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Evolution of curvature tensors under mean curvature flow

Evolución de los tensores de curvatura bajo el flujo de curvatura media

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Abstract. We obtain the evolution equations for the Riemann tensor, the Ricci tensor and the scalar curvature induced by the mean curvature flow. The evolution of the scalar curvature is similar to the Ricci flow, however, negative, rather than positive, curvature is preserved. Our results are valid in any dimension.

Key words and phrases. Curvature tensors, mean curvature flow.

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Resumen. Se obtienen las ecuaciones de evolución para el tensor de Riemann, el tensor de Ricci y el escalar de curvatura inducidas por el flujo de curvatura media. La evolución de la curvatura escalar es similar al flujo de Ricci, sin embargo, la curvatura negativa, en vez de la positiva, es favorecida. Nuestros resultados son válidos en cualquier dimensión.

Palabras y frases clave. Tensores de curvatura, flujo de curvatura media.

1. Introduction

Due to a series of remarkable theorems [1, 2, 3, 7, 8, 9, 10, 13, 14] any riemannian space can be considered as a space embedded into a higher–dimensional flat space. Therefore, the extrinsic approach to riemannian geometry is equivalent to the usual intrinsic approach in which no reference to an embedding space is done. In the intrinsic approach the metric tensor is the basic geometrical object, while in the extrinsic approach the basic geometrical objects are the embedding

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functions describing the situation of the embedded space as a subspace of the ambient space.

One of the most interesting problems in geometric analysis is the description of the evolution of riemannian spaces under particular flows. The best known examples are the Ricci flow and the mean curvature flow. The Ricci flow [11] describes how a riemannian space evolves, when the initial data are close enough to that of a sphere, towards a highly symmetric, homogeneous, configuration. The Ricci flow was initially developed for three–dimensional spaces having in mind its application in the demostration of the Thurston geometrisation conjecture [20] and, consequently, the resolution of the Poincaré conjecture [15, 16, 17]. On the other hand, the mean curvature flow [12, 21] describes the evolution of surfaces embedded in a higher dimensional, possibly flat, space. The mean curvature flow was initially developed for n dimensional surfaces embedded in R^{n+1}, but it can be generalized to an ambient space of any dimension $N > n$.

The Ricci flow is based on intrinsic riemannian geometry but, in spite of the fact that the evolution of a riemannian space under the Ricci flow is described intrinsically, these riemannian spaces are often pictured as embedded spaces.

On the other hand, the mean curvature flow is based on extrinsic riemannian geometry. In this case, due to the equivalence of both approaches to riemannian geometry, the elements of intrinsic riemannian geometry (metric tensor, Riemann tensor, Ricci tensor, scalar curvature) also evolve under the mean curvature flow. To obtain these evolution equations is the purpose of this work. Our main result is that the scalar curvature evolves according to an equation similar to the Ricci flow and therefore has similar properties, that is, for adequate initial data, spaces evolve towards highly symmetric, homogeneous, configurations. The main difference with respect to the Ricci flow is that, due to the presence of a minus sign, negative, rather than positive, curvature is preserved.

Section 2 contains preliminaries for Riemannian geometry and geometric flows. Section 3 contains our main results, namely, the evolution equations for the curvature tensors. Section 4 contains our conclusions.

2. Preliminaries

2.1. Riemannian geometry

Let us start with some fundamentals of riemannian geometry [18]. The basic geometrical object in riemannian geometry is the metric tensor $\mathbf{g}$ with components $g_{ij}(\mathbf{x})$. The Christoffel symbol is given by:

$$\Gamma^k{}_{ij} = \frac{1}{2}\, g^{k\ell} \left[\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right] , \tag{1}$$

and the covariant derivative of a covariant vector is given by

$$\nabla_i v_j = \frac{\partial v_j}{\partial x^i} - \Gamma^k{}_{ij}\, v_k\,. \tag{2}$$

