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Superquadratic convergence of a Hummel-Seebeck type method

Convergencia supercuadrática de un método tipo Hummel-Seebeck

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Abstract. The cubic convergence of a method inspired by a Hummel and Seebeck for solving variational inclusions, has been showed when the second order Fréchet derivative of some function f satisfies a Lipschitz condition. Here, we prove the superquadratic convergence of this method whenever this second order Fréchet derivative satisfies a Hölder condition.

Key words and phrases. Set-valued mappings, M-pseudo-Lipschitzness, superquadratic convergence, Hölder-type condition.

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Resumen. La convergencia cúbica de un método de Hummel y Seebeck para resolver inclusiones variacionales ha sido probado cuando la derivada de Fréchet de segundo orden de alguna función f satisface una condición de Lipschitz. Aquí probamos la convergencia supercuadrática de este método siempre que esta derivada de Fréchet de segundo orden satisfaga una condición de Hölder.

Palabras y frases clave. Aplicaciones conjunto-valoradas, pseudo-Lipschitz, convergencia supercuadrática, condición de tipo Hölder.

1. Introduction

In [11], the following cubic method

$$0 \in f(x_k) + \frac{1}{2}\big(\nabla f(x_k) + \nabla f(x_{k+1})\big)(x_{k+1} - x_k) + F(x_{k+1})\,, \tag{1}$$

has been studied for solving variational inclusions of the form

$$0 \in f(x) + F(x)\,, \tag{2}$$

where f is a function and F is a set-valued mapping. It was proved in [11] that there exists a sequence (x_k) defined by (1) and converging to a solution x^* of (2). Following this work, in [7], Geoffroy *et al.* have proved the stability of this method inspired by the work of Hummel-Seebeck [9]. In this study, X and Y are two Banach spaces, $f : X \to Y$ is a function whose second Fréchet derivative is Lipschitz continuous, the set-valued mapping $F : X \to 2^Y$ is such that its graph defined by graphF $= \{(x, y) \in X \times Y \mid y \in F(x)\}$ is closed and the set-valued mapping $(f + F)^{-1}$ is pseudo-Lipschitz around $(0, x^*)$.

Our aim, in this paper, is to extend this study to the functions f whose second Fréchet derivative $\nabla^2 f$ satisfies a Hölder condition on a neighborhood Ω of x^*:

$$\exists\ K > 0,\ \alpha \in (0, 1],\ \text{such that} \left\|\nabla^2 f(x) - \nabla^2 f(y)\right\| \le K\|x - y\|^{\alpha},\ \forall x, y \in \Omega\,.$$

Note that when $\alpha = 1$, we have the Lipschitz condition for $\nabla^2 f$. This kind of assumption has been used in [8, 12] for extending the convergence of others methods for solving variational inclusions of the form (2). Let us remark that the inclusion (2) is an abstract model for various problems including equations, inequalities and variational inequalities.

Throughout, we denote by $\mathbb{B}_r(x)$ the closed ball centered at x with radius r. The distance from a point $x \in X$ and a subset $A \subset X$ is defined as $\operatorname{dist}(x, A) = \inf\{\|x - y\| \mid y \in A\}$, the excess from the set A to the set C is defined by $e(C, A) = \sup_{x \in C}\ \operatorname{dist}(x, A)$.

Recall that a set-valued $\Gamma : X \longrightarrow 2^Y$ is said to be M-pseudo-Lipschitz around $(x_0, y_0) \in \operatorname{graph}\Gamma$ if there exist neighborhoods V of x_0 and U of y_0 such that

$$e\big(\Gamma(x_1) \cap U, \Gamma(x_2)\big) \le M\|x_1 - x_2\|\,, \quad \forall x_1, x_2 \in V. \tag{3}$$

The pseudo-Lipschitz property has been introduced by J. P. Aubin and one refers to it as Aubin-continuity. For more details, the reader could refer to [1, 2]. For recent advances on this concept, one can refer to the works of Rockafellar [13] and those of Dontchev et al. [4, 5, 6]. We say that a function f from a metric space (X, ρ) into a metric space (Y, d) is strictly stationary at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(f(x_1), f(x_2)) \le \varepsilon\rho(x_1, x_2)\,, \tag{4}$$

whenever $\rho(x_i, x_0) \le \delta$, $i = 1, 2$.

