepsilon) > 0$ *on* $R \times [0,T]$, *where* $c(t, c_0, \varepsilon)$ *could tend to zero as* c_0, ε *tend to zero or* t *tends to infinity.*

Proof. Let $\omega = \log v$, we rewrite the second equation of the related parabolic system (2.1) as follows:

$$\omega_t + \phi(r)\omega_x + \phi(r)_x + h(u,v) = \varepsilon\left(\omega_{xx} + \omega_x^2\right)\,, \tag{3.5}$$

then

$$\omega_t = \varepsilon\omega_{xx} + \varepsilon\left(\omega_x - \frac{\phi(r)}{2\varepsilon}\right)^2 - \phi(r)_x - \frac{\phi^2(r)}{4\varepsilon} - h(u,v)\,.$$

The solution ω^ε of (3.5) with initial data $\omega_0(x) = \log v_0(x)$ can be represented by a Green function $G^\varepsilon(x-y,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \exp\left\{-\frac{(x-y)^2}{4\varepsilon t}\right\}$:

$$\begin{aligned} \omega^\varepsilon = \int_{-\infty}^{\infty} G^\varepsilon(x-y,t)\omega_0(y)dy + \int_0^t \int_{-\infty}^{\infty} \Big[\varepsilon(\omega_x - \frac{\phi(r)}{2\varepsilon})^2 - \phi(r)_x \\ - \frac{\phi^2(r)}{4\varepsilon} - h(u,v)\Big]\left[G^\varepsilon(x-y,t-s)\right] dyds\,. \end{aligned} \tag{3.6}$$

Since,

$$\int_{-\infty}^{\infty} G^\varepsilon(x-y,t)dy = 1\,, \quad \int_0^t \int_{-\infty}^{\infty} \left|G_y^\varepsilon(x-y,t-s)\right| dyds = 2\sqrt{\frac{t}{\pi\varepsilon}}\,, \quad (t>0),$$

it follows from (3.6) that

$$\begin{aligned}\omega^\varepsilon &\geq \log c_0 + \int_0^t \int_{-\infty}^{\infty} \left(-\phi(r)_x - \frac{\phi^2(r)}{4\varepsilon} - h(u,v)\right) G^\varepsilon(x-y,t-s)dyds \\ &= \log c_0 + \int_0^t \int_{-\infty}^{\infty} \Bigg[\phi(r)G_y^\varepsilon(x-y,t-s) \\ &\quad - \left(\frac{\phi^2(r)}{4\varepsilon} + h(u,v)\right) G^\varepsilon(x-y,t-s)\Bigg]dyds \\ &\geq \log c_0 - 2M_1\sqrt{\frac{t}{\pi\varepsilon}} - M_2 t = -C(t,c_0,\varepsilon) > -\infty\,.\end{aligned}$$

Thus $v^\varepsilon(x,t)$ has a positive lower bound $c(t,c_0,\varepsilon)$. ☑

Let us consider the Cauchy problem for scalar conservation laws

$$u_t + f(u)_x = 0\,, \tag{3.7}$$

and bounded measurable initial data

$$u(x,0) = u_0(x)\,. \tag{3.8}$$

The following lemmas are about the BV compactness and the compensated compactness frameworks for Cauchy problem of scalar conservation laws (3.7)-(3.8).

Lemma 3.4. *(See* [12]*) Suppose that a sequence of function $u^\varepsilon(x,t)$ satisfies*

$$|u^\varepsilon|_{L^\infty} \leq C|u_0|_{L^\infty}\,, \quad TV(u^\varepsilon) \leq CTV(u_0)\,,$$

where $u^\varepsilon(x,t)$ is a viscosity approximate solution of Cauchy problem (3.7)-(3.8), the constant C is independent of ε and $TV(u)$ is the total variation of u. Then there exists a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ such that

$$u^{\varepsilon_k}(t,x) \to u(t,x)\,, \quad a.e.,\ as, \quad k\to\infty\,.$$

This limiting function $u(t,x)$ is a bounded weak solutions for Cauchy problem of scalar conservation laws (3.7)-(3.8)*.*

Remark 3.1. *The role of BV norms is indicated by Glimm's theorem* [12] *for hyperbolic systems (also see* [10, 4, 14]*).*