The covariant derivative extends in a natural way to contravariant vectors and tensors of any covariance so as to preserve the Leibniz rule. In particular

$$\nabla_k g_{ij} \equiv 0\,. \tag{3}$$

The Riemann tensor is given by

$$R^k{}_{\ell ij} = \frac{\partial \Gamma^k{}_{j\ell}}{\partial x^i} - \frac{\partial \Gamma^k{}_{i\ell}}{\partial x^j} + \Gamma^k{}_{im}\,\Gamma^m{}_{j\ell} - \Gamma^k{}_{jm}\,\Gamma^m{}_{i\ell}\,. \tag{4}$$

Covariant derivatives do not commute, and we obtain the Ricci identity for a covariant vector

$$\nabla_i \nabla_j v_k - \nabla_j \nabla_i v_k = -R^\ell{}_{kij}\, v_\ell\,. \tag{5}$$

The Ricci identity extends in a natural way to contravariant vectors and tensors of any covariance.

The completely covariant Riemann tensor, $R_{ijk\ell} = g_{im} R^m{}_{jk\ell}$, is given by

$$\begin{aligned} R_{ijk\ell}(\mathbf{g}) = \frac{1}{2} & \left[\frac{\partial^2 g_{jk}}{\partial x^i \partial x^\ell} + \frac{\partial^2 g_{i\ell}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{j\ell}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^\ell}\right] \\ & + g_{mn}\left[\Gamma^m{}_{i\ell}\,\Gamma^n{}_{jk} - \Gamma^m{}_{ik}\,\Gamma^n{}_{j\ell}\right]. \end{aligned} \tag{6}$$

The Ricci tensor and the scalar curvature are given by the usual expressions

$$R_{j\ell}(\mathbf{g}) = g^{ik}\, R_{ijk\ell}(\mathbf{g})\,, \tag{7}$$

$$R(\mathbf{g}) = g^{j\ell}\, R_{j\ell}(\mathbf{g})\,. \tag{8}$$

Finally, the completely covariant Riemann tensor (6) satisfies the Bianchi identity

$$\nabla_i R_{jk\ell m}(\mathbf{g}) + \nabla_j R_{ki\ell m}(\mathbf{g}) + \nabla_k R_{ij\ell m}(\mathbf{g}) \equiv 0\,. \tag{9}$$

Now we remind some elementary results in extrinsic riemannian geometry [6]. Let V_n be a riemannian space with metric tensor $\mathbf{g}$ with components $g_{ij}(\mathbf{x})$. According to the local and global embedding theorems [1, 2, 3, 7, 8, 9, 10, 13, 14] always there exist functions $X^A = X^A(\mathbf{x})$, $A = 1, \cdots, N$, and G_{AB}, $N \geq n$, with $\mathrm{Rie}(\mathbf{G}) = 0$, such that the components of the metric tensor $\mathbf{g}$ can be written as

$$g_{ij} = G_{AB}\, X^A{}_i\, X^B{}_j\,, \tag{10}$$

where $X^A{}_i = \partial X^A / \partial x^i$. Therefore, any riemannian space V_n can be considered as a space embedded in a higher–dimensional flat space E_N, $N \geq n$, with metric tensor $\mathbf{G}$ with components G_{AB} in local coordinates X^A. Then, g_{ij} as given by

(10) is the induced metric tensor. For later convenience we choose Minkowskian coordinates in E_N such that $G_{AB} = \eta_{AB} = \mathrm{diag}(\pm 1, \cdots, \pm 1)$. Denoting the coordinates X^A by $\mathbf{X}$ we rewrite (10) as

$$g_{ij} = \mathbf{X}_i \cdot \mathbf{X}_j \,, \tag{11}$$

where $\mathbf{X}_i = \partial \mathbf{X}/\partial x^i$ and the dot '$\cdot$' denotes the inner product with η_{AB}.

