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A direct proof of Noether's second isomorphism theorem for abelian categories

Prueba directa del segundo teorema de isomorfismo de Noether para categorías abelianas

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ABSTRACT. A direct and simple proof of Noether's second isomorphism theorem for abelian categories is obtained.

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RESUMEN. Obtenemos una demonstración directa y simple del segundo teorema de isomorfismo de Noether para categorías abelianas.

Palabras y frases clave. Categorías abelianas, teorema del isomorfismo de Noether.

Noether's first isomorphism theorem for modules [6] asserts that, if R is a unitary ring, A is a unitary left R-module and A_1, A_2 are two submodules of A such that $A_1 \subset A_2$, then the quotient R-modules $(A/A_1)/(A/A_2)$ and A_2/A_1 are isomorphic; proofs of its extension to arbitrary abelian categories may be found in [1], [2] and [4].

Noether's second isomorphism theorem for modules [6] asserts that if R is a unitary ring, A is a unitary left R-module and A_1, A_2 are two submodules of A, then the quotient R-modules $A_2/(A_1 \cap A_2)$ and $(A_1 + A_2)/A_1$ are isomorphic;

proofs of its extension to arbitrary abelian categories may be found in [1], [2] and [4]. In this note we present a proof of Noether's second isomorphism theorem for abelian categories, which only presupposes the rudiments on abelian categories and is inspired by that of the classical case.

For the sake of clarity let us begin with some basic notions and facts concerning categories, to be found in [1], [2], [3], [4] and [7], which will be needed in the sequel.

Let $\mathcal{C}$ be a category and $Ob(\mathcal{C})$ the class of objects of $\mathcal{C}$. For $A, B \in Ob(\mathcal{C})$, 1_A shall denote the identity morphism of A and $\mathrm{Mor}_{\mathcal{C}}(A, B)$ the set of morphisms from A to B. Let $u \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, u is *injective* (resp. *surjective*) if the relations $C \in Ob(\mathcal{C})$, $v_1, v_2 \in \mathrm{Mor}_{\mathcal{C}}(C, A)$ (resp. $w_1, w_2 \in \mathrm{Mor}_{\mathcal{C}}(B, C)$), $uv_1 = uv_2$ (resp. $w_1 u = w_2 u$) imply $v_1 = v_2$ (resp. $w_1 = w_2$); u is *bijective* if u is injective and surjective; u is an *isomorphism* if there exists (necessarily unique) $u' \in \mathrm{Mor}_{\mathcal{C}}(B, A)$ such that $u'u = 1_A$ and $uu' = 1_B$; A and B are *isomorphic* if there exists an isomorphism $u : A \to B$. Every isomorphism is bijective, but the converse is not true in general; see Example 3b below.

Let $A \in Ob(\mathcal{C})$ be fixed. If $u_1 \in \mathrm{Mor}_{\mathcal{C}}(A_1, A)$ and $u_2 \in \mathrm{Mor}_{\mathcal{C}}(A_2, A)$ are injective, we write $(A_1, u_1) \leq (A_2, u_2)$ (or $A_1 \leq A_2$) to indicate the existence of a $v \in \mathrm{Mor}_{\mathcal{C}}(A_1, A_2)$ such that $u_1 = u_2 v$; $\leq$ is a partial order in the class of all such pairs (A_1, u_1). (A_1, u_1) and (A_2, u_2) as above are equivalent if $(A_1, u_1) \leq (A_2, u_2)$ and $(A_2, u_2) \leq (A_1, u_1)$; in this case, A_1 and A_2 are isomorphic. In each class of equivalent pairs we choose a pair, called a *subobject* of A. The class of subobjects of A is an ordered class under the relation $\leq$. Dually, we consider a partial order $\leq$ in the class of all pairs (P, w), where $w \in \mathrm{Mor}_{\mathcal{C}}(A, P)$ is surjective, and we choose a pair in each class of equivalent pairs, called a *quotient* of A. The class of quotients of A is an ordered class under the relation $\leq$.

A category $\mathcal{C}$ is *additive* if:

(a) for all $A, B \in Ob(\mathcal{C})$, the product $A \times B$ and the direct sum $A \oplus B$ exist;

(b) for all $A, B \in Ob(\mathcal{C})$, $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is an abelian group, whose identity element shall be denoted by 0_{AB};

(c) for all $A, B, C \in Ob(\mathcal{C})$, the composition of morphisms

$$(u, v) \in \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_{\mathcal{C}}(B, C) \to vu \in \mathrm{Mor}_{\mathcal{C}}(A, C)$$

is a $\mathbb{Z}$-bilinear mapping;

(d) there exists $A \in Ob(\mathcal{C})$ such that $1_A = 0_{AA}$.

