On a general type of $p$-adic parabolic equations

Un tipo general de ecuaciones parabólicas $p$-ádicas

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Abstract. In this paper we study the existence and uniqueness of the Cauchy problem for a general type of $p$-adic parabolic pseudo-differential operators constructed using the Taibleson operator. The results presented here constitute an extension of some results obtained by Zúñiga-Galindo and the author [13].

Key words and phrases. Parabolic equations, Markov processes, $p$-adic numbers, ultrametric diffusion.

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Resumen. En este artículo se estudia la existencia y unicidad de soluciones del problema de Cauchy asociado a un tipo general de ecuación parabólica $p$-ádica, construida usando el operador de Taibleson. Los resultados presentados aquí constituyen una extensión de algunos de los resultados obtenidos por Zúñiga-Galindo y el autor en [13].

Palabras y frases clave. Ecuaciones parabólicas, procesos de Markov, números $p$-ádicos, difusión ultramétrica.

1. Introduction

In recent years $p$-adic analysis has received a lot of attention due to its applications in mathematical physics, see e.g. [1], [2], [3], [4], [6], [7], [10], [12], [15] and the references therein. In particular, stochastic models involving Markov processes have appeared in several physical models describing complex systems such as proteins and macromolecules.
In [13] Zúñiga-Galindo and the author studied the following Cauchy problem:

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} + a(D_T^\alpha u)(x,t) &= f(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \in (0,T], \\
u(x,0) &= \varphi(x),
\end{aligned}
\]

(1)

where \( a > 0, \alpha > 0 \) and \( D_T^\alpha \) is the Taibleson operator of order \( \alpha \) defined as

\[
(D_T^\alpha u)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} (||\xi||_p^\alpha \mathcal{F}_{x \rightarrow \xi} u),
\]

(2)

where \( ||\xi||_p = \max\{||\xi||_p, \ldots, ||\xi_n||_p\} \).

The existence and uniqueness of a solution for (1) was established when the initial datum \( \varphi \) belongs to a class of increasing functions (see [13, Thm 1]). Also, it is shown that the fundamental solution is the transition density of a Markov process with space state \( \mathbb{Q}_p^n \) (see [13, Thm. 2]). These results continue Kochubei’s work on \( p \)-adic parabolic equations [9], [10, Sec. 4].

In this paper we consider the following initial value problem:

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} + a_0(x,t) (D_T^\alpha u) (x,t) &+ \sum_{k=1}^{n} a_k (x,t) (D_T^{\alpha_k} u) (x,t) + \\
&+ b(x,t) u(x,t) = f(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \in (0,T], \\
u(x,0) &= \varphi(x).
\end{aligned}
\]

(3)

here \( \alpha > 1, 0 < \alpha_1 < \ldots < \alpha_n < \alpha \), the coefficients \( a_0(x,t), a_1(x,t), \ldots, a_n(x,t), b(x,t) \), are real functions and \( D_T^{\alpha_k} \) is the Taibleson operator of order \( \beta \).

Denote by \( \mathcal{M}_\lambda (\lambda \geq 0) \) the class of complex-valued locally constant functions \( \varphi(x) \) on \( \mathbb{Q}_p^n \), satisfying

\[|\varphi(x)| \leq C \left(1 + ||x||_p^3\right).
\]

We solve (3) in the class \( \mathcal{M}_\lambda \) for a suitable \( \lambda \) (see Thm. 2 ahead) following the ideas introduced by Kochubei in [9] (see also [10, Sec. 4], [8]).

In the case \( n = 1 \), our main result, (see Thm. 2), agrees with Kochubei’s results (see [9, Thm. 1], [10]).

A different generalization of the \( p \)-adic parabolic equations and its Markov processes was given recently by Zúñiga-Galindo in [16].