Lemma 3.5. *(See* [2]*) Suppose that a sequence of function $u^\varepsilon(x,t)$ satisfies*

$$|u^\varepsilon(x,t)|_{L^\infty} \leq M\,,$$

where $u^\varepsilon(x,t)$ *is a viscosity approximate solution of Cauchy problem* (3.7)-(3.8), *and for two entropy pairs*

$$(\eta_1(u), q_1(u)) = (u-k, f(u)-f(k))$$

$$(\eta_2(u), q_2(u)) = \left(f(u)-f(k), \int_k^u f'(y)^2 dy \right),$$

satisfies

$$\eta_i\left(u^\varepsilon(x,t)\right)_t + q_i\left(u^\varepsilon(x,t)\right)_x \quad \text{is compact in} \quad W^{-1,2}_{loc}\left(R\times R^+\right),$$

where $k \in R, i = 1,2$. *Then*

(1) There exists a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ *such that*

$$w^* - \lim_{k\to\infty} u^{\varepsilon_k} = u, \qquad w^* - \lim_{k\to\infty} f(u^{\varepsilon_k}) = f(u).$$

(2) Furthermore, if there is no interval in which the flux function $f(u)$ *is linear, then the sequence* $u^\varepsilon(t,x)$ *is compact in* $L^1_{loc}(R\times R^+)$. *That is, if* $f \in C^2(R\times R^+)$ *and* meas $\{u : f''(u) = 0\} = 0$, *then* $u^{\varepsilon_k}(t,x) \to u(t,x)$, a.e *as* $k\to\infty$. *This limiting function* $u(t,x)$ *is a bounded weak solutions for Cauchy problem of scalar conservation laws* (3.7)-(3.8).

Remark 3.2. *The simple proof of this lemma can see* [2, 8]. *A rigorous proof by using infinite entropy pairs was first given by Tartar* [13].

4. Proof of Theorem 2.1

In this section we proof Theorem 2.1 by using the compensated compactness method and BV compactness frameworks.

Proof. (1) According to Lemma 3.1, to prove the existence of the viscosity solution in Theorem 2.1, it is sufficient to get the uniform L^∞ bound. Multiplying the first and second equations of the parabolic system (2.1) by $2u$ and $2v$, respectively, then adding the result, we have

$$r_t + \phi(r)r_x + 2r(\phi(r))_x + 2ug_1(u,v) + 2vg_2(u,v) = \varepsilon r_{xx} - 2\varepsilon\left(u_x^2 + v_x^2\right). \quad (4.1)$$

By using the condition (H2), we get the following inequality

$$r_t + f(r)_x + 2Cr + 2\widetilde{C} \le \varepsilon r_{xx}, \quad (4.2)$$

where $f(r) = \int_0^r \phi(s) + 2r\phi'(s)ds$.

Since the initial data $(u_0(x), v_0(x))$ is bounded measurable, we have

$$r(x,0) = u^2(x,0) + v^2(x,0) \le M.$$

Using Corollary 3.1 to (4.2), we obtain the following L^∞ estimates of $r^\varepsilon(x,t)$

$$r^\varepsilon = (u^\varepsilon)^2 + (v^\varepsilon)^2 \le M(T), \quad (x,t)\in R\times[0,T], \quad (4.3)$$

where $M(T)$ is a positive constant, being independent of ε, this implies the uniform boundedness of $(u^\varepsilon, v^\varepsilon)$. According to Lemma 3.1, the viscosity solutions exists on $R\times[0,T]$.

(2) To prove the strong convergence of r^ε, we multiply (4.1) by a test function Φ, where $\Phi \in C_0^\infty(R \times R^+)$ satisfies $\Phi_K = 1$, $0 \le \Phi \le 1$, and $S = \operatorname{supp} \Phi$ for an arbitrary compact set $K \subset S \subset R \times R^+$. Then, we have that

