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Continuity of the quenching time in a semilinear heat equation with a potential

Continuidad del tiempo de extinción en una ecuación semilineal de calor con potencial

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Abstract. In this paper, we consider a semilinear heat equation with a potential subject to Neumann boundary conditions and positive initial data. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of the potential and the initial data. Finally, we give some numerical results to illustrate our analysis.

Key words and phrases. Quenching, semilinear heat equation, numerical quenching time.

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Resumen. En este trabajo consideramos una ecuación semilineal de calor con potencial, sujeta a condiciones de Neumann de frontera y datos iniciales positivos. Bajo ciertos supuestos mostramos que la solución de dicha ecuación se apaga en tiempo finito y estimamos el tiempo en que lo hace. También probamos la continuidad del tiempo de extinción en función del potencial y de los datos iniciales. Finalmente damos algunos resultados numéricos que ilustran nuestro análisis.

Palabras y frases clave. Apagamiento, ecuación de calor semilineal, tiempo de apagamiento numérico.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t = Lu - a(x)f(u) \text{ in } \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) > 0 \text{ in } \overline{\Omega}, \tag{3}$$

where $f : (0,\infty) \longrightarrow (0,\infty)$ is a C^1 convex, nonincreasing function, $\int_0^\gamma \frac{ds}{f(s)} < \infty$ for any positive real γ, $\lim_{s\to 0^+} f(s) = \infty$, $a \in C^0\left(\overline{\Omega}\right)$, $a(x) > 0$ in $\overline{\Omega}$. The operators L and $\frac{\partial}{\partial \eta}$ are defined as follows

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right), \quad \frac{\partial u}{\partial \eta} = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i\,,$$

where $\nu = (\nu_1, \dots, \nu_N)$ is the exterior normal unit vector on $\partial\Omega$, $a_{ij} : \overline{\Omega} \longrightarrow \mathbb{R}$, $a_{ij} \in C^1\left(\overline{\Omega}\right)$, $a_{ij} = a_{ji}$, $1 \le i,j \le N$, and there exists a positive constant C such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge C\|\xi\|^2\,, \quad \forall\, x \in \overline{\Omega}\,, \quad \forall\, \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N,$$

where $\|\cdot\|$ stands for the Euclidean norm of $\mathbb{R}^N$. The initial data $u_0 \in C^1\left(\overline{\Omega}\right)$, $u_0(x) > 0$ in $\overline{\Omega}$ and satisfies the compatibility condition $\frac{\partial u_0}{\partial \eta} = 0$ on $\partial\Omega$. Here $(0,T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T} u_{\min}(t) = 0\,,$$

where $u_{\min}(t) = \min_{x\in\overline{\Omega}} u(x,t)$. In this last case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u. Consequently, in this paper, with the definition of the time T, we have

$$u(x,t) > 0 \text{ in } \overline{\Omega} \times [0,T)\,.$$

Solutions of semilinear heat equations which quench in a finite time have been the subject of investigation of many authors (see [1], [2], [3], [5], [7], [8], [9], [10], [11], [17], [19], [20], [21], [22], [23], [25], [26], and the references cited therein). In particular, in [5], the problem (1)-(3) has been studied. The local in time existence of a classical solution has been proved and this solution is unique (see [5]). It is also shown that the solution of (1)-(3) quenches in a finite time, and its quenching time has been estimated (see [5]). In this paper, we are interested in the continuity of the quenching time as a function of the potential a and the initial data u_0. More precisely, we consider the following initial-boundary value problem

$$v_t = Lv - a_h(x) f(v) \text{ in } \Omega \times \left(0, T_h^k\right), \tag{4}$$

$$\frac{\partial v}{\partial \eta} = 0 \text{ on } \partial\Omega \times \left(0, T_h^k\right), \tag{5}$$