2. Preliminary results

Let \( \mathbb{Q}_p \) be the field of the \( p \)-adic numbers. For \( x \in \mathbb{Q}_p \), let \( v(x) \) denote the valuation of \( x \) normalized by the condition \( v(p) = 1 \), and \( |x|_p = p^{-v(x)} \) the normalized absolute value. We extend the \( p \)-adic norm to \( \mathbb{Q}_p^n \) as follows:

\[||x||_p := \max\{|x_1|_p, \ldots, |x_n|_p\}, \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.\]
Let $S \left( \mathbb{Q}_p^n \right)$ denote the $\mathbb{C}$-vector space of Schwartz-Bruhat functions over $\mathbb{Q}_p^n$. Its dual space $S' \left( \mathbb{Q}_p^n \right)$ is the space of distribution over $\mathbb{Q}_p^n$.

If $\varphi(x) \in S \left( \mathbb{Q}_p^n \right)$, we define its exponent of local constancy as the smallest integer $l \geq 0$ with the property that for any $x \in \mathbb{Q}_p^n$

$$\varphi(x + x') = \varphi(x), \quad \text{if } ||x'||_p \leq p^{-l}.$$ 

For $x, y$ in $\mathbb{Q}_p^n$ we put $x \cdot y = \sum_{i=1}^n x_i y_i$.

Let $\Psi$ denote an additive character of $\mathbb{Q}_p$, trivial on $\mathbb{Z}_p$ but no on $p^{-1} \mathbb{Z}_p$

$$\mathcal{F} \varphi(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-x \cdot \xi) \varphi(x) \, d^nx,$$

where $d^nx$ denotes the Haar measure of $\mathbb{Q}_p^n$ normalized in such a way that $\mathbb{Z}_p^n$ has measure 1.

### 2.1. The taibleson operator

We set

$$\Gamma_p^{(n)}(\alpha) := \frac{1 - p^{\alpha-n}}{1 - p^{-\alpha}}, \quad \alpha \neq 0.$$ 

This function is called the $p$-adic Gamma function. The function

$$k_\alpha(x) = \frac{||x||_p^{\alpha-n}}{\Gamma_p^{(n)}(\alpha)}, \quad \alpha \in \mathbb{R} \setminus \{0, n\}, \quad x \in \mathbb{Q}_p^n,$$

is called the multi-dimensional Riesz kernel. It determines a distribution on $S \left( \mathbb{Q}_p^n \right)$ as follows. If $\alpha \neq 0, n$, and $\varphi \in S \left( \mathbb{Q}_p^n \right)$,

$$\langle k_\alpha(x), \varphi(x) \rangle = \frac{1 - p^{\alpha-n}}{1 - p^{-\alpha}} \varphi(0) + \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} \int_{||x||_p \geq 1} ||x||_p^{\alpha-n} \varphi(x) \, d^n x + \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} \int_{||x||_p \leq 1} ||x||_p^{\alpha-n} (\varphi(x) - \varphi(0)) \, d^n x.$$ 

Thus $k_\alpha \in S' \left( \mathbb{Q}_p^n \right)$, for $\mathbb{R} \setminus \{0, n\}$. In the case $\alpha = 0$, by passing to the limit, we obtain

$$\langle k_0(x), \varphi(x) \rangle := \lim_{\alpha \to 0} \langle k_\alpha(x), \varphi(x) \rangle = \varphi(0),$$ 

i.e., $k_0(x) = \delta(x)$, the Dirac delta function, and therefore $k_\alpha \in S' \left( \mathbb{Q}_p^n \right)$, for $\mathbb{R} \setminus \{n\}$.

It follows that, for $\alpha > 0$,

$$\langle k_{-\alpha}(x), \varphi(x) \rangle = \frac{1 - p^\alpha}{1 - p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} ||x||_p^{\alpha-n} (\varphi(x) - \varphi(0)) \, d^n x. \quad (4)$$

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Lemma 1. [14, Chap. III, Theorem 4.5] As elements of $S'(\mathbb{Q}_p^n)$, $(\mathcal{F}k_n)(x)$ equals $||x||_p^{-\alpha}$, $\alpha \neq n$.