An important object in extrinsic riemannian geometry is the Gauss tensor, which is given by

$$\nabla_{ij}\mathbf{X} = \mathbf{X}_{ij} - \Gamma^k{}_{ij}\, \mathbf{X}_k \,, \tag{12}$$

where $\mathbf{X}_{ij} = \partial^2 \mathbf{X}/\partial x^i \partial x^j$ and $\nabla_{ij} = \nabla_i \nabla_j$ are second–order covariant derivatives. The Gauss tensor satisfies the important identity

$$\nabla_{ij}\mathbf{X} \cdot \nabla_k \mathbf{X} \equiv 0 \,, \tag{13}$$

where $\nabla_k \mathbf{X} = \mathbf{X}_k$. Identity (13) is used extensively in the rest of this work. The completely covariant Riemann tensor (6) is given by

$$R_{ijk\ell} = \nabla_{ik}\mathbf{X} \cdot \nabla_{j\ell}\mathbf{X} - \nabla_{i\ell}\mathbf{X} \cdot \nabla_{jk}\mathbf{X} \,. \tag{14}$$

This is the Gauss equation of the embedding.

2.2. Geometric flows

In 1982 Hamilton [11] introduced the Ricci flow, which is a differential equation generalizing some features of the heat equation for the components of a positive definite metric tensor. The Ricci flow is given by

$$\partial_t g_{ij} = -2\, R_{ij}(\mathbf{g}) \,. \tag{15}$$

Due to the minus sign, spaces are contracted in the direction of positive Ricci curvature, while are expanded in the direction of negative Ricci curvature. The scalar curvature evolves according to

$$\partial_t R = \nabla^2 R + 2\, R_{ij}\, R^{ij} \,, \tag{16}$$

where $\nabla^2 = g^{ij} \nabla_{ij}$ is the Laplacian and $R_{ij} R^{ij} \geq 0$. When the initial data are close enough to that of a sphere, the Ricci flow makes a riemannian space to evolve towards stationary solutions which are constant curvature configurations. Furthermore, spaces with positive curvature keep on having positive curvature.

The Ricci flow (16) was initially developed for a three–dimensional space having in mind its application in the demostration of the Thurston geometrisation conjecture [20] and, consequently, the resolution of the Poincaré conjecture [15, 16, 17]. In three dimensions several simplifications are possible, however, it must be observed that the Ricci flow can be defined in any dimension.

One can also consider extrinsic flows closer in spirit to what one has in mind when picturing evolving surfaces. In the extrinsic approach to riemannian geometry the basic geometrical objects are the embedding functions. Therefore, it would be natural to describe the evolution of embedded spaces in term of these functions. In analogy with the heat equation of thermodynamics one considers the mean curvature flow [12, 21]

$$\partial_t \mathbf{X} = \nabla^2 \mathbf{X}\,, \tag{17}$$

where now the Laplacian is with respect to the induced metric tensor $\mathbf{g}$ as given by (11). The stationary solution of equation (17), $\partial_t \mathbf{X} = 0$, is given by the condition $\nabla^2 \mathbf{X} = 0$, which corresponds to minimal surfaces [5, 19].

The mean curvature flow has been widely studied in the literature [12, 21]. For example, the influence of the signature of the ambient space has been considered in [4]. However, our forthcoming analysis is new and different. In the first place, Huisken [12] considers only n–dimensional surfaces smoothly embedded in R^{n+1}. We are instead considering n–dimensional spaces embedded in some R^N, with N sufficiently large as to guarantee that the embedding theorems hold, locally and, if necessary, also globally. This is necessary because a smooth surface, which can be embedded locally in some R^N, may develop singularities, and in this case a larger dimensionality would be necessary to guarantee that the global embedding theorems hold.