$$v(x,0) = u_0^k(x) > 0 \text{ in } \overline{\Omega}, \tag{6}$$

where $a_h \in C^0\left(\overline{\Omega}\right)$, $0 < a_h(x) \leq a(x)$ in $\overline{\Omega}$, $\lim_{h\to 0} a_h = a$. The initial data $u_0^k \in C^1\left(\overline{\Omega}\right)$, $u_0^k(x) \geq u_0(x)$ in $\overline{\Omega}$, u_0^k obeys the compatibility condition $\frac{\partial u_0^k}{\partial \eta} = 0$ on $\partial\Omega$, $\lim_{k\to 0} u_0^k = u_0$. Here $\left(0, T_h^k\right)$ is the maximal time interval on which the solution v of (4)-(6) exists. When T_h^k is finite, then we say that the solution v of (4)-(6) quenches in a finite time, and the time T_h^k is called the quenching time of the solution v. The definition of the time T_h^k renders

$$v(x,t) > 0 \text{ in } \overline{\Omega} \times \left[0, T_h^k\right).$$

By a little transformation, it is not hard to see

$$v_t - Lv + a(x) f(v) \geq 0 \text{ in } \Omega \times \left(0, T_h^k\right),$$
$$v_0(x) \geq u_0(x) \,\text{in}\, \overline{\Omega}.$$

From the maximum principle, we have $v \geq u$ as long as all of them are defined. We deduce that $T_h^k \geq T$. In the present paper, under some hypotheses, we prove that if h and k are small enough, then the solution v of (4)-(6) quenches in a finite time, and its quenching time T_h^k goes to T as h and k go to zero, where T is the quenching time of the solution u of (1)-(3) . Similar results have been obtained in [4], [8], [13], [14], [15], [16], [18], [24], [27], where the authors have considered both the phenomenon of blow-up and the continuity of the blow-up time as a function of the initial data (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). Recently, in [7], Boni and N'gohisse have handled the continuity of the quenching time as a function of the initial data for the problem (1)-(3) in the case where the operator L is replaced by the Laplacian, $a(x) = 1$ and $f(s) = s^{-p}$ with p a positive constant. The rest of the paper is organized as follows. In the next section, under some assumptions, we show that the solution v of (4)-(6) quenches in a finite time and estimate its quenching time. In the third section, we prove the continuity of the quenching time and finally in the last section, we give some numerical results to illustrate our analysis.

2. Quenching time

In this section, under some assumptions, we show that the solution v of (4)-(6) quenches in a finite time and estimate its quenching time.
We borrow an idea of Friedman and McLeod in [12], and prove the following result.

Theorem 2.1. *Suppose that there exists a constant $A \in (0,1]$ such that the initial data at (6) satisfies*

$$Lu_0^k(x) - a_h(x) f\left(u_0^k(x)\right) \leq -Af\left(u_0^k(x)\right) \quad in \ \Omega\,. \tag{7}$$

Then, the solution v of (4)-(6) quenches in a finite time T_h^k which obeys the following estimate

$$T_h^k \leq \frac{1}{A}\int_0^{u^k_{0min}} \frac{ds}{f(s)}\,,$$

where $u^k_{0min} = \min_{x\in\overline{\Omega}} u_0^k(x)$.

Proof. Since $\left(0, T_h^k\right)$ is the maximal time interval of existence of the solution v, our aim is to show that T_h^k is finite and satisfies the above inequality. Introduce the function $J(x,t)$ defined as follows

$$(x,t) = v_t(x,t) + Af(v(x,t)) \text{ in } \overline{\Omega}\times\left[0, T_h^k\right).$$

A straightforward computation reveals that

$$J_t - LJ = \left(v_t - Lv\right)_t + Af'(v)v_t - ALf(v) \text{ in } \Omega\times\left(0,T_h^k\right). \tag{8}$$

Again, by a direct calculation, it is easy to check that

$$Lf(v) = f''(v)\sum_{i,j=1}^{N} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + f'(v)Lv \text{ in } \Omega\times\left(0,T_h^k\right).$$