Definition 1. The Taibleson pseudo-differential operator $D_T^\alpha$, $\alpha > 0$, is defined as

$$(D_T^\alpha \varphi)(x) = \mathcal{F}^{-1}_{\xi \to x} (||\xi||_p^\alpha \mathcal{F}x \to \xi \varphi), \quad \text{for } \varphi \in S(\mathbb{Q}_p^n).$$

As a consequence of the previous Lemma and (4), we get

$$(D_T^\alpha \varphi)(x) = (k_\alpha * \varphi)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} ||y||_p^{-\alpha-n} (\varphi(x-y) - \varphi(x)) \, d^n y. \quad (5)$$

Let us remark that the right-hand side of (5) makes sense for a wider class of functions, for example, for locally constant functions $\varphi(x)$ satisfying

$$\int_{||x||_p \geq 1} ||x||_p^{-\alpha-n} |\varphi(x)| \, d^n x < \infty.$$

Definition 2. Denote by $\mathcal{M}_\lambda$ ($\lambda \geq 0$) the class of complex-valued locally constant functions $\varphi(x)$ on $\mathbb{Q}_p^n$, such that

$$|\varphi(x)| \leq C (1 + ||x||_p^\lambda).$$

If a function $\varphi$ depends also on a parameter $t$, we shall say that $\varphi \in \mathcal{M}_\lambda$ uniformly with respect to $t$, if $C$ and the corresponding exponent of local constancy do not depend on $t$.

2.2. The parametrized equation

As in the Euclidean case, the first step is the study of the parametrized fundamental solution $Z(x, t, y, \theta)$ of the Cauchy problem

$$\left\{ \begin{array}{l}
\frac{\partial u(x, t)}{\partial t} + a_0(y, \theta)(D_T^\alpha u)(x, t) = 0, \quad x \in \mathbb{Q}_p^n, \quad t \in (0, T], \\
u(x, 0) = \varphi(x),
\end{array} \right. \quad (6)$$

where $y \in \mathbb{Q}_p^n$ and $\theta > 0$ are parameters. This equation was studied in the recent paper [13] by Zúñiga-Galindo and the author.

In this article we consider the following fundamental solution:

$$Z(x, t, y, \theta) = \int_{\mathbb{Q}_p^n} \Psi(x \cdot \xi) e^{-a_t(y, \theta)||\xi||_p^\alpha} \, d^n \xi,$$
Lemma 2. The fundamental solution of (6) $Z(x,t,y,\theta)$, has the following properties

\[ Z(x,t,y,\theta) \leq Ct \left( t^{1/\alpha + ||x||_p} \right)^{-\alpha-n}, \quad (7) \]

\[ \left| \frac{\partial Z}{\partial t}(x,t,y,\theta) \right| \leq C \left( t^{1/\alpha + ||x||_p} \right)^{-\alpha-n}, \quad (8) \]

\[ |(D^\gamma_T Z)(x,t,y,\theta)| \leq C \left( t^{1/\alpha + ||x||_p} \right)^{-\gamma-n}, \quad (9) \]

where the constants do not depend on $y, \theta$.

Proof. These results where established in Lemmas 3 and 8 of [13]. □✓

As an [13], we get the identities

Lemma 3.

\[ \int_{\mathbb{Q}_p^n} Z(x,t,y,\theta) \, d^n x = 1, \quad (10) \]

\[ \frac{\partial Z}{\partial t}(x,t,y,\theta) = -a_0(y,\theta) \int_{\mathbb{Q}_p^n} \psi(x \cdot \xi) ||\xi||_p^\alpha e^{-a_0(y,\theta)t} ||\xi||_p^\alpha \, d^n \xi, \quad (11) \]

\[ (D^\gamma_T Z)(x,t,y,\theta) = \int_{\mathbb{Q}_p^n} \psi(x \cdot \xi) ||\xi||_p^\gamma e^{-a_0(y,\theta)t} ||\xi||_p^\gamma \, d^n \xi, \quad (12) \]

\[ \int_{\mathbb{Q}_p^n} (D^\gamma_T Z)(x,t,y,\theta) \, d^n x = 0. \quad (13) \]

