Revista Colombiana de Matemáticas
Volumen 44(2010)1, páginas 15-21

Rigidity of the Stable Norm on Tori

Rigidez de la norma estable sobre toros

Osvaldo Osuna[a]

Universidad Michoacana, Morelia, México

Abstract. Given a closed, orientable Riemannian manifold, we study the stable norm on the real homology groups. In particular, for $n \geq 2$ we prove that a Riemannian n-torus, which has the same stable norms as a flat n-torus on the first and $n-1$ homology groups, is in fact isometric to the flat torus.

Key words and phrases. Stable norm, p-norm, Poincaré duality.

2000 Mathematics Subject Classification. 53C23, 53D25, 53C24.

Resumen. Dada una variedad Riemanniana, cerrada y orientable, estudiamos la norma estable sobre sus grupos de homología real. En particular, para $n \geq 2$ demostramos que si un n-toro Riemanniano tiene normas estables iguales a las normas estables de un n-toro plano sobre el primer y $n-1$ grupos de homología; entonces es isométrico a dicho toro plano.

Palabras y frases clave. Norma estable, p-norma, dualidad de Poincaré.

1. Introduction and Results

The real homology groups of a compact Riemannian manifold (M, g) are naturally endowed with a norm introduced by Federer ([7]). More precisely, we define the volume $\mathrm{vol}_k(\sigma)$ of a Lipschitz k-simplex $\sigma : \Delta^k \to M$ as the integral over the k-simplex Δ^k of the volume form of the pullback $\sigma^*(g)$. Now, given $h \in H_k(M, \mathbb{R})$, we take

$$|h|_{s,g} := \inf \left\{ \sum_i |r_i| \mathrm{vol}_k(\sigma_i) \right\}, \tag{1}$$

[a]The author was partially supported by C.I.C.-UMSNH. and thanks the referee for various comments that helped improving the paper.

where σ_i are k-simplexes, $r_i \in \mathbb{R}$, and $\sum r_i\sigma_i$ is a real Lipschitz cycle representing h. Note that this function defines a norm on $H_k(M,\mathbb{R})$ which we call the stable norm, and by duality it induces a stable norm $|h|^*_{s,g}$ on $H^k(M,\mathbb{R})$.

The stable norm has been studied to some extent (see [3], [6], [11], [10], [12], [7], [14], [9], [1], [13], [5], [4]) but important questions are still open. The goal of this note is to study the natural question:

> Let g_0 and g be two Riemannian metrics on $\mathbb{T}^n$, $n \geq 2$. Suppose that g_0 is flat and that the stable norms associated with g_0 and g on the homology group $H_1(\mathbb{T}^n,\mathbb{R})$ are equal. Are g_0 and g isometric?

This question is answered affirmatively under the additional assumption that the stable norms coincide on the homology group $H_{n-1}(\mathbb{T}^n,\mathbb{R})$. From now, unless otherwise stated, we will suppose that (M,g) is orientable, $\mathrm{vol}_g(M) = 1$ and $n \geq 2$. Our main result is the following one:

Theorem 1. *Let g_0 and g be metrics on $\mathbb{T}^n$. If g_0 is flat and the stable norms of g_0 and g on $H_1(\mathbb{T}^n,\mathbb{R})$ and $H_{n-1}(\mathbb{T}^n,\mathbb{R})$ are equal, then g_0 and g are isometric.*

From Theorem 1, we can recover a result of Bangert [2] (Theorem 6.1), indeed we have

Corollary 1. *Suppose that a Riemannian metric g on the 2-torus $\mathbb{T}^2$ has the same stable norm on $H_1(\mathbb{T}^2,\mathbb{R})$ as a flat metric g_0, then g_0 and g are isometric.*

Proof. We have $H_{2-1}(\mathbb{T}^2,\mathbb{R}) = H_1(\mathbb{T}^2,\mathbb{R})$, then the corollary follows from the above theorem. ☑

There is extensive literature with results on the stable norm in the case of surfaces, our main theorem gives in particular a result for dimension > 2 where little is known (with valuable exceptions see [1], [5], [9]), the methods for proving our results are an adaptation of some ideas that were used for surfaces in [15]. An key point in our arguments is to study certain relationships between the stable norm and Poincaré duality with the underlying geometry of the manifold, which is of independent interest.