2. Main result

The main theorem of this study is:

Theorem 2.1. *Let x^* be a solution of (2) and f a function whose second Fréchet derivative $\nabla^2 f$ satisfies a Hölder condition on a neighborhood Ω of x^**

with positive constants K and α. If the set-valued mapping $(f+F)^{-1}$ is pseudo-Lipschitz around $(0,x^*)$, then there exists a positive constant M such that, for every $c > \frac{MK(\alpha+4)}{2(\alpha+1)(\alpha+2)}$, one can find $\delta > 0$ such that, for every starting point $x_0 \in B_\delta(x^*)$, there exists a sequence (x_k) for (2), defined by (1), which satisfies

$$\|x_{k+1} - x^*\| \leq c\,\|x_k - x^*\|^{\alpha+2}\,, \tag{5}$$

that is, (x_k) is superquadratically convergent to x^*.

For the proof of Theorem 2.1, we need two lemmas:

Lemma 2.2. *If $f : X \to Y$ is a function such that ∇f is Lipschitz continuous then the following are equivalent:*

(i) The mapping $(f+F)^{-1}$ is pseudo-Lipschitz around (y^*, x^*).

(ii) The mapping $\left[f(x^*) + \frac{1}{2}(\nabla f(x^*) + \nabla f(\cdot))(\cdot - x^*) + F(\cdot)\right]^{-1}$ is pseudo-Lipschitz around (y^*, x^*).

In the sequel, we denote by M the modulus of regularity of the mapping defined in *(ii)*.

Proof of Lemma 2.2. According to [3, Corolary 2] proving the equivalence in Lemma 2.2 reduces to showing that the function $g : X \to Y$ defined by

$$g(x) = -f(x) + f(x^*) + \frac{1}{2}(\nabla f(x^*) + \nabla f(x))(x - x^*)\,,$$

is strictly stationary at x^*. Fix $\varepsilon > 0$; then there exists $\delta > 0$ such that for every $x_1, x_2 \in B_\delta(x^*)$,

$$\|f(x_2) - f(x_1) - \nabla f(x_1)(x_2 - x_1)\| \leq \frac{\varepsilon}{2}\,,$$

and

$$\begin{aligned}
\|g(x_1) - g(x_2)\| &= \Bigg\| f(x_2) - f(x_1) - \nabla f(x_1)(x_2 - x_1) + \nabla f(x_1)(x_2 - x_1) \\
&\quad - \frac{1}{2}\nabla f(x^*)(x_2 - x_1) + \frac{1}{2}\nabla f(x_1)(x_1 - x_2) + \frac{1}{2}\nabla f(x_1)(x_2 - x^*) \\
&\quad - \frac{1}{2}\nabla f(x_2)(x_2 - x^*)\Bigg\| \\
&\leq \Bigg\| f(x_2) - f(x_1) - \nabla f(x_1)(x_2 - x_1)\Bigg\| + \frac{1}{2}\big\|\nabla f(x_1) - \nabla f(x^*)\big\|\|x_2 - x_1\| \\
&\quad + \frac{1}{2}\big\|\nabla f(x_1) - \nabla f(x_2)\big\|\|x_2 - x^*\| \\
&\leq \frac{\varepsilon}{2}\|x_2 - x_1\| + \frac{K}{2}\|x_1 - x^*\|\,\|x_2 - x_1\| + \frac{K}{2}\|x_2 - x_1\|\,\|x_2 - x^*\|
\end{aligned}$$

$$\leq \left(\frac{\varepsilon}{2} + K\delta\right) \|x_2 - x_1\| .$$

Without loss of generality, we can take $\delta \leq \frac{\varepsilon}{2K}$, thus, $\|g(x_1) - g(x_2)\| \leq \varepsilon\|x_2 - x_1\|$. Hence, g is strictly stationary at x^* and the proof is complete. ☑

Lemma 2.3. *Let (X,ρ) be a complete metric space, let ϕ a map from X into the closed subsets of X, let $\eta_0 \in X$ and let r and λ be such that $0 < \lambda < 1$ and*