3. Uniqueness of the solution

In this section we assume that the coefficients $a_k(x,t)$, $k = 0, 1, \ldots, n$ are non-negative bounded continuous functions, and that $b(x,t)$ is a continuous bounded function. Let $0 \leq \gamma < \alpha_1$ (if $a_1(x,t) = \cdots = a_n(x,t) = 0$, we shall assume that $0 \leq \gamma < \alpha$). The proof of the following Theorem is a simple variation of the one given by Kochubei in [10, Thm. 4.5] for the case $n = 1$.

Theorem 1. [10, Thm. 4.5] If $u(x,t)$ is a solution of (3) with $f(x,t) = 0$, and such that $u \in \mathcal{M}_\gamma$ uniformly with respect to $t$, and $u(x,0) = 0$, then $u(x,t) = 0$ for any $x \in \mathbb{Q}_p^n$ and $t \in (0,T]$.  

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We now consider the heat potential

\[ u(x,t,\tau) := \int_{\tau}^{t} \int_{Q^p_n} Z(x-y,t-\theta, y, \theta) f(y, \theta) \, d^n y \, d\theta, \]

where \( \tau < t \), \( f(x,t) \) is uniformly locally constant in \( x \in Q^p_n \), continuous in \( (x,t) \in Q^p_n \times (0,T] \), and

\[ |f(x,t)| \leq Ct^{-\rho} \left( 1 + ||x||^\lambda_p \right), \]

for some \( 0 \leq \rho < 1 \), and \( 0 \leq \lambda < \alpha \).

Next we calculate the derivative with respect to \( t \) and the action of the \( \mathcal{T} \)bilsen operator on this potentials. This can be achieved using the techniques presented in [10, Sec. 4.5]. We formally summarize these facts for future reference as follows

**Lemma 4.** With the above notations,

i) \( \frac{\partial u}{\partial t}(x,t,\tau) = f(x,t) + \int_{\tau}^{t} \int_{Q^p_n} \frac{\partial Z}{\partial t}(x-y,t-\theta, y, \theta)(f(y, \theta) - f(x, \theta)) \, d^n y \, d\theta \)

\[ + \int_{\tau}^{t} f(x, \theta) \int_{Q^p_n} \frac{\partial Z}{\partial t}(x-y,t-\theta, y, \theta) \, d^n y \, d\theta. \]

ii) If \( \lambda < \gamma < \alpha \), then

\( (D^\gamma_T u)(x,t,\tau) = \int_{\tau}^{t} \int_{Q^p_n} Z_\gamma(x-y,t-\theta, y, \theta) f(y, \theta) \, d^n y \, d\theta, \quad \lambda < \gamma < \alpha. \)

iii) \( (D^\alpha_T u)(x,t,\tau) = \int_{\tau}^{t} \int_{Q^p_n} Z_\alpha(x-y,t-\theta, y, \theta)(f(y, \theta) - f(x, \theta)) \, d^n y \, d\theta \)

\[ + \int_{\tau}^{t} f(x, \theta) \int_{Q^p_n} (Z_\alpha(x-y,t-\theta, x, \theta) - Z_\alpha(x-y,t-\theta, y, \theta)) \, d^n y \, d\theta. \]
5. The Cauchy problem

In this section we construct a fundamental solution for the following Cauchy problem

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} + a_0(x,t)(D^a Tu)(x,t) &+ \sum_{k=1}^{n} a_k(x,t)(D^{a_k} u)(x,t) + \\
+ b(x,t)u(x,t) &= f(x,t), \quad x \in Q^n_p, \quad t \in (0,T], \\
u(x,0) &= \varphi(x).
\end{aligned}
\]