Obviously, every A', A'' as in (d) are isomorphic.

If $\mathcal{C}$ is a category satisfying conditions (b) and (c) above, then, for all $A, B \in Ob(\mathcal{C})$, the assumptions "$A \times B$ exists" and "$A \oplus B$ exists" are equivalent.

If $\mathcal{C}$ is an additive category and $u \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, to say that u is injective (resp. surjective) is equivalent to saying that the relations $C \in Ob(\mathcal{C})$, $v \in \mathrm{Mor}_{\mathcal{C}}(C, A)$ (resp. $w \in \mathrm{Mor}_{\mathcal{C}}(B, C)$), $uv = 0_{CB}$ (resp. $wu = 0_{AC}$) imply $v = 0_{CA}$ (resp. $w = 0_{BC}$).

Let $\mathcal{C}$ be an additive category and $u \in \mathrm{Mor}_{\mathcal{C}}(A, B)$. A pair (I, i) (where $i \in \mathrm{Mor}_{\mathcal{C}}(I, A)$) is a generalized kernel of u if the following conditions hold:

(a) i is injective;

(b) $ui = 0_{IB}$;

(c) for each $C \in Ob(\mathcal{C})$ and for each $v \in \mathrm{Mor}_{\mathcal{C}}(C, A)$ with $uv = 0_{CB}$, there exists $w \in \mathrm{Mor}_{\mathcal{C}}(C, I)$ so that $iw = v$.

Two generalized kernels of u are equivalent. Therefore among them (if they do exist) there is exactly one, denoted by $(\mathrm{Ker}(u), i)$ and called the *kernel* of u, which is a subobject of A (the morphism $i : \mathrm{Ker}(u) \to A$ is called the canonical injection).

Dually, a pair (J, j) (where $j \in \mathrm{Mor}_{\mathcal{C}}(B, J)$) is a generalized cokernel of u if the following conditions hold:

(a) j is surjective;

(b) $ju = 0_{AJ}$;

(c) for each $C \in Ob(\mathcal{C})$ and for each $w \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ with $wu = 0_{AC}$, there exists $v \in \mathrm{Mor}_{\mathcal{C}}(J, C)$ so that $w = vj$.

Two generalized cokernels of u are equivalent. Therefore among them (if they do exist) there is exactly one, denoted by $(\mathrm{Coker}(u), j)$ and called the *cokernel* of u, which is a quotient of B (the morphism $j : B \to \mathrm{Coker}(u)$ is called the canonical surjection). If $\mathrm{Coker}(u)$ exists, we define the *image* of u as $\mathrm{Im}(u) = \mathrm{Ker}(\mathrm{Coker}(u))$, if $\mathrm{Ker}(\mathrm{Coker}(u))$ exists. And, if $\mathrm{Ker}(u)$ exists, we define the *coimage* of u as $\mathrm{Coim}(u) = \mathrm{Coker}(\mathrm{Ker}(u))$, if $\mathrm{Coker}(\mathrm{Ker}(u))$ exists.

Proposition 1. *Let $\mathcal{C}$ be an additive category and let $u \in Mor_{\mathcal{C}}(A, B)$ be such that $Coim(u)$ and $Im(u)$ exist. Then there exists an unique*

$$\overline{u} \in Mor_{\mathcal{C}}(Coim(u), Im(u))$$

such that $u = i\overline{u}j$, where $i : Im(u) \to B$ is the canonical injection and $j : A \to Coim(u)$ is the canonical surjection.

A category $\mathcal{C}$ is *abelian* if it is additive and the following conditions hold:

(AB1) for all $u \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, $\mathrm{Ker}(u)$ and $\mathrm{Coker}(u)$ exist;

(AB2) for all $u \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, the above morphism $\overline{u}$ is an isomorphism.

If $\mathcal{C}$ is an additive category satisfying (AB1), then $\mathcal{C}$ is an abelian category if, and only if, the conditions (α) and (β) below hold:

(α) for all $u \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, $\overline{u}$ is bijective;

(β) every bijection is an isomorphism.

Example 2. *(a) Let R be a unitary ring. Then Mod_R, the category whose objects are unitary left R-modules and whose morphisms are R-linear mappings, is abelian. In particular, the category of abelian groups is abelian.*

(b) If p is a positive prime number, the category of finite abelian p-groups is abelian.

(c) The category of vector bundles [8] is abelian.

(d) The category of sheaves of abelian groups on a topological space [5] is abelian.