(a) $\mathrm{dist}(\eta_0, \phi(\eta_0)) < \mathrm{r}(1-\lambda)$,

(b) $e(\phi(x_1) \cap B_r(\eta_0)\,, \quad \phi(x_2)) \leq \lambda\, \rho(x_1, x_2)\,, \quad \forall x_1, x_2 \in B_r(\eta_0)$,

then ϕ has a fixed point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single-valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

This lemma is a generalization of a fixed-point theorem proved in [10] where, in (b), the excess e is replaced by the Haussdorff distance. The reader can consult the proof of this lemma in [3].

Proof of Theorem 2.1. Denote

$$P(x) = f(x^*) + \frac{1}{2}\big(\nabla f(x^*) + \nabla f(x)\big)(x - x^*) + F(x)$$

and

$$Z_k(x) = f(x^*) + \frac{1}{2}\Big(\nabla f(x^*) + \nabla f(x)\Big)(x - x^*) - f(x_k) - \frac{1}{2}\Big(\nabla f(x_k) + \nabla f(x)\Big)(x - x_k)\,, \quad \text{for } k \in \mathbb{N}\,.$$

Finally, we define the map $\phi_k : X \to 2^X$ by

$$\phi_k(x) = P^{-1}[Z_k(x)]\,. \tag{6}$$

The map $(f+F)^{-1}$ is pseudo-Lipschitz around $(0, x^*)$ then there exist positive numbers a, b and M such that

$$e\Big(P^{-1}(y') \cap B_a(x^*)\,, P^{-1}(y'')\Big) \leq M\,\|y' - y''\|\,, \quad \forall y', y'' \in B_b(0)\,. \tag{7}$$

Since $c > \frac{MK(\alpha+4)}{2(\alpha+1)(\alpha+2)}$, there exists $\lambda \in]0,1[$ such that $c(1-\lambda) > \frac{MK(\alpha+4)}{2(\alpha+1)(\alpha+2)}$.

Choose $\delta > 0$ such that

$$\delta < \min\left\{ a, \left[\frac{2b(\alpha+1)(\alpha+2)}{K(\alpha+4)}\right]^{\frac{1}{\alpha+2}}, \frac{1}{\sqrt[\alpha+1]{c}}, \left[\frac{2b(\alpha+1)(\alpha+2)}{K(\alpha 2^{\alpha+2}+\alpha+2^{\alpha+4}+4)}\right]^{\frac{1}{\alpha+2}}, \left[\frac{2\lambda(\alpha+1)}{MK(2^{\alpha+1}+\alpha 2^{\alpha}+1)}\right]^{\frac{1}{\alpha+1}} \right\}. \tag{8}$$

We apply Lemma 2.3 to the map ϕ_0 with $\eta_0 = x^*$ and r is to be set. Let us check that assertions (a) and (b) of Lemma 2.3 are satisfied. From the definition of the excess e, we have

$$\operatorname{dist}\left(x^*, \phi_0\left(x^*\right)\right) \leq e\left(P^{-1}(0) \cap B_\delta\left(x^*\right), \phi_0\left(x^*\right)\right). \tag{9}$$