We shall assume that \( \alpha > 1 \) and that \( 0 < \alpha_1 < \ldots < \alpha_n < \alpha \), and that the coefficients \( a_0(x,t), a_1(x,t), \ldots, a_n(x,t), b(x,t) \) belong (with respect to \( x \in Q^n_p \)) to the class \( \mathcal{M}_0 \) uniformly with respect to \( t \in [0,T] \), and satisfy the Hölder condition in \( t \), with an exponent \( \nu \in (0,1] \), uniformly with respect to \( x \in Q^n_p \). We also assume the uniform parabolicity condition \( a_0(x,t) \geq \mu > 0 \), and that \( a_{n+1} = \alpha(1-\nu) > \alpha_n \).

As in [10, Sec. 4.5] we look for a fundamental solution of (14) of the form

\[
\Gamma(x,t,\xi,\tau) = \int_{Q^n_p} R(x,t,\xi,\tau) + \int_{Q^n_p} R(x,t,\eta,\theta) \Phi(\eta,\theta,\xi,\tau) d^n \eta d\theta.
\]

Thus we formally require that

\[
\frac{\partial \Gamma}{\partial t}(x,t,\xi,\tau) + a_0(x,t)(D^a T \Gamma)(x,t,\xi,\tau) + \\
+ \sum_{k=1}^{n} a_k(x,t)(D^{a_k} \Gamma)(x,t,\xi,\tau) + b(x,t)\Gamma(x,t,\xi,\tau) = 0.
\]

By using formally the formulas given in the Lemma (4), we can see that \( \Phi(x,t,\xi,\tau) \) is a solution of the integral equation

\[
\Phi(x,t,\xi,\tau) = R(x,t,\xi,\tau) + \int_{Q^n_p} R(x,t,\eta,\theta) \Phi(\eta,\theta,\xi,\tau) d^n \eta d\theta,
\]

where

\[
R(x,t,\xi,\tau) = (a_0(\xi,\tau) - a_0(x,t))Z_0(x - \xi, t - \tau, \xi, \tau) \\
- \sum_{k=1}^{n} a_k(x,t)Z_0(x - \xi, t - \tau, \xi, \tau) - b(x,t)Z(x - \xi, t - \tau, \xi, \tau).
\]

In order to solve the integral equation (15) we use the method of successive approximations (see e.g. [5], [11]). We set

\[
R_1(x,t,\xi,\tau) := R(x,t,\xi,\tau),
\]

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and

\[ R_{m+1}(x, t, \xi, \tau) := \int_{Q^n_p} \int_{Q^n_p} R(x, t, \eta, \theta) R_m(\eta, \theta, \xi, \tau) d^n \eta d\theta, \quad m \in \mathbb{N} \setminus \{0\}. \]

We claim that

\[ \Phi(x, t, \xi, \tau) = \sum_{m=1}^{\infty} R_m(x, t, \xi, \tau) \]

is a solution of (15). In order to prove the convergence of the series we need the followings two Lemmas, whose proof is a simple variation of those given by Kochubei in [10, Sec. 4.5] for the case \( n = 1 \).

**Lemma 5.** [10, Eq 4.64] With the above notation,

\[ |R(x, t, \xi, \tau)| \leq C \sum_{k=1}^{n+1} ((t - \tau)^{1/\alpha} + ||x - \xi||_p)^{-\alpha_k - \alpha}, \]

where \( C \) is a positive constant.