Example 3. *(a) The category of free abelian groups is additive, but is not abelian; see [6, p. 110].*

(b) It is easily verified that Gt, the category whose objects are abelian topological groups and whose morphisms are continuous group homomorphisms, is additive and satisfies condition (α). But Gt is not abelian. In fact, let A be the additive group of real numbers endowed with the discrete topology, B the additive group of real numbers endowed with the usual topology and $u : A \to B$ the identity mapping. Then $A, B \in Ob(Gt)$, $u \in Mor_{Gt}(A, B)$, u is bijective, but u is not an isomorphism. Hence condition (β) is not satisfied and Gt is not abelian.

Proposition 4. *Let $\mathcal{C}$ be an abelian category, $A, B, C \in Ob(\mathcal{C})$, $u \in Mor_{\mathcal{C}}(A, B)$ and $v \in Mor_{\mathcal{C}}(B, C)$. Then the following assertions hold:*

(a) u is surjective if, and only if, $Im(u) = B$ (that is, the canonical injection $Im(u) \to B$ is an isomorphism);

(b) $Ker(vu) \geq Ker(u)$;

(c) $vu = 0_{AC}$ if and only if, $Im(u) \leq Ker(v)$;

(d) If (A, u) is a subobject of B, then $A = Im(u)$, that is, the morphism $A \to Coim(u) \xrightarrow{\overline{u}} Im(u)$ is an isomorphism.

Proposition 5. *Let $\mathcal{C}$ be an abelian category and $A \in Ob(\mathcal{C})$. Let S be the class of subobjects of A and Q the class of quotients of A. For $(A', i) \in S$ and $(A'', j) \in Q$, the relations Coker$(i) = A''$ and Ker$(j) = A'$ are equivalent and establish a one-to-one correspondence between S and Q.*

For each $A' \in S$, A/A' shall denote the corresponding element of Q.

If $\mathcal{C}$ is an abelian category and $A \in Ob(\mathcal{C})$, it is well-known that the ordered class of subobjects of A is a lattice. If A_1, A_2 are two subobjects of A, we put $A_1 \cap A_2 := \inf(A_1, A_2)$ and $A_1 \cup A_2 := \sup(A_1, A_2)$. The next proposition is Theorem 2.13 of [2]. We recall its proof (here in a slightly modified version) since it will be used later on.

Proposition 6. *Let $\mathcal{C}$ be an abelian category and $A \in Ob(\mathcal{C})$. Then any two subobjects of A admit an infimum.*

Proof. Let (A_1, i_1) and (A_2, i_2) be two subobjects of A and let $j_1 : A \to A/A_1$ be the canonical surjection. Put $u = j_1 i_2$ and let $(\mathrm{Ker}(u), i)$ be the kernel of u. Then $(\mathrm{Ker}(u), i_2 i)$ is a subobject of A such that $\mathrm{Ker}(u) \le A_2$. We claim that $\mathrm{Ker}(u) \le A_1$. Indeed, since

$$0_{\mathrm{Ker}(u)A/A_1} = ui = (j_1 i_2)i = j_1(i_2 i)\,,$$

and since $\mathrm{Ker}(j_1) = A_1$ by Proposition 5, there exists a morphism $w : \mathrm{Ker}(u) \to A_1$ such that the diagram

$$\begin{tikzcd} \mathrm{Ker}(u) \arrow[r, "i"] \arrow[d, "w"'] & A_2 \arrow[d, "i_2"] \\ A_1 \arrow[r, "i_1"'] & A \end{tikzcd}$$

is commutative. Thus $\mathrm{Ker}(u) \le A_1$.

Now, let (X, k) be a subobject of A such that $X \le A_1$ and $X \le A_2$. We claim that $X \le \mathrm{Ker}(u)$. Indeed, since $X \le A_1$, there exists a morphism $\theta_1 : X \to A_1$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "\theta_1"'] & A \\ A_1 \arrow[ur, "i_1"'] & \end{tikzcd}$$

is commutative. And, since $X \le A_2$, there exists a morphism $\theta_2 : X \to A_2$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "\theta_2"'] & A \\ A_2 \arrow[ur, "i_2"'] & \end{tikzcd}$$

is commutative. On the other hand,

$$u\theta_2 = (j_1 i_2)\theta_2 = j_1(i_2\theta_2) = j_1(i_1\theta_1) = (j_1 i_1)\theta_1 = 0_{A_1 A/A_1}\theta_1 = 0_{XA/A_1}\,.$$