3. Evolution of curvature tensors

The evolution of intrinsic quantities under the mean curvature flow is obtained just by combining (17) with the corresponding definitions, (11) and (14). For the metric tensor we obtain

$$\partial_t g_{ij} = -2\,\nabla^2 \mathbf{X}\,\cdot\,\nabla_{ij}\mathbf{X}\,. \tag{18}$$

This equation was considered by Huisken [12], but only in the case of an n–dimensional surface embedded in R^{n+1}, while we are considering the general case of an n dimensional surface embedded in R^N with N large enough as to guarantee that the embedding theorems, local or global, as necessary, hold.

Therefore, the evolution of the metric tensor is driven by the Gauss tensor (12). The stationary solutions of equation (18), $\partial_t g_{ij} = 0$, are given by the condition $\nabla^2 \mathbf{X} \cdot \nabla_{ij}\mathbf{X} = 0$. Two possible solutions are: $\nabla^2 \mathbf{X} = 0$ which corresponds to minimal surfaces as above, and $\nabla_{ij}\mathbf{X} = 0$, but this implies, according to (14), a flat space.

Now we consider the evolution of the Riemann tensor, the Ricci tensor and the scalar curvature. The time derivative of the Riemann tensor can be written, using (6), completely in terms of derivatives of the metric tensor. Then, replacing from (18) we would obtain the evolution equation for the Riemann tensor induced by the mean curvature flow. However, we can also combine (12) and (17) which is the way in which we proceed now.

The time derivative of the Gauss tensor is given by

$$\begin{aligned}\partial_t \nabla_{ij}\mathbf{X} &= \partial_t \left(\partial_{ij}\mathbf{X} - \Gamma^k{}_{ij}\,\partial_k\mathbf{X}\right)\\ &= \partial_t\partial_{ij}\mathbf{X} - \Gamma^k{}_{ij}\,\partial_t\partial_k\mathbf{X} - \partial_t\Gamma^k{}_{ij}\,\partial_k\mathbf{X}\\ &= \partial_{ij}\partial_t\mathbf{X} - \Gamma^k{}_{ij}\,\partial_k\partial_t\mathbf{X} - \partial_t\Gamma^k{}_{ij}\,\partial_k\mathbf{X}\\ &= \nabla_{ij}\partial_t\mathbf{X} - \partial_t\Gamma^k{}_{ij}\,\partial_k\mathbf{X}\\ &= \nabla_{ij}\nabla^2\mathbf{X} - \partial_t\Gamma^k{}_{ij}\,\partial_k\mathbf{X}\,. \end{aligned} \tag{19}$$

The time derivative of the Riemann tensor is given by

$$\begin{aligned}\partial_t R_{ijk\ell} &= \partial_t\nabla_{ik}\mathbf{X}\,\cdot\,\nabla_{j\ell}\mathbf{X} + \nabla_{ik}\mathbf{X}\,\cdot\,\partial_t\nabla_{j\ell}\mathbf{X}\\ &\quad - \partial_t\nabla_{i\ell}\mathbf{X}\,\cdot\,\nabla_{jk}\mathbf{X} - \nabla_{i\ell}\mathbf{X}\,\cdot\,\partial_t\nabla_{jk}\mathbf{X}\\ &= \nabla_{ik}\nabla^2\mathbf{X}\,\cdot\,\nabla_{j\ell}\mathbf{X} + \nabla_{ik}\mathbf{X}\,\cdot\,\nabla_{j\ell}\nabla^2\mathbf{X}\\ &\quad - \nabla_{i\ell}\nabla^2\mathbf{X}\,\cdot\,\nabla_{jk}\mathbf{X} - \nabla_{i\ell}\mathbf{X}\,\cdot\,\nabla_{jk}\nabla^2\mathbf{X}\,. \end{aligned} \tag{20}$$

where we have used (19) and the identity (13). We now attempt to write the right–hand side of equation (20) in terms of intrinsic geometrical objects. A look at equation (20) indicates that it is likely that its right–hand side contains terms involving second–order covariant derivatives of the Riemann and Ricci tensors.