2. Preliminaries

Recall a pairing between two finite dimensional vector spaces

$$\langle\cdot,\cdot\rangle : W \times V \longrightarrow \mathbb{R}$$

is non-degenerate if $\langle w, v\rangle = 0, \forall w \in W \Rightarrow v = 0$ and $\langle w, v\rangle = 0, \forall v \in V \Rightarrow w = 0$, from linear algebra we have $W^* \cong V$.

Recall that for an oriented, closed manifold M, the exterior product on forms induces a bilinear map

$$H^k(M,\mathbb{R}) \times H^{n-k}(M,\mathbb{R}) \longrightarrow H^n(M,\mathbb{R}).$$

Using the integral we obtain a bilinear, non-degenerate pairing

$$H^k(M,\mathbb{R}) \times H^{n-k}(M,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad \big([\omega_1],[\omega_2]\big) := \int_M \omega_1 \wedge \omega_2,$$

and from the above observation $H^k(M,\mathbb{R})^* \cong H^{n-k}(M,\mathbb{R})$, which in turn defines a linear map $P : H_k(M,\mathbb{R}) \to H^{n-k}(M,\mathbb{R})$ (using the non-degenerate Kronecker pairing) which is called *the operator of the Poincaré duality.*

We briefly recall an alternative definition of the stable norm which is more adequate for our objectives. It is based on the notion of comass of a k-form. Denote by $\Omega^k(M)$, the space of closed k-forms on (M,g) a closed oriented Riemannian manifold, and by dvol the volume form induced by the Riemannian metric g. For $\omega \in \Omega^k(M)$ and $1 \le p \le \infty$, we define the L^p-norm as

$$|\omega|_{p,g} = \begin{cases} \big(\int_M \|\omega_x\|_g^p \,\mathrm{dvol}(x)\big)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty \\ \max\big\{\|\omega_x\|_g \mid x \in M\big\}, & \text{if } p = \infty, \end{cases} \tag{2}$$

where

$$\|\omega_x\|_g := \max\big\{\omega_x(v_1,\dots,v_k) \mid |v_i|_g \le 1, 1 \le i \le k\big\} \tag{3}$$

is the called *comass norm* of the corresponding multilinear map ω_x on T_xM.

Now, for $\alpha \in H^k(M,\mathbb{R})$ and $1 \le p \le \infty$, we consider

$$\|\alpha\|^*_{p,g} := \inf\big\{|\omega|_{p,g} \mid \text{is a closed } k\text{-form representing } \alpha\big\}. \tag{4}$$

On the other hand, considering the integration of closed forms over cycles, we have the non-degenerate Kronecker pairing

$$\langle\cdot,\cdot\rangle : H_k(M,\mathbb{R}) \times H^k(M,\mathbb{R}) \longrightarrow \mathbb{R}.$$

So, we can define the dual norm $\|\cdot\|_{p,g}$ on $H_k(M,\mathbb{R})$, more precisely, given $h \in H_k(M,\mathbb{R})$ and $1 \le p \le \infty$, we take

$$\|h\|_{p,g} := \sup\big\{\langle h,\alpha\rangle \mid \alpha \in H^k(M,\mathbb{R}),\ \|\alpha\|^*_{p,g} \le 1\big\}. \tag{5}$$

As was mentioned (see [7], 4.10) we have $\|\cdot\|_{\infty,g} = |\cdot|_{s,g}$ and therefore $|\cdot|^*_{s,g} = \|\cdot\|^*_{\infty,g}$.

Now we consider the L^2-product on $H^k(M,\mathbb{R})$ defined as

$$\langle\omega,\eta\rangle = \int_M \omega \wedge *\eta,$$

where $*$ is the Hodge star operator of the metric g and we denote by $\|\omega\|_{L^2,g}$ the norm induced. Moreover, if $p = 2$ and $k \in \{1, n-1\}$ we have $\|\cdot\|_{L^2,g} = \|\cdot\|^*_{2,g}$. The importance of this result is that it allows us to use the Hodge theory on harmonic forms [8]. For instance, if $p = 2$ Hodge theory implies that every cohomology class $\alpha \in H^k(M, \mathbb{R})$ contains a unique harmonic form ω representing α; moreover, if $k \in \{1, n-1\}$, then this harmonic form ω is characterized by the equality

$$|\omega|_{2,g} = \|\alpha\|^*_{2,g},$$

and the operator of the Poincaré duality $P : H_k(M, \mathbb{R}) \to H^{n-k}(M, \mathbb{R})$ is an L^2-isometry. In the next section, we will analyze some results when P is an $|\cdot|_{s,g}$-isometry.