For all $x_0 \in B_\delta(x^*)$ such that $x_0 \neq x^*$ we have

$$\begin{aligned}
\|Z_0\left(x^*\right)\| &= \left\| f\left(x^*\right) - f(x_0) - \frac{1}{2}\Big(\nabla f\left(x_0\right) + \nabla f\left(x^*\right)\Big)\left(x^* - x_0\right)\right\| \\
&= \Big\| f\left(x^*\right) - f(x_0) - \nabla f(x_0)\left(x^* - x_0\right) - \frac{1}{2}\nabla^2 f(x_0)\left(x^* - x_0\right)^2 \\
&\quad - \frac{1}{2}\Big(\nabla f\left(x^*\right) - \nabla f(x_0) - \nabla^2 f(x_0)\left(x^* - x_0\right)\Big)\left(x^* - x_0\right)\Big\| \\
&\leq \left\| \int_0^1 (1-t)\nabla^2 f\left(x_0 + t\left(x^* - x_0\right)\right)\left(x^* - x_0\right)^2 dt - \frac{1}{2}\nabla^2 f(x_0)(x^* - x_0)^2 \right\| \\
&\quad + \frac{1}{2}\left\| \int_0^1 \nabla^2 f\left(tx^* + (1-t)x_0\right) dt\left(x^* - x_0\right) - \int_0^1 \nabla^2 f(x_0) dt (x^* - x_0)\right\| \\
&\quad \|x^* - x_0\| \\
&\leq \|x^* - x_0\|^2 \cdot \int_0^1 (1-t)\left\|\nabla^2 f(x_0 + t(x^* - x_0)) - \nabla^2 f(x_0)\right\| dt \\
&\quad + \frac{1}{2}\|x^* - x_0\|^2 \cdot \int_0^1 \|\nabla^2 f(tx^* + (1-t)x_0) - \nabla^2 f(x_0)\| dt \\
&\leq K\|x^* - x_0\|^2 \cdot \int_0^1 (1-t)\|x_0 + t\left(x^* - x_0\right) - x_0\|^\alpha dt \\
&\quad + \frac{K}{2}\|x^* - x_0\|^2 \cdot \int_0^1 \|tx^* + (1-t)x_0 - x_0\|^\alpha dt \\
&\leq K\|x^* - x_0\|^{\alpha+2} \cdot \left|\int_0^1 (1-t)t^\alpha dt\right| + \frac{K}{2}\|x^* - x_0\|^{\alpha+2} \cdot \left|\int_0^1 t^\alpha dt\right| \\
&\leq \frac{K}{(\alpha+1)(\alpha+2)}\|x^* - x_0\|^{\alpha+2} + \frac{K}{2(\alpha+1)}\|x^* - x_0\|^{\alpha+2} \\
&\leq \frac{K(\alpha+4)}{2(\alpha+1)(\alpha+2)}\|x^* - x_0\|^{\alpha+2} < b\,.
\end{aligned}$$

This last inequality holds true thanks to (8). From this result and the definition of ϕ_0, we get

$$\begin{aligned} \operatorname{dist}\left(\mathrm{x}^*, \phi_0\left(\mathrm{x}^*\right)\right) &\leq e\left(P^{-1}(0) \cap B_\delta(x^*), P^{-1}[Z_0\left(x^*\right)]\right) \\ &\leq \frac{MK(\alpha+4)}{2(\alpha+1)(\alpha+2)}\|x^*-x_0\|^{\alpha+2} \\ &< c(1-\lambda)\|x^*-x_0\|^{\alpha+2}. \end{aligned} \tag{10}$$

By setting $r = r_0 = c\|x^* - x_0\|^{\alpha+2}$, condition (a) of Lemma 2.3 is fulfilled.

Let us observe that from (8), $r_0 < \delta < a$. For $x \in B_\delta\left(x^*\right)$, using (8), we have