In order to proceed it is convenient to introduce a notation more adequate for indices manipulation. We write the expression above as

$$\begin{aligned}\partial_t R_{ijk\ell} &= \nabla_{ik\bullet\bullet}\mathbf{X}\,\cdot\,\nabla_{j\ell}\mathbf{X} + \nabla_{ik}\mathbf{X}\,\cdot\,\nabla_{j\ell\bullet\bullet}\mathbf{X}\\ &\quad - \nabla_{i\ell\bullet\bullet}\mathbf{X}\,\cdot\,\nabla_{jk}\mathbf{X} - \nabla_{i\ell}\mathbf{X}\,\cdot\,\nabla_{jk\bullet\bullet}\mathbf{X}\,, \end{aligned} \tag{21}$$

where $\bullet\bullet$ means that these two indices are contracted with the metric tensor; there is no ambiguity in this notation since the metric tensor and the covariant derivative commute, equation (3).

Next we consider a series of identities which allow to interchange indices in several relevant expressions. The Ricci identity for $\nabla_k\mathbf{X}$, is given by

$$\nabla_{ijk}\mathbf{X} - \nabla_{jik}\mathbf{X} = -R^m{}_{kij}\,\nabla_m\mathbf{X}\,. \tag{22}$$

Contracting this equation with $\nabla_{ab}\mathbf{X}$ we obtain

$$\nabla_{ijk}\mathbf{X}\,\cdot\,\nabla_{ab}\mathbf{X} - \nabla_{jik}\mathbf{X}\,\cdot\,\nabla_{ab}\mathbf{X} \equiv 0\,, \tag{23}$$

This relation allows to interchange indices in expressions of the type $\nabla^3\mathbf{X}\cdot\nabla^3\mathbf{X}$; for example

$$\nabla_{abc}\mathbf{X}\,\cdot\,\nabla_{ijk}\mathbf{X} = \nabla_{bac}\mathbf{X}\,\cdot\,\nabla_{ijk}\mathbf{X} - R^d{}_{cba}\,\nabla_{id}\mathbf{X}\,\cdot\,\nabla_{jk}\mathbf{X}\,. \tag{24}$$

where we have used (13).

The Bianchi identity for $\nabla_{k\ell}\mathbf{X}$ is given by

$$\nabla_{ijk\ell}\mathbf{X} - \nabla_{jik\ell}\mathbf{X} = -R^m{}_{kij}\,\nabla_{m\ell}\mathbf{X} - R^m{}_{\ell ij}\,\nabla_{km}\mathbf{X}\,. \tag{25}$$

A similar equation is obtained by considering the covariant derivative of equation (22), namely

$$\nabla_{\ell ijk}\mathbf{X} - \nabla_{\ell jik}\mathbf{X} = -R^m{}_{kij}\,\nabla_{\ell m}\mathbf{X} - \nabla_\ell R^m{}_{kij}\,\nabla_m\mathbf{X}\,. \tag{26}$$

These last two expressions allow to interchange indices in expressions of the type $\nabla^4\mathbf{X} = \nabla_{k\ell ij}\mathbf{X}$. Indices 3 and 4 can be interchanged without any difficulty, indices 2 and 3 can be interchanged using (26), and indices 1 and 2 can be interchanged using (25). For instance, we obtain

$$\begin{aligned}\nabla_{\bullet\bullet ik}\mathbf{X} = \nabla_{ik\bullet\bullet}\mathbf{X}&\\ &+ [R^m{}_i\,\nabla_{km}\mathbf{X} + R^m{}_k\,\nabla_{im}\mathbf{X}] - (R^m{}_{k\bullet i} + R^m{}_{i\bullet k})\,\nabla_{\bullet m}\mathbf{X}\\ &- \frac{1}{2}\,[\nabla_i R^m{}_{\bullet\bullet k} + \nabla_k R^m{}_{\bullet\bullet i} + \nabla_\bullet\,(R^m{}_{k\bullet i} + R^m{}_{i\bullet k})]\,\nabla_m\mathbf{X}\,.\end{aligned} \tag{27}$$