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty 2\varepsilon \left((u_x^\varepsilon)^2 + (v_x^\varepsilon)^2\right) \Phi dx dt &= \int_0^\infty \int_{-\infty}^\infty \Big[\varepsilon r_{xx} - r_t - f(r)_x \\ &\quad - u g_1(u,v) - v g_2(u,v)\Big] \Phi dx dt \\ &= \int_0^\infty \int_{-\infty}^\infty \varepsilon r \Phi_{xx} + r\Phi_t + f(r)\Phi_x dx dt \\ &\quad + \int_0^\infty \int_{-\infty}^\infty (-u g_1(u,v) - v g_2(u,v))\Phi dx dt \\ &\le M(\Phi)\,, \end{aligned} \tag{4.4}$$

and hence

$$\varepsilon (u_x^\varepsilon)^2 \text{ and } \varepsilon (v_x^\varepsilon)^2 \quad \text{are bounded in} \quad L^1_{loc}\left(R \times R^+\right)\,. \tag{4.5}$$

Let $(\eta(r), q(r))$ be any pair of entropy-entropy fluxes of the scalar equation

$$r_t + f(r)_x + 2u g_1(u,v) + 2v g_2(u,v) = 0\,,$$

and multiply (4.1) by $\eta'(r)$. Then

$$\begin{aligned}\eta(r)_t + q(r)_x &= \varepsilon\left(\eta'(r) r_x\right)_x - \varepsilon \eta''(r) r_x^2 - 2\varepsilon\eta'(r)\left(u_x^2 + v_x^2\right) \\ &\quad - \eta'(r)(2u g_1(u,v) + 2v g_2(u,v)) \\ &= I_1 - I_2 - I_3 - I_4\,,\end{aligned} \tag{4.6}$$

where $I_2 + I_3$ are bounded in $L^1_{loc}(R \times R^+)$, I_4 is in $L^\infty(R \times [0,T])$, and since I_1 is compact in $W_{loc}^{-1,2}(R \times R^+)$, then I_4 is bounded in $L^1_{loc}(R \times R^+)$, and hence $I_1 - I_2 - I_3 - I_4$ are compact in $W_{loc}^{-1,\alpha}(R \times R^+)$ for $\alpha \in (1,2)$, by (4.5). Noticing that $\eta(r)_t + q(r)_x$ is bounded in $W^{-1,\infty}$, and using Murat's theorem (cf. [13]), we get the proof that

$$\eta_i\left(r^\varepsilon(x,t)\right)_t + q_i(r^\varepsilon(x,t))_x \text{ are compact in } W_{loc}^{-1,2}\left(R \times R^+\right)\,, \tag{4.7}$$

for $i = 1, 2$, where

$$(\eta_1(r), q_1(r)) = (r - k, f(r) - f(k))\,, \tag{4.8}$$

and

$$(\eta_2(r), q_2(r)) = \left(f(r) - f(k), \int_k^r (f'(s))^2\, ds\right)\,, \tag{4.9}$$

and k is an arbitrary constant. Similar Lemma 3.5, if we consider that r is an independent variable, noticing the condition (1.6) on f, we get the proof of $r^\varepsilon(x,t) \to l(x,t)$, almost everywhere.

(3) Now we are going to prove the third part of Theorem 2.1. First, using Lemma 3.3, we get $v^\varepsilon \geq c(t, c_0, \varepsilon) > 0$ when the conditions $v_0(x) \geq c_0 > 0$, (H1)and (H3) are hold. Second, we prove the strong convergence of $(u^\varepsilon, v^\varepsilon) \to (u, v)$. By simple calculations, from system (2.1) we have that

$$\begin{aligned}\left(\frac{u}{v}\right)_t + \lambda_1 \left(\frac{u}{v}\right)_x &= \varepsilon \left(\frac{u}{v}\right)_{xx} - \varepsilon \left(\frac{2u}{v^3} v_x^2 - \frac{2}{v^2} u_x v_x\right) \\ &\quad - \left(\frac{v g_1(u, v) - u g_2(u, v)}{v^2}\right) \\ &= \varepsilon \left(\frac{u}{v}\right)_{xx} + 2\varepsilon \frac{v_x}{v} \left(\frac{u}{v}\right)_x - \left(\frac{v g_1(u, v) - u g_2(u, v)}{v^2}\right),\end{aligned} \tag{4.10}$$

where, for simplicity, we omit the superscript ε in the viscosity solutions $(u^\varepsilon, v^\varepsilon)$. Using the condition (H4) and the maximum principle for (4.10), similar the proof of Lemma 3.2, we get that $\frac{u^\varepsilon}{v^\varepsilon}$ is uniformly bounded with respect to ε.