This implies that $Lf(v) \geq f'(v)Lv$ in $\Omega\times\left(0,T_h^k\right)$, because the first term on the right hand side of the above equality is nonnegative. Using this estimate and (8), we arrive at

$$J_t - LJ \leq \left(v_t - Lv\right)_t + Af'(v)(v_t - Lv) \text{ in } \Omega\times\left(0,T_h^k\right). \tag{9}$$

According to (4) and (9), it is not hard to see that

$$J_t - LJ \leq -a_h(x)f'(v)v_t - Aa_h(x)f(v)f'(v) \text{ in } \Omega\times\left(0,T_h^k\right).$$

Taking into account the expression of J, we find that

$$J_t - LJ \leq -a_h(x)f'(v)J \text{ in } \Omega\times\left(0,T_h^k\right).$$

We also have

$$\frac{\partial J}{\partial \eta} = \left(\frac{\partial v}{\partial \eta}\right)_t + Af'(v)\frac{\partial v}{\partial \eta} = 0 \text{ on } \partial\Omega \times \left(0, T_h^k\right),$$

and due to (7), we discover that

$$J(x,0) = Lu_0^k(x) - a_h(x)f\left(u_0^k(x)\right) + Af\left(u_0^k(x)\right) \leq 0 \text{ in } \Omega\,.$$

It follows from the maximum principle that

$$J(x,t) \leq 0 \text{ in } \Omega \times \left(0, T_h^k\right),$$

which implies that

$$v_t(x,t) + Af(v(x,t)) \leq 0 \text{ in } \Omega \times \left(0, T_h^k\right).$$

This estimate may be rewritten in the following manner

$$\frac{dv}{f(s)} \leq -Adt \text{ in } \Omega \times \left(0, T_h^k\right). \tag{10}$$

Integrate the above inequality over $\left(0, T_h^k\right)$ to obtain

$$T_h^k \leq \frac{1}{A}\int_0^{v(x,0)} \frac{ds}{f(s)} \text{ for } x \in \Omega\,.$$

We deduce that

$$T_h^k \leq \frac{1}{A}\int_0^{u_{0min}^k} \frac{ds}{f(s)}\,.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. ☑

Remark 2.1. *Let $t_0 \in \left(0, T_h^k\right)$. Integrating the inequality in (10) from t_0 to T_h^k, we get*

$$T_h^k - t_0 \leq \frac{1}{A}\int_0^{v(x,t_0)} \frac{ds}{f(s)} \text{ for } x \in \Omega\,.$$

We deduce that

$$T_h^k - t_0 \leq \frac{1}{A}\int_0^{v_{\min}(t_0)} \frac{ds}{f(s)}\,.$$

3. Continuity of the quenching time

In this section, under some assumptions, we show that the solution v of (4)–(6) quenches in a finite time, and its quenching time goes to that of the solution u of (1)–(3) when h and k go to zero. Firstly, we show that the solution v approaches the solution u in $\overline{\Omega} \times [0, T-\tau]$ with $\tau \in (0,T)$ when h and k tend to zero. This result is stated in the following theorem.

Theorem 3.1. *Let u be the solution of problem (1)-(3). Suppose that $u \in C^{2,1}\left(\overline{\Omega} \times [0, T-\tau]\right)$ and $\min_{t\in[0,T-\tau]} u_{\min}(t) = \alpha > 0$ with $\tau \in (0,T)$. Assume that*

$$\|a_h - a\|_\infty = o(1) \ \text{ as } \ h \to 0\,, \tag{11}$$

$$\left\|u_0^k - u_0\right\|_\infty = o(1) \ \text{ as } \ k \to 0\,, \tag{12}$$

then, the problem (4)-(6) admits a unique solution $v \in C^{2,1}\left(\overline{\Omega} \times \left[0, T_h^k\right)\right)$ and the following relation holds

$$\sup_{t\in[0,T-\tau]} \|v(\cdot,t) - u(\cdot,t)\|_\infty = O\left(\|a_h - a\|_\infty + \left\|u_0^k - u_0\right\|_\infty\right) \ as\, (h,k) \to (0,0)\,.$$