In the previous expression we have taken care of keeping this expression symmetric in i and k. Contracting with $\nabla_{j\ell}\mathbf{X}$ we obtain

$$\begin{aligned}\nabla_{\bullet\bullet ik}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X} = \nabla_{ik\bullet\bullet}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X}&\\ &+ R^m{}_k\,\nabla_{im}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X} + R^m{}_i\,\nabla_{km}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X}\\ &- R^m{}_{k\bullet i}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X} - R^m{}_{i\bullet k}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X}\,.\end{aligned} \tag{28}$$

The right–hand side of equation (20) has the same symmetries as the Riemann tensor. The only terms, involving second–order covariant derivatives of the Riemann and Ricci tensors, with the proper symmetries are

$$\begin{aligned}A_{ijk\ell} &= \nabla^2 R_{ijk\ell}\,,\\ B_{ijk\ell} &= \nabla_{\bullet i}R_{k\ell j\bullet} - \nabla_{\bullet j}R_{k\ell i\bullet} + \nabla_{\bullet k}R_{ij\ell\bullet} - \nabla_{\bullet\ell}R_{ijk\bullet}\,,\\ C_{ijk\ell} &= \nabla_{ik}R_{\ell j} - \nabla_{jk}R_{\ell i} - \nabla_{i\ell}R_{kj} + \nabla_{j\ell}R_{ki}\\ &\quad + \nabla_{ki}R_{j\ell} - \nabla_{\ell i}R_{jk} - \nabla_{kj}R_{i\ell} + \nabla_{\ell j}R_{ik}\,.\end{aligned} \tag{29}$$

However, using the Bianchi identity (9) and the Ricci identity (5) it can be shown that the second term coincides with the first one. Therefore, we can consider only the first and the third terms.

The first term in (29) can be rewritten, using (28), as

$$\begin{aligned}\nabla^2 R_{ijk\ell} = {}& \nabla_{ik\bullet\bullet}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X} + \nabla_{j\ell\bullet\bullet}\mathbf{X}\cdot\nabla_{ik}\mathbf{X}\\
&- \nabla_{jk\bullet\bullet}\mathbf{X}\cdot\nabla_{i\ell}\mathbf{X} - \nabla_{i\ell\bullet\bullet}\mathbf{X}\cdot\nabla_{jk}\mathbf{X}\\
&- R^m{}_{k\bullet i}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X} - R^m{}_{i\bullet k}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X}\\
&+ R^m{}_k\,\nabla_{im}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X} + R^m{}_i\,\nabla_{km}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X}\\
&+ R^m{}_{k\bullet j}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{i\ell}\mathbf{X} + R^m{}_{j\bullet k}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{i\ell}\mathbf{X}\\
&- R^m{}_k\,\nabla_{jm}\mathbf{X}\cdot\nabla_{i\ell}\mathbf{X} - R^m{}_j\,\nabla_{km}\mathbf{X}\cdot\nabla_{i\ell}\mathbf{X}\\
&+ R^m{}_{\ell\bullet i}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{jk}\mathbf{X} + R^m{}_{i\bullet\ell}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{jk}\mathbf{X}\\
&- R^m{}_\ell\,\nabla_{im}\mathbf{X}\cdot\nabla_{jk}\mathbf{X} - R^m{}_i\,\nabla_{\ell m}\mathbf{X}\cdot\nabla_{jk}\mathbf{X}\\
&- R^m{}_{\ell\bullet j}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{ik}\mathbf{X} - R^m{}_{j\bullet\ell}\,\nabla_{m\bullet}\mathbf{X}\cdot\nabla_{ik}\mathbf{X}\\
&+ R^m{}_\ell\,\nabla_{jm}\mathbf{X}\cdot\nabla_{ik}\mathbf{X} + R^m{}_j\,\nabla_{\ell m}\mathbf{X}\cdot\nabla_{ik}\mathbf{X}\\
&+ 2\,\nabla_{\bullet ik}\mathbf{X}\cdot\nabla_{\bullet j\ell}\mathbf{X} - 2\,\nabla_{\bullet jk}\mathbf{X}\cdot\nabla_{\bullet i\ell}\mathbf{X}\,, \end{aligned}\tag{30}$$