According to the BV compactness frameworks Lemma 3.4, for obtaining that the total variation of $\left(\frac{u^\varepsilon}{v^\varepsilon}\right)_x$ is bounded in $(-\infty, \infty)$, we differentiate (4.10) with respect to x and then multiplying the sequence of smooth functions $m'(\theta, \alpha)$ by the result, where $\theta = \left(\frac{u^\varepsilon}{v^\varepsilon}\right)_x$ and α is a parameter, we have

$$\begin{aligned}m(\theta, \alpha)_t + (\lambda_1 m(\theta, \alpha))_x + (m'(\theta, \alpha)\theta - m(\theta, \alpha))\lambda_{1x} &= \\ = \varepsilon m(\theta, \alpha)_{xx} - \varepsilon m''(\theta, \alpha)\theta_x^2 &+ \left(2\varepsilon \frac{v_x^\varepsilon}{v^\varepsilon} m(\theta, \alpha)\right)_x \\ + \left(2\varepsilon \frac{v_x^\varepsilon}{v^\varepsilon}\right)_x &\big(m'(\theta, \alpha)\theta - m(\theta, \alpha)\big) \\ - m'(\theta, \alpha) &\left(\frac{v^\varepsilon g_1(u^\varepsilon, v^\varepsilon) - u^\varepsilon g_2(u^\varepsilon, v^\varepsilon)}{(v^\varepsilon)^2}\right)_x.\end{aligned} \tag{4.11}$$

Choosing $m(\theta, \alpha)$ such that $m''(\theta, \alpha) \geq 0$, $m'(\theta, \alpha) \to \operatorname{sign}\theta$, $m(\theta, \alpha) \to |\theta|$ as $\alpha \to 0$, we have from (4.11)

$$\begin{aligned}|\theta|_t + (\lambda_1|\theta|)_x \leq \varepsilon|\theta|_{xx} &+ \left(2\varepsilon \frac{v_x^\varepsilon}{v^\varepsilon}|\theta|\right)_x \\ &- \operatorname{sign}\theta \left(\frac{v^\varepsilon g_1(u^\varepsilon, v^\varepsilon) - u^\varepsilon g_2(u^\varepsilon, v^\varepsilon)}{(v^\varepsilon)^2}\right)_x.\end{aligned} \tag{4.12}$$

Using the condition (H5), we have

$$|\theta|_t + (\lambda_1|\theta|)_x \leq \varepsilon|\theta|_{xx} + \left(2\varepsilon \frac{v_x^\varepsilon}{v^\varepsilon}|\theta|\right)_x - G'\left(\frac{u^\varepsilon}{v^\varepsilon}\right)|\theta|, \tag{4.13}$$

hence

$$|\theta|_t + (\lambda_1|\theta|)_x \leq \varepsilon|\theta|_{xx} + \left(2\varepsilon \frac{v_x^\varepsilon}{v^\varepsilon}|\theta|\right)_x. \tag{4.14}$$

Integrating (4.14) in $R \times [0,t]$, we have

$$\int_{-\infty}^{\infty} |\theta(x,t)|dx \le \int_{-\infty}^{\infty} |\theta(x,0)|dx \le M\,. \tag{4.15}$$

Since

$$TV\left(\frac{u^\varepsilon(x,t)}{v^\varepsilon(x,t)}\right) = \int_{-\infty}^{\infty} |\theta(x,t)|dx\,,$$

we have

$$TV\left(\frac{u^\varepsilon(x,t)}{v^\varepsilon(x,t)}\right) \le TV\left(\frac{u^\varepsilon(x,0)}{v^\varepsilon(x,0)}\right) \le M\,.$$

According to Lemma 3.4, this implies the pointwise convergence of a subsequence of $\frac{v^\varepsilon}{v^\varepsilon}$. Combining this with the result in the second part of Theorem 2.1, we get the pointwise convergence of a subsequence of $(v^\varepsilon, v^\varepsilon) \to (u,v)$, where the limit (u,v) is a global bounded weak solution of the Cauchy problem (1.1)-(1.2). Thus we complete the proof of Theorem 2.1. ☑

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