Proof. The problem (4)-(6) has a unique solution $v \in C^{2,1}\left(\overline{\Omega} \times \left[0, T_h^k\right)\right)$, for each h. In the introduction of the paper, we have seen that $T_h^k \geq T$. Let $t(h,k) \leq T$ be the greatest value of $t > 0$ such that

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty \leq \frac{\alpha}{2} \ \text{ for } \ t \in (0, t(h,k))\,. \tag{13}$$

Obviously, we see that $\|v(\cdot,0) - u(\cdot,0)\|_\infty = \left\|u_0^k - u_0\right\|_\infty$. Due to this fact, we deduce from (12) and (13) that $t(h,k) > 0$ for k sufficiently small. By the triangle inequality, we find that

$$v_{\min}(t) \geq u_{\min}(t) - \|v(\cdot,t) - u(\cdot,t)\|_\infty \ \text{ for } \ t \in (0, t(h,k))\,, \tag{14}$$

which leads us to

$$v_{\min}(t) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \ \text{ for } \ t \in (0, t(h,k))\,.$$

Introduce the function $e(x,t)$ defined as follows

$$e(x,t) = v(x,t) - u(x,t) \ \text{ in } \ \overline{\Omega} \times [0, t(h,k))\,.$$

A routine computation reveals that

$$\begin{aligned} e_t - Le &= -a(x) f'(\theta) e + (a(x) - a_h(x)) f(v) \ \text{ in } \ \Omega \times (0, t(h,k))\,, \\ \frac{\partial e}{\partial \eta} &= 0 \ \text{ on } \ \partial\Omega \times (0, t(h,k)), \\ e(x,0) &= u_0^k(x) - u_0(x) \ \text{ in } \ \Omega\,, \end{aligned}$$

where θ is an intermediate value between u and v. According to (14), we find that $f(v) \leq M$, where $M = f(\frac{\alpha}{2})$. We deduce that

$$e_t - Le \leq -a(x)f'(\theta)e + M\|a - a_h\|_\infty \ \text{ in } \ \Omega \times (0, t(h,k))\,.$$

Introduce the function z defined as follows

$$z(x,t) = e^{(L+M+1)t}\left(\|a_h - a\|_\infty + \left\|u_0^k - u_0\right\|_\infty\right) \ \text{ in } \ \overline{\Omega} \times [0,T]\,,$$

where $L = -\|a\|_\infty f'(\frac{\alpha}{2})$. Due to (14), it is not hard to see that $L = -\|a\|_\infty f'(\frac{\alpha}{2})$ $\geq -a(x)f'(\theta)$ in $\Omega \times (0, t(h,k))$. Thanks to this observation, a straightforward calculation yields

$$\begin{aligned} z_t - Lz &\geq -a(x)f'(\theta)z + M\|a - a_h\|_\infty \ \text{ in } \ \Omega \times (0, t(h,k))\,,\\ \frac{\partial z}{\partial \eta} &= 0 \ \text{ on } \ \partial\Omega \times (0, t(h,k))\,,\\ z(x,0) &\geq e(x,0) \ \text{ in } \ \Omega\,. \end{aligned}$$

It follows from the maximum principle that

$$z(x,t) \geq e(x,t) \ \text{ in } \ \Omega \times (0, t(h,k))\,.$$

In the same way, we also prove that

$$z(x,t) \geq -e(x,t) \ \text{ in } \ \Omega \times (0, t(h,k))\,,$$

which implies that

$$\|e(\cdot,t)\|_\infty \leq e^{(L+M+1)t}\left(\|a_h - a\|_\infty + \left\|u_0^k - u_0\right\|_\infty\right) \ \text{ for } \ t \in (0, t(h,k))\,.$$

Let us show that $t(h,k) = T$. Suppose that $t(h,k) < T$. From (13), we obtain

$$\frac{\alpha}{2} = \|v(\cdot, t(h,k)) - u(\cdot, t(h,k))\|_\infty \leq e^{(L+M+1)T}\left(\|a_h - a\|_\infty + \left\|u_0^k - u_0\right\|_\infty\right).$$

Since the term on the right hand side of the above inequality goes to zero as h and k go to zero, we deduce that $\frac{\alpha}{2} \leq 0$, which is impossible. Consequently, $t(h,k) = T$, and the proof is complete. ☑

Now, we are in a position to prove the main result of the paper.