while the third term can be rewritten as

$$\begin{aligned}\nabla_{[i|[k}R_{\ell]|j]} = {}& \nabla_{ik\bullet\bullet}\mathbf{X}\cdot\nabla_{j\ell}\mathbf{X} + \nabla_{j\ell\bullet\bullet}\mathbf{X}\cdot\nabla_{ik}\mathbf{X}\\
&- \nabla_{jk\bullet\bullet}\mathbf{X}\cdot\nabla_{i\ell}\mathbf{X} - \nabla_{i\ell\bullet\bullet}\mathbf{X}\cdot\nabla_{jk}\mathbf{X}\\
&+ 4\,\nabla_{\bullet j\ell}\mathbf{X}\cdot\nabla_{\bullet ki}\mathbf{X} - 4\,\nabla_{\bullet i\ell}\mathbf{X}\cdot\nabla_{\bullet kj}\mathbf{X}\\
&- R^m{}_{j\ell k}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet\bullet}\mathbf{X} + R^m{}_{i\ell k}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet\bullet}\mathbf{X}\\
&+ R^m{}_{jk\ell}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet\bullet}\mathbf{X} - R^m{}_{ik\ell}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet\bullet}\mathbf{X}\\
&- R^m{}_{\ell\bullet j}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{ki}\mathbf{X} - R^m{}_{k\bullet i}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{\ell j}\mathbf{X}\\
&- R^m{}_{i\bullet k}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet\ell}\mathbf{X} - R^m{}_{j\bullet\ell}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet k}\mathbf{X}\\
&+ R^m{}_{\ell\bullet i}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{kj}\mathbf{X} + R^m{}_{k\bullet j}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{\ell i}\mathbf{X}\\
&+ R^m{}_{j\bullet k}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet\ell}\mathbf{X} + R^m{}_{i\bullet\ell}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet k}\mathbf{X}\\
&+ R^m{}_{k\bullet j}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{\ell i}\mathbf{X} + R^m{}_{\ell\bullet i}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{kj}\mathbf{X}\\
&+ R^m{}_{i\bullet\ell}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet k}\mathbf{X} + R^m{}_{j\bullet k}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet\ell}\mathbf{X}\\
&- R^m{}_{k\bullet i}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{\ell j}\mathbf{X} - R^m{}_{\ell\bullet j}\,\nabla_{\bullet m}\mathbf{X}\cdot\nabla_{ki}\mathbf{X}\\
&- R^m{}_{j\bullet\ell}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet k}\mathbf{X} - R^m{}_{i\bullet k}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet\ell}\mathbf{X}\\
&- R^m{}_{jk\ell}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet\bullet}\mathbf{X} + R^m{}_{ik\ell}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet\bullet}\mathbf{X}\\
&- R^m{}_{\bullet k\ell}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet i}\mathbf{X} + R^m{}_{\bullet k\ell}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet j}\mathbf{X}\\
&+ R^m{}_{\bullet\ell k}\,\nabla_{jm}\mathbf{X}\cdot\nabla_{\bullet i}\mathbf{X} - R^m{}_{\bullet\ell k}\,\nabla_{im}\mathbf{X}\cdot\nabla_{\bullet j}\mathbf{X}\,. \end{aligned}\tag{31}$$