Hence there exists a morphism $t : X \to \mathrm{Ker}(u)$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "\theta_2"'] & \mathrm{Ker}(u) \arrow[dl, "i"] \\ A_2 & \end{tikzcd}$$

is commutative. Consequently,

$$k = i_2\theta_2 = i_2(it) = (i_2 i)t\,,$$

proving that $X \leq \mathrm{Ker}(u)$. Therefore the subobjects (A_1, i_1) and (A_2, i_2) of A admit an infimum, namely, $(\mathrm{Ker}(u), i_2 i)$. This completes the proof. ☑

Now, let us state Noether's second isomorphism theorem for abelian categories [2, p. 59, 2.67]:

Theorem 7. *Let $\mathcal{C}$ be an abelian category and $A \in Ob(\mathcal{C})$. If A_1, A_2 are two subobjects of A, then $A_2/(A_1 \cap A_2)$ and $(A_1 \cup A_2)/A_1$ are isomorphic.*

In order to prove Theorem 7 we shall need two auxiliary lemmas.

Lemma 8. *Let $\mathcal{C}$ be an abelian category. If $u \in Mor_{\mathcal{C}}(A, B)$ is such that $Ker(u) = A$, that is, if the canonical injection $i : Ker(u) \to A$ is an isomorphism, then $u = 0_{AB}$.*

Proof. Let $i' \in \mathrm{Mor}_{\mathcal{C}}(A, \mathrm{Ker}(u))$ be such that $ii' = 1_A$ and $i'i = 1_{\mathrm{Ker}(u)}$. Since $ui = 0_{\mathrm{Ker}(u)B}$, we obtain

$$u = u1_A = u\,(ii') = (ui)i' = 0_{\mathrm{Ker}(u)B}i' = 0_{AB}\,.$$

☑

Lemma 9. *Let $\mathcal{C}$ be an abelian category and $A \in Ob(\mathcal{C})$. If A_1, A_2 are two subobjects of A, consider the sequence*

$$A_2 \xrightarrow{k} A_1 \cup A_2 \xrightarrow{l} (A_1 \cup A_2)/A_1\,,$$

where k is the canonical injection and l is the canonical surjection. Then $v = lk$ is surjective.

Proof. Let $C \in Ob(\mathcal{C})$ and $w \in \mathrm{Mor}_{\mathcal{C}}((A_1 \cup A_2)/A_1, C)$ be such that $wv = 0_{A_2C}$. We have to show that $w = 0_{(A_1\cup A_2)/A_1C}$. But, since l is surjective, it suffices to show that $wl = 0_{(A_1\cup A_2)C}$. So, let us prove that $wl = 0_{(A_1\cup A_2)C}$. Indeed, the relation

$$(wl)k = w(lk) = wv = 0_{A_2C}$$

and Proposition 4(c) furnish $\mathrm{Im}(k) \leq \mathrm{Ker}(wl)$. Thus, by Proposition 4(d), $A_2 \leq \mathrm{Ker}(wl)$. On the other hand, $\mathrm{Ker}(wl) \geq \mathrm{Ker}(l) = A_1$, in view of Propositions 4(b) and 5. Consequently, $A_1 \cup A_2 \leq \mathrm{Ker}(wl)$. Since $\mathrm{Ker}(wl) \leq A_1 \cup A_2$, we get $\mathrm{Ker}(wl) = A_1 \cup A_2$, and therefore $wl = 0_{(A_1\cup A_2)C}$ by Lemma 8. This completes the proof. ☑

Now, let us turn to the proof of Theorem 7:

Proof. Clearly we may suppose that $A = A_1 \cup A_2$. Let v be as in the proof of Lemma 9. By the proof of Proposition 6, $\mathrm{Ker}(v) = A_1 \cap A_2$, and hence

$$\mathrm{Coim}(v) = \mathrm{Coker}(\mathrm{Ker}(v)) = \mathrm{Coker}(A_1 \cap A_2) = A_2/(A_1 \cap A_2)\,.$$

Since $\mathcal{C}$ is abelian,

$$\overline{v} : A_2/(A_1 \cap A_2) \to \mathrm{Im}(v)$$

is an isomorphism. Moreover, $\mathrm{Im}(v) = (A_1 \cup A_2)/A_1$, in view of Lemma 9 and Proposition 4(a). Then $A_2/(A_1 \cap A_2)$ and $(A_1 \cup A_2)/A_1$ are isomorphic, as was to be shown. ☑

Corolary 10. *Let R be a unitary ring, $A \in Ob(Mod_R)$ and A_1, A_2 two submodules of A. Then the quotient R-modules $A_2/(A_1 \cap A_2)$ and $(A_1 + A_2)/A_1$ are isomorphic.*

Proof. The result follows immediately from Theorem 7, because $A_1 + A_2 = A_1 \cup A_2$. ☑

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