**Lemma 6.** [10, Lemma 4.6] Let

\[ J(x, \xi, t, \tau) = \int_{\mathbb{Q}^n_p} (t - \mu)^{-\rho/\alpha} (\mu - \tau)^{-\sigma/\alpha}
\left( \int_{\mathbb{Q}^n_p} ((t - \mu)^{1/\alpha} + ||x - \eta||_p)^{-n-b_1} \right)^{-n-b_1} \]

\[ \left( (\mu - \tau)^{1/\alpha} + ||\eta - \xi||_p \right)^{-n-b_2} d^n \eta d\mu, \]

where \( 0 \leq \tau < t, x, \xi \in \mathbb{Q}^n_p, b_1, b_2 > 0, \rho + b_1 < \alpha, \sigma + b_2 < \alpha \). Then

\[ J(x, \xi, t, \tau) \leq C \left( (t - \tau)^{\rho} B \left( 1 - \frac{\rho}{\alpha}, 1 - \frac{\sigma + b_2}{\alpha} \right) (t - \tau)^{1/\alpha} + ||x - \xi||_p \right)^{-n-b_1}
\]

\[ + C \left( (t - \tau)^{\rho} B \left( 1 - \frac{\rho + b_1}{\alpha}, 1 - \frac{\sigma}{\alpha} \right) (t - \tau)^{1/\alpha} + ||x - \xi||_p \right)^{-n-b_2}, \]

where \( \kappa = \frac{\rho + \sigma + b_2 - \alpha}{\alpha}, \theta = \frac{\rho + \sigma + b_1 - \alpha}{\alpha} \), \( C \) is a positive constant depends only on \( b_1, b_2 \) and \( B(z_1, z_2) \) is the Archimedean Beta function.
Lemma 7. With the above notation,

\[ |R_m(x, t, \xi, \tau)| \leq CM^m(t - \tau)^{(m - 1)\nu/\alpha} \frac{(\Gamma(\nu/\alpha))^m}{\Gamma(m\nu/\alpha)} \sum_{k=1}^{n+1} ((t - \tau)^{1/\alpha} + \|x - \xi\|_p)^{-\alpha_k - n}, \]

where \( C \) is a positive constant.

Proof. We use induction on \( m \). The case \( m = 1 \) is clear. We assume the case \( m \) as induction hypothesis, then by Lemmas (5), (6) and (7) we have

\[
|R_{m+1}(x, t, \eta, \theta)| \leq \int_{Q^n_p} |R(x, t, \eta, \theta)| \cdot |R_m(\eta, \theta, \xi, \tau)| \, d^n\eta \, d\theta
\]

\[
= CM^m \frac{(\Gamma(\nu/\alpha))^m}{\Gamma(m\nu/\alpha)} \sum_{k,l=1}^{n+1} \int_{Q^n_p} ((t - \eta)^{1/\alpha} + \|x - \eta\|_p)^{-\alpha_k - n} \int_{Q^n_p} ((t - \theta)^{1/\alpha} + \|x - \xi\|_p)^{-\alpha_l - n} \, d^n\eta \, d\theta.
\]

Thus it is sufficient to bound the integral

\[
I_{k,l}(x, \xi, t, \tau) = \int_{Q^n_p} ((t - \eta)^{1/\alpha} + \|x - \eta\|_p)^{-\alpha_k - n} \int_{Q^n_p} ((t - \theta)^{1/\alpha} + \|x - \xi\|_p)^{-\alpha_l - n} \, d^n\eta \, d\theta.
\]