3. Proofs and Consequences

Before going to the proof of Theorem 1, we will establish a result with respect to the norm of the operator of the Poincaré duality, which is of independent interest.

Lemma 1. *Let $(\mathbb{T}^n, g)$ be Riemannian torus. The operator of the Poincaré duality $P : \big(H_1(\mathbb{T}^n, \mathbb{R}), |\cdot|_{s,g}\big) \to \big(H^{n-1}(\mathbb{T}^n, \mathbb{R}), |\cdot|^*_{s,g}\big)$ is an isometry if and only if $|\cdot|^*_{s,g} = \|\cdot\|^*_{2,g}$ in $H^l(\mathbb{T}^n, \mathbb{R})$ for $l = 1, n-1$.*

Proof. First we will prove that if $P : H_1(\mathbb{T}^n, \mathbb{R}) \to H^{n-1}(\mathbb{T}^n, \mathbb{R})$ is an isometry with respect to the stable norms, then $|\cdot|^*_{s,g} = \|\cdot\|^*_{2,g}$.

Indeed, given $h \in H_1(\mathbb{T}^n, \mathbb{R})$, by hypothesis $|Ph|^*_{s,g} = |h|_{s,g}$. Now, as it was mentioned P is an L^2-isometry, i.e., $\|Ph\|^*_{2,g} = \|h\|_{2,g}$.

Using the Hölder's inequality, for $\alpha \in H^k(\mathbb{T}^n, \mathbb{R})$ and $1 \le r \le t \le \infty$ we have

$$\|\alpha\|^*_{r,g} \le \|\alpha\|^*_{t,g} \tag{6}$$

Therefore

$$\{\langle h, \alpha\rangle \mid \alpha \in H^k(\mathbb{T}^n, \mathbb{R}), \|\alpha\|^*_{2,g} \le 1\} \supseteq \{\langle h, \alpha\rangle \mid \alpha \in H^k(\mathbb{T}^n, \mathbb{R}), \|\alpha\|^*_{\infty,g} \le 1\}. \tag{7}$$

By duality this yields

$$\|h\|_{2,g} \ge \|h\|_{\infty,g}.$$

Now, combining the above inequalities we have

$$|h|_{s,g} \equiv \|h\|_{\infty,g} \le \|h\|_{2,g} = \|Ph\|^*_{2,g} \le \|Ph\|^*_{\infty,g} \equiv |Ph|^*_{s,g}. \tag{8}$$

Therefore $\|Ph\|^*_{2,g} = |Ph|^*_{s,g}$. This finishes the proof in one direction.

Now, if

$$|Ph|^*_{s,g} = \|Ph\|^*_{2,g}, \forall h \in H_l(\mathbb{T}^n, \mathbb{R}), \quad l = 1, n-1,$$

then the converse follows from the fact that P is an L^2-isometry. The proof of the lemma is complete. ☑

Proof of the Theorem 1:

Proof. First note that:

***Remark* 1.** if $(\mathbb{T}^n, \rho)$ is flat, then any differential k-form on $\mathbb{T}^n$ can be written as

$$\eta = \sum_{i_1,\ldots,i_k} \eta_{i_1,\ldots,i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Since $\mathbb{T}^n$ is flat, then η is harmonic if and only if the functions $\eta_{i_1,\ldots,i_k}$ are harmonic, and therefore constant. So the comass norm of η_x is constant for all $x \in \mathbb{T}^n$, then from the definition of L^p-norm, and the hypothesis $\mathrm{vol}_\rho(\mathbb{T}^n) = 1$, we have

$$\big\|[\eta]\big\|^*_{p,\rho} = \big\|[\eta]\big\|^*_{\infty,\rho}, \qquad \forall [\eta] \in H^k(\mathbb{T}^n, \mathbb{R}),\ \forall p \geq 1.$$

So, by the above lemma $P : H_1(\mathbb{T}^n, \mathbb{R}, |\cdot|_{s,\rho}) \to H^{n-1}(\mathbb{T}^n, \mathbb{R}, |\cdot|^*_{s,\rho})$ is an isometry.