$$\begin{aligned} \|Z_0(x)\| &= \Bigg\| f\left(x^*\right) + \frac{1}{2}\Big(\nabla f\left(x^*\right) + \nabla f(x)\Big)(x - x^*) - f(x_0) \\ &\quad - \frac{1}{2}\Big(\nabla f(x_0) + \nabla f(x)\Big)(x - x_0)\Bigg\| \\ &\leq \left\| f\left(x^*\right) - f(x) - \nabla f(x)(x^* - x) - \frac{1}{2}\nabla^2 f(x)(x^* - x)^2 \right\| \\ &\quad + \left\| f(x) - f(x_0) - \nabla f(x_0)(x - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x - x_0)^2 \right\| \\ &\quad + \frac{1}{2}\left\|\nabla f\left(x^*\right) - \nabla f(x) - \nabla^2 f(x)(x^* - x)\right\| \cdot \|x^* - x\| \\ &\quad + \frac{1}{2}\left\|\nabla f(x) - \nabla f(x_0) - \nabla^2 f(x_0)(x - x_0)\right\| \cdot \|x - x_0\| \\ &\leq \|x^* - x\|^2 \cdot \int_0^1 (1-t)\left\|\nabla^2 f(x + t(x^* - x)) - \nabla^2 f(x)\right\| dt \\ &\quad + \|x - x_0\|^2 \cdot \int_0^1 (1-t)\left\|\nabla^2 f(x_0 + t(x - x_0)) - \nabla^2 f(x_0)\right\| dt \\ &\quad + \frac{1}{2}\|x^* - x\|^2 \cdot \int_0^1 \left\|\nabla^2 f(tx^* + (1-t)x) - \nabla^2 f(x)\right\| dt \\ &\quad + \frac{1}{2}\|x - x_0\|^2 \cdot \int_0^1 \left\|\nabla^2 f(tx + (1-t)x_0) - \nabla^2 f(x_0)\right\| dt \\ &\leq \frac{K}{(\alpha+1)(\alpha+2)}\|x - x^*\|^{\alpha+2} + \frac{K}{(\alpha+1)(\alpha+2)}\|x - x_0\|^{\alpha+2} \\ &\quad + \frac{K}{2(\alpha+1)}\|x - x^*\|^{\alpha+2} + \frac{K}{2(\alpha+1)}\|x - x_0\|^{\alpha+2} \\ &\leq \frac{K\delta^{\alpha+2}}{2(\alpha+1)(\alpha+2)}\left(\alpha 2^{\alpha+2} + \alpha + 2^{\alpha+4} + 4\right) < b. \end{aligned}$$

It follows that for all $x', x'' \in B_{r_0}(x^*)$, we have, by (8)

$$\begin{aligned}
& e\left(\phi_0\left(x'\right) \cap B_{r_0}\left(x^*\right), \phi_0\left(x''\right)\right) \\
& \leq e(\phi_0\left(x'\right) \cap B_{\delta}\left(x^*\right), \phi_0\left(x''\right)) \\
& \leq M\|Z_0\left(x'\right) - Z_0\left(x''\right)\| \\
& \leq \frac{M}{2}\left[\|\nabla f\left(x^*\right) - \nabla f(x_0)\| \cdot \|x' - x''\| + \|\nabla f\left(x'\right) - \nabla f\left(x''\right)\| \cdot \|x_0 - x^*\|\right] \\
& \leq \frac{M}{2}\Bigg[\left\|\nabla f\left(x^*\right) - \nabla f(x_0) - \nabla^2 f(x_0)\left(x^* - x_0\right)\right\| \cdot \|x' - x''\| \\
& \quad + \left\|\nabla f\left(x'\right) - \nabla f\left(x''\right) - \nabla^2 f\left(x''\right)\left(x' - x''\right)\right\| \cdot \|x_0 - x^*\| \\
& \quad + \left\|\nabla^2 f\left(x''\right) - \nabla^2 f(x_0)\right\| \|x' - x''\| \cdot \|x_0 - x^*\|\Bigg] \\
& \leq \frac{MK\delta^{\alpha+1}}{2(\alpha+1)}(2^{\alpha+1} + \alpha 2^{\alpha} + 1)\|x' - x''\| \\
& \leq \lambda\|x' - x''\|.
\end{aligned}$$

Thus condition (b) of Lemma is satisfied. Since both conditions of Lemma 2.3 are fulfilled, we can deduce that ϕ_0 has a fixed point $x_1 \in B_{r_0}(x^*)$, that is, $x_1 \in \phi_0(x_1)$ then $0 \in f(x_0) + \frac{1}{2}\left(\nabla f(x_0) + \nabla f(x_1)\right)(x_1 - x_0) + F(x_1)$. Thus x_1 is obtained by the method (1) from x_0 and $x_1 \in B_{r_0}(x^*)$, that is

$$\|x_1 - x^*\| \leq c\|x_0 - x^*\|^{\alpha+2}.$$

Proceeding by induction, keeping $\eta_0 = x^*$ and setting $r_k = c\|x_k - x^*\|^{\alpha+2}$, we have the existence of a fixed point x_{k+1} for ϕ_k, which is an element of $B_{r_k}(x^*)$. Then

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^{\alpha+2}. \tag{11}$$

In others words, (x_k) is superquadratically convergent to x^* then the proof of Theorem 2.1 is complete. ☑

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