Now, we can combine equations (21), (30) and (31) to eliminate all terms of the form $\nabla^4\mathbf{X}\cdot\nabla^2\mathbf{X}$ and $\nabla^3\mathbf{X}\cdot\nabla^3\mathbf{X}$. The result is

$$\begin{aligned} 2\partial_t R_{ijk\ell} = 4\,\nabla^2 R_{ijk\ell} &- [\nabla_{ik}R_{\ell j} - \nabla_{jk}R_{\ell i} - \nabla_{i\ell}R_{kj} + \nabla_{j\ell}R_{ki} \\ &+\nabla_{ki}R_{j\ell} - \nabla_{\ell i}R_{jk} - \nabla_{kj}R_{i\ell} + \nabla_{\ell j}R_{ik}] \\ &- 4\,[R_{mjk\ell}\,R^m{}_i + R_{imk\ell}\,R^m{}_j + R_{ijm\ell}\,R^m{}_k + R_{ijkm}\,R^m{}_\ell] \\ &+ 8\,R^m{}_k{}^n{}_i\,R_{mjn\ell} - 8\,R^m{}_k{}^n{}_j\,R_{min\ell} + 4\,R^{mn}{}_{ij}\,R_{mnk\ell} \\ &+ R^m{}_{jk\ell}\,\partial_t g_{im} - R^m{}_{ik\ell}\,\partial_t g_{jm} + R^m{}_{\ell ij}\,\partial_t g_{km} - R^m{}_{kij}\,\partial_t g_{\ell m}\,. \end{aligned} \tag{32}$$

The time derivatives of the Ricci tensor and of the scalar curvature are given by

$$\begin{aligned} 2\,\partial_t R_{j\ell} = 2\,\nabla^2 R_{j\ell} &+ 2\,[\nabla_{mj}R^m{}_\ell + \nabla_{m\ell}R^m{}_j] - 2\,\nabla_{j\ell}R \\ &- 8\,R_{mj}\,R^m{}_\ell - 4\,R_{mnpj}\,R^{mnp}{}_\ell \\ &+ [R^m{}_j\,\partial_t g_{\ell m} + R^m{}_\ell\,\partial_t g_{jm}]\,, \end{aligned} \tag{33}$$

$$\partial_t R = \nabla^2 R - 4\,R_{ij}\,R^{ij} - 2\,R_{ijk\ell}\,R^{ijk\ell}\,. \tag{34}$$

The last equation, (34), is similar to the Ricci flow (16), except for the presence of minus signs in the last terms. For positive definite metric tensors we have $R_{ij}R^{ij} > 0$ and $R_{ijk\ell}R^{ijk\ell} > 0$. Therefore, if we consider $S = -R$ as our unknown function, then we have that negative, rather than positive, curvature is preserved under this flow.

Equation (34) can be rewritten in terms of extrinsic quantities using, for example, the Gauss equation (14). However, the resulting expression is not very illuminating, except for an embedding in R^{n+1}, which is the case which was considered by Huisken [12].

Equations (32), (33) and (34) are valid in any dimension. However, for two and three dimensions the Riemann tensor acquires particularly simple forms and equation (34) reduces to

$$\partial_t R^{(2)} = \nabla^2 R^{(2)} - 4\left[R^{(2)}\right]^2. \tag{35}$$

$$\partial_t R^{(3)} = \nabla^2 R^{(3)} - 12\left[\mathrm{Ric}^{(3)}\right]^2 + 2\left[R^{(3)}\right]^2. \tag{36}$$

In dimension $n \geq 4$ we obtain

$$\begin{aligned} \partial_t R^{(n)} = \nabla^2 R^{(n)} &- \frac{4n}{(n-2)}\left[\mathrm{Ric}^{(n)}\right]^2 \\ &+ \frac{4}{(n-2)(n-1)}\left[R^{(n)}\right]^2 - 2\left[\mathrm{Weyl}^{(n)}\right]^2. \end{aligned} \tag{37}$$

4. Conclusions

The evolution induced by the mean curvature flow on the curvature tensor has several interesting properties. Firstly, all developments are valid for any dimension and for any signature. Secondly, for positive definite metric tensors, the evolution equation for the scalar curvature, (34), has the same generic properties as the Ricci flow. The evolution is towards constant curvature spaces and negative, rather than positive, curvature is preserved.

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