***Remark* 2.** The converse also is valid i.e., if for every cohomology class on $H^1(\mathbb{T}^n, \mathbb{R})$ the L^2 norm coincides with its stable norm, then $(\mathbb{T}^n, \rho)$ is flat, see [16], Corollary 2 or [13], Proposition 5, for a proof

Now, using Remark 1 and Lemma 1, the operator of the Poincaré duality P is an isometry with respect to the stable norm of g_0, thus

$$1 = |P|_{g_0} := \sup_{h\neq 0} \left\{ \frac{|Ph|^*_{s,g_0}}{|h|_{s,g_0}} \right\} = \sup_{h\neq 0} \left\{ \frac{|Ph|^*_{s,g}}{|h|_{s,g}} \right\} =: |P|_g,$$

the second relation follows from the equality of the stable norms, therefore P is an $|\cdot|_{s,g}$-isometry, then by applying Lemma 1 and Remark 2 g is flat.

On the other hand, taking an orthogonal basis of harmonic 1-forms $\omega_1, \ldots, \omega_n$ for the metric g_0, this is an orthogonal basis of harmonic 1-forms for the flat metric g, moreover the comass norms satisfy

$$\|\omega_{i,x}\|_g = \|\omega_{i,x}\|_{g_0}$$

for $i = 1, \ldots, n, \quad \forall x \in \mathbb{T}^n$.

If we denote by $\big\{X_i(x) := (x, u_i)\big\}$ the vector fields induced by $\{\omega_i\}$ via g_0 and likewise we take vector fields $\big\{Y_i(x) := (x, v_i)\big\}$ induced by $\{\omega_i\}$ via g, then the function $h : (\mathbb{T}^n, g_0) \to (\mathbb{T}^n, g)$ defined as

$$h\left(\sum_{i=1}^{n} x_i u_i\right) = \sum_{i=1}^{n} x_i v_i, \quad (x_i \bmod 1),$$

is an isometry. This completes the proof. ☑

Of course it is of interest to know if it is possible or not to remove the additional condition of the stable norms on $H_{n-1}(\mathbb{T}^n, g_0)$ in this proposition.

References

[1] I. Babenko and F. Balacheff, *Sur la forme de la boule unité de la norme stable unidimensionnelle*, Manuscripta Math. **119** (2006), no. 3, 347–358 (fr).

[2] V. Bangert, *Geodesic rays, Busemann Functions and Monotone Twist Maps*, Calc. Var. Partial Differential Equations **2** (1994), no. 1, 49–63.

[3] ______, *Minimal Measures and Minimizing Closed Normal One-Currents*, GAFA **9** (1999), 413–427.

[4] V. Bangert and M. Katz, *Stable Systolic Inequalities and Cohomology Products*, Comm. Pure Appl. Math. **56** (2003), 979–997.

[5] ______, *An Optimal Loewner-Type Systolic Inequality and Harmonic One-Forms of Constant Norm*, Comm. Anal. Geom. **12** (2004), no. 3, 703–732.

[6] D. Y. Burago and S. Ivanov, *Riemannian Tori without Conjugate Points are Flat*, GAFA **4** (1994), no. 3, 259–269.

[7] H. Federer, *Real Flat Chains, Cochains and Variational Problems*, Indiana Univ. J. **86** (1964), 351–407.

[8] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, 1994.

[9] M. Jotz, *Hedlund Metrics and the Stable Norm*, Differential Geometry and its Applications **27** (2009), no. 4, 543–550.

[10] R. Mañé, *On the Minimizing Measures of the Lagrangian Dynamical Systems*, Non Linearity **5** (1992), no. 3, 623–638.

[11] D. Massart, *Stable Norms of Surfaces: Local Structure of the Unit Ball at Rational Directions*, Geom. Funct. Anal. **7** (1997), 996–1010.

[12] G. McShane and I. Rivin, *A Norm on Homology of Surface and Counting Simple Geodesic*, Int. Math. Res. Not. **2** (1995), 61–69.

[13] P. A. Nagy and C. Vernicos, *The Length of Harmonic Forms on a Compact Riemannian Manifold*, Trans. Amer. Math. Soc. **356** (2004), 2501–2513.

[14] O. Osuna, *Vertices of Mather's Beta Function*, Erg. Th. and Dyn. Syst. **25** (2005), 949–955.

[15] ______, *On the Stable Norm of Surfaces*, Bol. Soc. Mat. Mexicana **12** (2006), 75–80.

[16] G. Paternain, *Schrodinger Operators with Magnetic Fields and Minimal Action Functional*, Israel J. Math. **123** (2001), 1–27.

(Recibido en julio de 2008. Aceptado en abril de 2010)

Instituto de Física y Matemáticas
Universidad Michoacana
Edif. C-3, Cd. Universitaria, C.P. 58040
Morelia, Michoacan
México
e-mail: osvaldo@ifm.umich.mx