Theorem 3.2. *Suppose that problem (1)-(3) has a solution u which quenches at the time T and $u \in C^{2,1}\left(\overline{\Omega} \times [0,T)\right)$. Assume that*

$$\begin{aligned} \|a_h - a\|_\infty &= o(1) \ \text{ as } \ h \to 0\,,\\ \left\|u_0^k - u_0\right\|_\infty &= o(1) \ \text{ as } \ k \to 0\,. \end{aligned}$$

Under the assumption of Theorem 2.1, the problem (4)-(6) admits a unique solution v which quenches in a finite time T_h^k and the following relation holds

$$\lim_{(h,k)\to(0,0)} T_h^k = T.$$

Proof. Let $0 < \varepsilon \le T/2$. There exists $\rho > 0$ such that

$$\frac{1}{A}\int_0^y \frac{ds}{f(s)} \le \frac{\varepsilon}{2}, \quad 0 \le y \le \rho\,. \tag{15}$$

Since u quenches in a finite time T, there exists $T_0 \in \left(T - \frac{\varepsilon}{2}, T\right)$ such that

$$0 < u_{\min}(t) < \frac{\rho}{2} \ \text{ for } \ t \in [T_0, T)\,.$$

Set $T_1 = \frac{T_0+T}{2}$. It is not hard to see that

$$u_{\min}(t) > 0 \ \text{ for } \ t \in [0, T_1]\,.$$

From Theorem 3.1, for h and k small enough, the problem (4)-(6) admits a unique solution v, and the following estimate holds

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty < \frac{\rho}{2} \ \text{ for } \ t \in [0, T_1]\,,$$

which implies that $\|v(\cdot,T_1) - u(\cdot,T_1)\|_\infty \le \frac{\rho}{2}$ for h and k small enough. An application of the triangle inequality leads us to

$$v_{\min}(T_1) \le \|v(\cdot,T_1) - u(\cdot,T_1)\|_\infty + u_{\min}(T_1) \le \frac{\rho}{2} + \frac{\rho}{2} = \rho\,,$$

for h and k small enough. On the other hand, in the introduction of the present paper, it has been mentioned that $T_h^k \ge T$, and from Theorem 2.1, we know that v quenches at the time T_h^k. We deduce from Remark 2.1 and (15) that

$$0 \le T_h^k - T = T_h^k - T_1 + T_1 - T \le \frac{1}{A}\int_0^{v_{\min}(T_1)} \frac{ds}{f(s)} \le \varepsilon\,.$$

This ends the proof. ☑

4. Numerical results

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

$$\begin{aligned} u_t &= \Delta u - a(x)u^{-p} \ \text{ in } \ B \times (0,T)\,, \\ \frac{\partial u}{\partial \nu} &= 0 \ \text{ on } \ S \times (0,T)\,, \\ u(x,0) &= u_0(x) \ \text{ in } \ \overline{B}\,, \end{aligned}$$

where $B = \left\{x \in \mathbb{R}^N;\ \|x\| < 1\right\}$, $S = \left\{x \in \mathbb{R}^N; \|x\| = 1\right\}$. The above problem may be rewritten in the following form

$$u_t = u_{rr} + \frac{N-1}{r}u_r - a(r)u^{-p}, \quad r \in (0,1), \quad t \in (0,T)\,, \tag{16}$$
$$u_r(0,t) = 0, \quad u_r(1,t) = 0, \quad t \in (0,T)\,, \tag{17}$$
$$u(r,0) = \varphi(r), \quad r \in [0,1]\,. \tag{18}$$