By using Lemma (6),

\[
I_{k,l}(x, \xi, t, \tau) \leq CB \left( \frac{\alpha - \alpha_k}{\alpha}, \frac{mv + \alpha - v}{\alpha} \right) (t - \tau)^{-\nu(mv + \alpha_k - \alpha)/\alpha} \left( (t - \tau)^{1/\alpha} + \|x - \xi\|_p \right)^{-\alpha_l - n} + CB \left( 1, \frac{mv + \alpha - v}{\alpha} \right) (t - \tau)^{-\nu(mv + \alpha_l - \alpha)/\alpha} \left( (t - \tau)^{1/\alpha} + \|x - \xi\|_p \right)^{-\alpha_k - n}.
\]
We now recall that if \( \epsilon, \delta > 0 \), then \( B(x + \epsilon, y + \delta) \leq B(x, y) \), thus
\[
B \left( \frac{\alpha - \alpha_k}{\alpha}, \frac{m \lambda + \alpha - \lambda}{\alpha} \right) \leq B \left( \frac{\lambda}{\alpha}, \frac{m \lambda}{\alpha} \right),
\]
\[
B \left( 1, \frac{m \lambda + \alpha - \lambda - \alpha \lambda}{\alpha} \right) \leq B \left( \frac{\lambda}{\alpha}, \frac{m \lambda}{\alpha} \right),
\]
and
\[
(t - \tau)^{-\left(\nu - m \nu + \alpha_k - \alpha\right) \lambda} \leq C'(t - \tau)^{(m + 1) \nu \lambda}.
\]
Therefore,
\[
|R_{m+1}(x, t, \xi, \tau)| \leq CM^{m+1}(t - \tau)^{m \nu / \alpha} \frac{\Gamma(\nu / \alpha)^{m+1}}{\Gamma((m+1)\nu / \alpha)} \sum_{k=1}^{n+1} \left( (t - \tau)^{1/\alpha} + ||x - \xi||_{\nu} \right)^{-\alpha_k - n}.
\]

By using Stirling’s formula we verify the absolute convergence of
\[
\Phi(x, t, \xi, \tau) = \sum_{m=1}^{\infty} R_m(x, t, \xi, \tau),
\]
and also that
\[
|\Phi(x, t, \xi, \tau)| \leq C \sum_{k=1}^{n+1} ((t - \tau)^{1/\alpha} + ||x - \xi||_{\nu})^{-\alpha_k - n} \quad (16)
\]

We now come to the main result. This result is an \( n \)-dimensional version of Theorem 4.6, p. 156 in [10]. Here we assume that \( 0 \leq \lambda < \alpha_1 \); if all the coefficients \( a_1(x, t), \ldots, a_n(x, t) \) vanish identically, then we may assume \( 0 \leq \lambda < \alpha \).

**Theorem 2.** The Cauchy problem
\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} + a_0(x, t) (D^\alpha T u)(x, t) + \sum_{k=1}^{n} a_k(x, t) (D^\alpha_k u)(x, t) \\
+ b(x, t) u(x, t) &= f(x, t), \quad x \in Q^n_T, \quad t \in (0, T], \\
\end{align*}
\]
\[
u(x, 0) = \varphi(x),
\]
has a solution
\[ u(x, t) = \int_0^t \int_{\mathbb{Q}_p^n} \Gamma(x, t, \xi, \tau) f(\xi, \tau) \, d^n \xi \, d\tau + \int_{\mathbb{Q}_p^n} \Gamma(x, t, \xi, 0) \varphi(\xi) \, d^n \xi, \tag{18} \]
which is continuous on \( \mathbb{Q}_p^n \times [0, T] \), continuously differentiable in \( t \), and belonging to \( \mathcal{M}_\lambda \) uniformly with respect to \( t \). The fundamental solution \( \Gamma(x, t, \xi, \tau) \), \( x, \xi \in \mathbb{Q}_p^n \), \( 0 \leq \tau < t \leq T \), is then of the form
\[ \Gamma(x, t, \xi, \tau) = Z(x - \xi, t - \tau, \xi, \tau) + W(x, t, \xi, \tau), \tag{19} \]
and finally
\[ |W(x, t, \xi, \tau)| \leq C \left\{ (t - \tau)^{n+\lambda} \left[ (t - \tau)^{1/\alpha} + ||x - \xi||_p \right]^{-\alpha-n} \right. \]
\[ + \left. (t - \tau) \sum_{k=1}^{n+1} \left[ (t - \tau)^{1/\alpha} + ||x - \xi||_p \right]^{-\alpha_{k,n}} \right\}. \tag{20} \]