Here, we take $\varphi(r) = \frac{2+\varepsilon\cos(\pi r)}{5}$ and $a(r) = 1+\varepsilon(3+\sin(\pi r))$, where $\varepsilon \in [0,1]$. We start by the construction of an adaptive scheme as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$, and approximate the solution u of (16)-(18) by the solution $U_h^{(n)} = \left(U_0^{(n)}, \dots, U_I^{(n)}\right)^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N\frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - a(x_0)\left(U_0^{(n)}\right)^{-p},$$
$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}$$
$$- a(x_i)\left(U_i^{(n)}\right)^{-p}, \quad 1 \le i \le I-1\,,$$
$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = N\frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} - a(x_I)\left(U_I^{(n)}\right)^{-p},$$
$$U_i^{(0)} = \varphi_i,\ 0 \le i \le I\,,$$

where $\varphi_i = \frac{2+\varepsilon\cos(i\pi h)}{5}$, $a(x_i) = 1+\varepsilon(3+\sin(i\pi h))$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\left\{\frac{(1-h^2)h^2}{4N}, h^2\left(U_{hmin}^{(n)}\right)^{p+1}\right\},$$

with $U_{hmin}^{(n)} = \min_{0\le i\le I} U_i^{(n)}$. Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution u of (16)-(18) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N\frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - a(x_0)\left(U_0^{(n)}\right)^{-p-1}U_0^{(n+1)}$$
$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h}$$
$$- a(x_i)(U_i^{(n)})^{-p-1}U_i^{(n+1)},$$

with $1 \leq i \leq I-1$, and

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = N \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} - a(x_I) \left(U_I^{(n)} \right)^{-p-1} U_I^{(n+1)} ,$$

with $U_i^{(0)} = \varphi_i$, $0 \leq i \leq I$.

As in the case of the explicit scheme, here, we also choose

$$\Delta t_n = h^2 \left(U_{hmin}^{(n)} \right)^{p+1} .$$

Let us again remark that for the above implicit scheme, existence and positivity of the discrete solution are also guaranteed using standard methods (see, for instance [6]). It is not hard to see that $u_{rr}(1,t) = \lim_{r \to 1} \frac{u_r(r,t)}{r}$ and $u_{rr}(0,t) = \lim_{r \to 0} \frac{u_r(r,t)}{r}$. Hence, if $r=0$ and $r=1$, then we see that

$$u_t(0,t) = N u_{rr}(0,t) - a(x_0) u^{-p}(0,t),\ t \in (0,T)\,,$$
$$u_t(1,t) = N u_{rr}(1,t) - a(x_I) u^{-p}(1,t),\ t \in (0,T)\,.$$

These observations have been taken into account in the construction of our schemes when $i=0$ and $i=I$. We need the following definition.

Definition 4.1. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n \to \infty} U_{hmin}^{(n)} = 0$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.*

In the sequel, in order to facilitate our discussion, let us define the notion of order of our method. In the vast majority of numerical methods, the error is expressible in the form of an asymptotic series as

$$e(h) = c_1 h^{p_1} + c_2 h^{p_2} + \cdots \tag{19}$$

where the positive integer or real exponents p_i have been arranged in an ascending order of magnitude, $p_1 < p_2 < \cdots$; c_i are constants. The value of p_1, in particular, defines the order of the numerical method. Let us perform a series of m computations with values of h that differ by a certain positive constant factor $q = 2 > 1$, forming the geometric sequence

$$h_1 = \varepsilon, \quad h_2 = \varepsilon/q, \quad h_3 = \varepsilon/q^2, \ldots, \quad h_m = \varepsilon/q^{m-1} . \tag{20}$$

We denote the corresponding values of the numerical solution by

$$b_1, \quad b_2, \ldots, \quad b_m , \tag{21}$$

where b_m is the most accurate value. Using Eq. (19), we write

$$\frac{b_{k+1}-b_k}{b_{k+2}-b_{k+1}}=\frac{e_{k+1}-e_k}{e_{k+2}-e_{k+1}}\cong\frac{\left(\frac{\varepsilon}{q^k}\right)^{p_1}-\left(\frac{\varepsilon}{q^{k-1}}\right)^{p_1}}{\left(\frac{\varepsilon}{q^{k+1}}\right)^{p_1}-\left(\frac{\varepsilon}{q^k}\right)^{p_1}}=q^{p_1}\,.$$

Consequently, we have

$$p_1\cong\frac{\log\left((b_{k+1}-b_k)\,/\,(b_{k+2}-b_{k+1})\right)}{\log(2)}.$$

The accuracy of this estimate improves as we use more advanced triplets in the sequence (21).

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n=\sum_{j=0}^{n-1}\Delta t_j$ which is computed at the first time when

$$\Delta t_n=|t_{n+1}-t_n|\leq 10^{-16}.$$

The order (s) of the method is computed from

$$s=\frac{\log((T_{2h}-T_h)/(T_{4h}-T_{2h}))}{\log(2)}.$$

Numerical experiments for $p=1, N=2$

First case: $\varepsilon=0$

Table 1. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	CPU time	s
16	0.080156	2088	3.4	-
32	0.080039	7654	17.2	-
64	0.080009	27786	125	1.96
128	0.080002	99795	785	2.09

Table 2. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	CPU time	s
16	0.080156	1794	3	-
32	0.080038	6475	13	-
64	0.080008	23071	131	1.97
128	0.079998	80933	2983	1.58

Second case: $\varepsilon = 1/10$

Table 3. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	CPU time	s
16	0.055760	1547	2.2	-
32	0.055641	5663	12.7	-
64	0.055609	20634	95	1.89
128	0.055601	74183	659	2.00

Table 4. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

I	t_n	n	CPU time	s
16	0.055697	1535	2.3	-
32	0.055610	5661	12.2	-
64	0.055595	20625	122	2.53
128	0.055593	74163	2829	2.91

Third case: $\varepsilon = 1/100$

Table 5. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	**CPU time**	s
16	0.076972	2014	3.1	-
32	0.076856	7386	17	-
64	0.076827	26821	121	1.04
128	0.075820	96347	862	1.02

Table 6. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

I	t_n	n	**CPU time**	s
16	0.076954	2014	3	-
32	0.076847	7385	16	-
64	0.076823	26820	211	2.15
128	0.076817	96343	3617	2.00

Fourth case: $\varepsilon = 1/1000$

Table 7. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	**CPU time**	s
16	0.079827	1261	3	-
32	0.079710	7626	17	-
64	0.079681	27686	177	2.01
128	0.079673	99438	904	1.85

Table 8. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

I	t_n	n	CPU time	s
16	0.079825	2080	3	-
32	0.079709	7626	16	-
64	0.079680	27686	149	2.00
128	0.079673	99438	3570	2.05

Remark 4.1. *If we consider the problem (16)-(18) in the case where the potential $a(r) = 1$, the initial data $\varphi(r) = \frac{2}{5}$ and p=1, then we see that the numerical quenching time of the discrete solution for the explicit scheme or the implicit scheme is approximately 0.08 (see Tables 1 and 2). Let us notice that theoretically, we know that value. In fact, since the initial value is constant, and the potential equals one, it is well known that the quenching time is that of the solution of the following differential equation $\alpha'(t) = -\alpha^{-p}(t)$, $t > 0$, $\alpha(0) = \frac{2}{5}$, with $p = 1$, and this quenching time is equal to 0.08. We observe from Tables 3, 4, 5, 6, 7 and 8 that if the above initial data increases slightly, then the numerical quenching time also increases slightly. This result confirms the theory established in the previous section.*

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