Proof. Denote by \( u_1(x, t) \) and \( u_2(x, t) \) the first and the second summands in the right hand side of (18). We find that
\[ u_1(x, t) = \int_0^t \int_{\mathbb{Q}_p^n} Z(x - \xi, t - \tau, \xi, \tau) f(\xi, \tau) \, d^n \xi \, d\tau \]
\[ + \int_0^t \int_{\mathbb{Q}_p^n} Z(x - \eta, t - \theta, \eta, \theta) F(\eta, \theta) \, d^n \eta \, d\theta, \]
and
\[ u_2(x, t) = \int_{\mathbb{Q}_p^n} Z(x - \xi, t, \xi, 0) \varphi(\xi) \, d^n \xi \]
\[ + \int_0^t \int_{\mathbb{Q}_p^n} Z(x - \eta, t - \theta, \eta, \theta) G(\eta, \theta) \, d^n \eta \, d\theta, \]
where
\[ F(\eta, \theta) = \int_0^\theta \int_{\mathbb{Q}_p^n} \Phi(\eta, \theta, \xi, \tau) f(\xi, \tau) \, d^n \xi \, d\tau, \]
\[ G(\eta, \theta) = \int_{\mathbb{Q}_p^n} \Phi(\eta, \theta, \xi, 0) \varphi(\xi) \, d^n \xi. \]
Now by (16) and Proposition 2 in [13],

\[ |F(\eta, \theta)| \leq C, \text{ and } |G(\eta, \theta)| \leq C\theta^{-\alpha_n+1/\alpha}, \]

for all \( \eta \in Q_n^p \text{ and } \theta \in [0, T] \). In addition the functions \( F \) and \( G \) are uniformly locally constant. Indeed, by the recursive definition of the function \( \Phi \) we see that if \( N \) is a local constancy exponent for all the functions \( a_i, b, Z_{\alpha_i} \), and \( Z \), and if \( |\delta| \leq q^{-N} \), then

\[ \phi(x + \delta, t, \xi + \delta, \tau) = \phi(x, t, \xi, \tau), \]

whence

\[ F(\eta + \delta, \theta) = F(\eta, \theta), \quad G(\eta + \delta, \theta) = G(\eta, \theta). \]

Thus the potentials in the expressions for \( u_1(x, t) \) and \( u_2(x, t) \) satisfy the conditions under which the differentiation formulas of the Lemmas (4) were obtained. By using these formulas one verifies after some simple transformations that \( u(x, t) \) is a solution of the equation (17).

Let us show that \( u(x, t) \to \varphi(x) \) as \( t \to 0 \). Due to (19) and (20), it is sufficient to verify that \( u_2(x, t) \to \varphi(x) \) as \( t \to 0 \). By virtue of (10) we have

\[
u_2(x, t) = \int_{Q_n^p} [Z(x - \xi, t, \xi, 0) - Z(x - \xi, t, 0, 0)] \varphi(\xi) \, d^n\xi
\]

\[ + \int_{Q_n^p} Z(x - \xi, t, x, 0)[\varphi(\xi) - \varphi(x)] \, d^n\xi + \varphi(x). \]

Since as functions of their third argument \( Z \) and \( \varphi \) are locally constant, both integrals in the previous expression are performed over the set

\[ \{\xi \mid ||x - \xi||_p \geq p^{-N}\}. \]

By applying (7) we see that both integrals tend to zero as \( t \to 0 \). \( \square \)

6. Markov processes

By using Theorems (1) and (2), we obtain a probabilistic interpretation for the function \( \Gamma(x, t, \xi, \tau) \).

**Theorem 3.** The fundamental solution \( \Gamma(x, t, \xi, \tau) \) is the transition density of a bounded right-continuous strict Markov process without second kind discontinuities. If \( b(x, t) = 0 \), then the process does not explode.

The proof uses the same argument given in [10, pg. 162].

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References


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